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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 74, Number 1, Pages 130–137 (2025)
https://doi.org/10.31801/cfsuasmas.1592998
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: November 28, 2024; Accepted: December 20, 2024

Forced vibrations of a thin viscoelastic shell immersed in fluid under the effect of damping

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Abstract. The plane strain problem for low-frequency forced vibrations of a fluid-loaded thin viscoelastic shell is considered. A small structural damping is incorporated using the concept of a complex Young's modulus. The two-term asymptotic expansion is derived assuming that the structural damping is of the same order as the small thickness of the shell. It is demonstrated that the effect of the structural damping is remarkably greater than that of the radiation damping and the latter can be neglected in the vast majority of the problems.

2020 Mathematics Subject Classification. 74B05, 74B10, 74F10, 34K25
Keywords. Asymptotics, thin shell, complex Young's modulus, viscoelastic

1. Introduction

Structural damping plays a significant role in the dynamic analysis of mechanical systems since it governs the mechanism of energy dissipation which is crucial for various technical applications in civil, mechanical, naval and automotive engineering, e.g. see [1], [2], [3], [4] and references therein. There is a great number of publications on the subject. In particular, the vibrations of viscoelastic fluid-loaded shells were treated in numerical contributions, including [5], [6], [7], [8], [9], [10] to mention a few. At the same time, the asymptotic methods widely spread in the thin shell theory have been mainly applied within the purely elastic framework, e.g., see [11], [12], [13], [14], [15], [16].

The recent asymptotic analysis in [17] and [18] show that the radiation damping of low-frequency resonant vibrations of purely elastic cylindrical shells is remarkably small. It is natural to question, in this case, whether the formulations not taking into consideration structural damping may provide adequate evaluation of dynamic behaviour. This observation motivates to extend the framework of [17] to viscoelastic shells.

In this paper, the viscoelastic properties are incorporated using the simplest model of the structural damping based on the concept of a complex Young's modulus, see [19]. The imaginary part of the latter stands for energy dissipation. It is assumed to be of the same order that the relative thickness of the shell.

Instead of scattering problem tackled in [17], below we deal with a radiation problem. A time-harmonic load is assumed to be specified along the inner surface of the shell, while the outer face is subject to fluid loading. The developed asymptotic procedure is oriented to a coupled fluid-structure interaction problem similar to the above mentioned publications [17], [18], and also [20], studying a flat, fluid-loaded elastic layer. It was noted that for a long time, the asymptotic results for thin-walled bodies with traction free faces were readily adapted for modeling of fluid-structure interaction ignoring, in a sense, the effect of coupling, e.g., see [11].

We expand displacement, stresses and fluid pressure in the Fourier series across the polar angle prior to the asymptotic integration across the shell thickness. A two-term asymptotic solution is derived. As might be expected, a small term corresponding to the structural damping does not appear at leading order. However, it is shown that it is significantly greater than the contribution of the damping caused by radiation. The most important result of the presented analysis is that the latter may usually be neglected.

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2. Statement of the Problem

Consider a thin cylindrical shell with thickness $2h$ and a mid-surface radius R immersed in a compressible fluid for which $\eta = h/R \ll 1$ is a small geometric parameter, see Fig.1. We specify curvilinear coordinates α_2 and α_3 for which $0 \leq \alpha_2 < 2\pi R$ and $-h \leq \alpha_3 \leq h$. The 2D plane strain equations

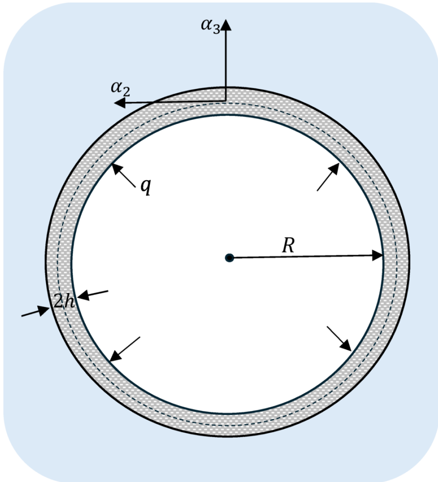

Figure 1. Schematic diagram of a thin cylindrical shell immersed in a fluid.

governing the time-harmonic vibrations of a shell, omitting the factor $\exp(-i\omega t)$, with ω representing the angular frequency and t denoting time, are given by, see [12],

$$\frac{R}{R+\alpha_3}\frac{\partial \sigma_{22}}{\partial \alpha_2} + \frac{\partial \sigma_{32}}{\partial \alpha_3} + \frac{2}{R+\alpha_3}\sigma_{32} + \rho\omega^2 v_2 = 0, \tag{1}$$

$$\frac{R}{R+\alpha_3}\frac{\partial \sigma_{32}}{\partial \alpha_2} + \frac{\partial \sigma_{33}}{\partial \alpha_3} - \frac{1}{R+\alpha_3}\sigma_{22} + \frac{1}{R+\alpha_3}\sigma_{33} + \rho\omega^2 v_3 = 0, \tag{2}$$

where σ_{ij} ($\sigma_{ij} = \sigma_{ji}$) and v_j, $i, j = 2, 3$, are the stresses and displacements, respectively and ρ is the mass density of the shell. The corresponding stress-displacement relations are also presented as

$$\sigma_{22} = \frac{E}{1-\nu^2}\left(\frac{R}{R+\alpha_3}\frac{\partial v_2}{\partial \alpha_2} + \frac{1}{R+\alpha_3}v_3\right) + \frac{\nu}{1-\nu}\sigma_{33}, \tag{3}$$

$$E\frac{\partial v_3}{\partial \alpha_3} = (1-\nu^2)\sigma_{33} - \nu(1+\nu)\sigma_{22}, \tag{4}$$

$$\sigma_{32} = \frac{E}{2(1+\nu)}\left(\frac{R}{R+\alpha_3}\frac{\partial v_3}{\partial \alpha_2} + \frac{\partial v_2}{\partial \alpha_3} - \frac{1}{R+\alpha_3}v_2\right). \tag{5}$$

The mechanical parameters of the considered problem are the Young's modulus E and Poisson's ratio ν. To incorporate the effect of viscosity in the simplest manner, we define the Young's modulus in a complex form, e.g., see [19]

$$E = E_0(1 + i\alpha), \tag{6}$$

where E_0 and α are real constants. The fluid pressure is governed by the 2D Helmholtz equation

$$\Delta p + \frac{\omega^2}{c_f^2}p = 0 \tag{7}$$

where p is fluid pressure and c_f is the wave speed in the fluid. 2D Laplace operator Δ is given by

$$\Delta = \frac{R^2}{(R+\alpha_3)^2}\frac{\partial^2}{\partial \alpha_2^2} + \frac{1}{R+\alpha_3}\frac{\partial}{\partial \alpha_3} + \frac{\partial^2}{\partial \alpha_3^2}. \tag{8}$$

The boundary conditions along the shell faces are given by

$$\sigma_{32} = 0, \qquad \sigma_{33} = q \qquad \text{at} \qquad \alpha_3 = -h, \tag{9}$$

$$\sigma_{32} = 0, \qquad \sigma_{33} = -p, \qquad \text{and} \qquad v_3 = \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial \alpha_3} \qquad \text{at} \qquad \alpha_3 = h, \tag{10}$$

where q is the force applied at the inner surface of the shell and ρ_f is the fluid density.

In the dimensionless coordinates $\theta = \alpha_2/R$ and $\zeta = \alpha_3/h$ ($0 \le \theta < 2\pi$ and $-1 \le \zeta \le 1$ inside the shell or $\zeta > 1$ outside the shell) the displacement and stress components of the shell, the acoustic pressure and the external force can be presented as

$$v_2(\theta, \zeta) = u_2(\zeta)\sin(n\theta), \qquad v_3(\theta, \zeta) = u_3(\zeta)\cos(n\theta), \tag{11}$$

$$\sigma_{22}(\theta, \zeta) = s_{22}(\zeta)\cos(n\theta), \quad \sigma_{32}(\theta, \zeta) = s_{32}(\zeta)\sin(n\theta), \quad \sigma_{33}(\theta, \zeta) = s_{33}(\zeta)\cos(n\theta), \tag{12}$$

$$p(\theta, \zeta) = P(\zeta)\cos(n\theta), \qquad q(\theta, \zeta) = Q(\zeta)\cos(n\theta). \tag{13}$$

3. Scaling

Let us now define the dimensionless equations in the previous section similar to those in [17] setting

$$u_2 = Ru_2^*, \quad u_3 = Ru_3^*, \tag{14}$$

$$s_{22} = E_0\eta s_{22}^*, \quad s_{32} = E_0\eta^2 s_{32}^*, \quad s_{33} = E_0\eta^2 s_{33}^* \tag{15}$$

$$P = E_0\eta^2 P^*, \quad Q = E_0\eta^3 Q^*. \tag{16}$$

where the starred quantities are assumed to be of order unity. In addition, we assume that the viscosity coefficient (6) can be taken as

$$\alpha = \eta\alpha_0^*. \tag{17}$$

We also specify the dimensionless frequency by

$$\Omega = \eta^{-3/2}\omega R\sqrt{\frac{\rho}{E_0}}, \qquad \Omega \sim 1. \tag{18}$$

The fluid pressure, subject to the radiation condition, e.g., see [21], is found from equation (7) and is given by

$$p = p_0 H_n^{(2)}\left(\frac{\omega R(1+\eta\zeta)}{c_f}\right). \tag{19}$$

where $H_n^{(2)}$ is the Hankel function of the second kind, see [22], and p_0 is an unknown constant.

Next, combining boundary conditions $(10)_2$ and $(10)_3$, and substituting (19) there, accounting $(11)_2$, $(12)_3$ and $(13)_1$, we have

$$u_3 + s_{33}\frac{\mathcal{H}}{c_f\rho_f} = 0 \tag{20}$$

where

$$\mathcal{H} = \frac{\left(H_n^{(2)}(z)\right)'}{H_n^{(2)}(z)} \qquad \text{at} \qquad z = \Omega\eta^{3/2}\frac{1}{c_f}\sqrt{\frac{E_0}{\rho}}(1+\eta). \tag{21}$$

Inserting the dimensionless quantities (14), (15) and (17) into the equations of motion (1)–(2) and the relations (3)–(5), we obtain

$$\frac{\partial s_{32}^*}{\partial\zeta} - \frac{n}{1+\eta\zeta}s_{22}^* + \frac{2\eta}{1+\eta\zeta}s_{32}^* + \eta^2\Omega^2 u_2^* = 0, \tag{22}$$

$$\frac{\partial s_{33}^*}{\partial\zeta} + \frac{n\eta}{1+\eta\zeta}s_{32}^* - \frac{1}{1+\eta\zeta}s_{22}^* + \frac{\eta}{1+\eta\zeta}s_{33}^* + \eta^2\Omega^2 u_3^* = 0 \tag{23}$$

and

$$\eta s_{22}^* = \frac{1+i\eta\alpha_0}{1-\nu^2}\frac{1}{1+\eta\zeta}(nu_2^* + u_3^*) + \frac{\nu}{1-\nu}\eta^2 s_{33}^*, \tag{24}$$

$$(1+i\eta\alpha_0)\frac{\partial u_3^*}{\partial\zeta} = (1-\nu^2)\eta^3 s_{33}^* - \nu(1+\nu)\eta^2 s_{22}^*, \tag{25}$$

$$\eta^3 s_{32}^* = \frac{1+i\eta\alpha_0}{2(1+\nu)}\left(\frac{\partial u_2^*}{\partial \zeta} - \frac{\eta}{1+\eta\zeta}(u_2^* + nu_3^*)\right). \tag{26}$$

In addition, we set

$$\mathcal{H} = \eta^{-3/2}\mathcal{H}^*, \tag{27}$$

where, according to [22],

$$\mathcal{H}^* = -\frac{nc_f}{\Omega(1+\eta)}\sqrt{\frac{\rho}{E_0}}\left(1 + \frac{n-2}{4(n-1)}\Omega^2\eta^3(1+\eta)^2\frac{E_0}{c_f^2\rho} + \ldots \right.$$
$$\left. \ldots + i\frac{\pi}{2^{2n-1}c_f^{2n}n\left((n-1)!\right)^2}\Omega^{2n}\eta^{3n}(1+\eta)^{2n}\left(\frac{E_0}{\rho}\right)^n + \ldots\right). \tag{28}$$

The contact conditions (9) and (10) become

$$s_{32}^* = 0, \quad \zeta = \pm 1 \qquad \text{and} \qquad s_{33}^* = \eta Q^*, \quad \zeta = -1, \tag{29}$$

$$\eta\Omega u_3^* + \frac{1}{c_f\rho_f}\sqrt{\frac{\rho}{E_0}}\mathcal{H}^* s_{33}^* = 0, \quad \zeta = 1. \tag{30}$$

In what follows, we expand all the started quantities in the asymptotic series as

$$f^* = f^{(0)} + \eta f^{(1)} + \eta^2 f^{(2)} + \ldots. \tag{31}$$

4. Asymptotic Solution

Let us start by integrating (25) and (26) with respect to the thickness coordinate ζ to get at leading order

$$u_3^{(0)} = U_3^{(0)} \qquad \text{and} \qquad u_2^{(0)} = U_2^{(0)}, \tag{32}$$

where the unknown constants $U_3^{(0)}$ and $U_2^{(0)}$ are, due to (24), related by

$$U_2^{(0)} = -\frac{1}{n}U_3^{(0)}. \tag{33}$$

Formula (33) corresponds to the circumferential inextensibility of the mid-surface of a cylindrical shell, see [23].

Then, integrating (22) and (23) with respect to the thickness variable, we obtain

$$s_{3m}^{(0)} = -n^{3-m}\int_\zeta^1 s_{22}^{(0)}\,ds, \quad m = 2, 3. \tag{34}$$

Applying the conditions (29), we deduce

$$\int_{-1}^1 s_{22}^{(0)}\,ds = 0. \tag{35}$$

At next order, first, we integrate (25) in the thickness variable having

$$u_3^{(1)} = U_3^{(1)} \tag{36}$$

where $U_3^{(1)}$ is an unknown constant. In the same manner, integrating (26) in ζ and employing the relation (33), we get

$$u_2^{(1)} = -\frac{1-n^2}{n}\zeta U_3^{(0)} + U_2^{(1)} \tag{37}$$

and

$$U_2^{(1)} = -\frac{1}{n}U_3^{(1)}. \tag{38}$$

Now, integrating (24) and taking into account the latter relation, we arrive at

$$s_{22}^{(0)} = -\frac{(1-n^2)}{1-\nu^2}\zeta U_3^{(0)}. \tag{39}$$

As a result, formulae (34) may be rewritten as

$$s_{3m}^{(0)} = n^{3-m}\frac{(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)U_3^{(0)}. \tag{40}$$

Next, we integrate (23) across the thickness and adopt formulae (39) and (40) together with the boundary condition $(29)_2$ to get

$$\int_{-1}^{1} s_{22}^{(1)} ds = \left(\frac{\rho_f}{\rho n}\Omega^2 + \frac{2n^2(1-n^2)}{3(1-\nu^2)} \right) U_3^{(0)} - Q^*. \tag{41}$$

We also integrate (22) in ζ and utilize formula (39). Then, we subject the resulting equation to condition $(29)_1$. As a result, we have

$$s_{32}^{(1)} = -n \int_{\zeta}^{1} s_{22}^{(1)} ds + \frac{2n(1-n^2)}{3(1-\nu^2)}. \tag{42}$$

Taking $\zeta = -1$ in the last equation, taking into consideration $(29)_1$ and (41), we derive

$$\left(\Omega^2 - \frac{2\rho(1-n^2)^2 n}{3\rho_f(1-\nu^2)} \right) U_3^{(0)} = \frac{\rho n}{\rho_f} Q^*. \tag{43}$$

It is clear that the effect of the material damping, i.e., the parameter α_0, on the stress components and the vertical displacement does not appear in this equation. To incorporate the effect of this parameter, we need to consider the next order approximation.

Following the same process carried out in the previous sections and omitting intermediate calculations, we get from (25)

$$u_3^{(2)} = \frac{\nu}{2(1-\nu)}(1-n^2)\zeta^2 U_3^{(0)} + U_3^{(2)}. \tag{44}$$

Similarly, it follows from (26) that

$$u_2^{(2)} = -\frac{1-n^2}{n}\zeta U_3^{(1)} + U_2^{(2)}. \tag{45}$$

Then, integration of equation (24), taking into consideration (32), (37), (38), (40), (42), (44) and (45), results in

$$U_2^{(2)} + \frac{1}{n}U_3^{(2)} = -\frac{\nu(1-n^2)}{2(1-\nu)n}U_3^{(0)}. \tag{46}$$

Inserting the last formula back into equation (23), we obtain

$$s_{22}^{(1)} = -\frac{1-n^2}{1-\nu^2}\zeta U_3^{(1)} + \frac{1-n^2}{1-\nu^2}\left(\zeta^2 - i\zeta\right) U_3^{(0)}. \tag{47}$$

Now, we revisit equation (23), using the dimensionless impenetrability condition (30) taken at first order, and also equations (39), (47) and (48). The result is

$$\begin{aligned} s_{33}^{(1)} =& \frac{(1-n^2)(1-\zeta^2)}{2(1-\nu^2)}U_3^{(1)} + \Bigg(\frac{5\zeta^3 - 3\zeta - 2 + 4n^4(1-\zeta^3) - n^4(2 - 3\zeta + \zeta^3)}{6(1-\nu^2)} \\ &+ \frac{\rho_f}{\rho n}\Omega^2 + i\alpha_0 \frac{(1-\zeta^2)(1-n^2)}{2(1-\nu^2)} \Bigg) U_3^{(0)}. \end{aligned} \tag{48}$$

This formula allows us to rewrite formula (42) as

$$s_{32}^{(1)} = \frac{n(1-n^2)}{2(1-\nu^2)}(1-\zeta^2)U_3^{(1)} - \frac{(1-n^2)(1-\zeta^2)}{1-\nu^2}\left(n\zeta - \frac{i\alpha_0}{2} \right) U_3^{(0)}. \tag{49}$$

Using (30) and integrating (22) and (23) along the thickness at second order, we derive, respectively,

$$\int_{-1}^{1} s_{22}^{(2)} d\zeta = \frac{2(1-n^2)}{3(1-\nu^2)}U_3^{(1)} - \left(\frac{2\Omega^2}{n^2} - i\alpha_0 \frac{2(1-n^2)}{3(1-\nu^2)} \right) U_3^{(0)} \tag{50}$$

and

$$\begin{aligned} \int_{-1}^{1} s_{22}^{(2)} d\zeta = & \left(\frac{\rho_f}{\rho n}\Omega^2 + \frac{2n^2(1-n^2)}{3(1-\nu^2)} \right) U_3^{(1)} \\ & \left(\frac{2\rho n + 3\rho_f}{\rho n}\Omega^2 - \frac{2(1-n^2)}{3(1-\nu^2)}\left(1 - n^2 - i\alpha_0 n^2\right) \right) U_3^{(0)}. \end{aligned} \tag{51}$$

Comparing (50) and (51), we finally have

$$\left(\Omega^2-\frac{2\rho(1-n^2)^2n}{3\rho_f(1-\nu^2)}\right)U_3^{(1)}=-\left(\Omega^2-\frac{2\rho(1-n^2)^2n}{3\rho_f(1-\nu^2)}\right)U_3^{(0)} \\ -2\left(\Omega^2+\frac{\rho}{\rho_f}\left(\frac{1+n^2}{n}\right)\Omega^2-i\alpha_0\frac{\rho(1-n^2)^2n}{3\rho_f(1-\nu^2)}\right)U_3^{(0)}. \tag{52}$$

5. Discussion

Let us set $W=U_3^{(0)}+\eta U_3^{(1)}$. Then, we obtain from (43) and (52)

$$W=\frac{Q^*}{g(\Omega)}\frac{\rho n}{\rho_f}(1-\eta), \tag{53}$$

where

$$g(\Omega)=\Omega^2-\frac{2\rho n(1-n^2)^2}{3\rho_f(1-\nu^2)}+\eta\left(2\Omega^2\left(1+\frac{\rho(1+n^2)}{\rho_f n}\right)-i\alpha_0\frac{2\rho n(1-n^2)^2}{3\rho_f(1-\nu^2)}\right). \tag{54}$$

The roots of the equation $g(\Omega)=0$ correspond to the resonance frequencies. Let us adapt a two-term expansion $\Omega^2=\Omega_0^2+\eta\Omega_1^2+\ldots$. In this case, we may rewrite (54) as

$$g(\Omega)=2\Omega_0\left(\Omega-\Omega_0+\eta\Omega_0\left(1+\frac{\rho(1+n^2)}{\rho_f n}-\frac{i\alpha_0}{2}\right)\right) \tag{55}$$

in which

$$\Omega_0^2=\frac{2\rho n(1-n^2)^2}{3\rho_f(1-\nu^2)}. \tag{56}$$

Thus, the approximate vertical displacement component takes the form

$$W=\frac{\rho n}{2\rho_f\Omega_0}\frac{Q^*}{\Omega-\Omega_0+\eta\Omega_0\left(1+\frac{\rho(1+n^2)}{\rho_f n}-\frac{i\alpha_0}{2}\right)} \tag{57}$$

predicting the complex resonance frequencies

$$\Omega=\Omega_0-\eta\Omega_0\left(1+\frac{\rho(1+n^2)}{\rho_f\, n}-\frac{i\alpha_0}{2}\right). \tag{58}$$

This formula demonstrates the role of the small viscosity of interest.

From the above derivation, it is clear that the damping due to the radiation corresponding to a small imaginary term in (28) is far beyond the accuracy of (57) taking into account the structural damping defined by the parameter α_0. As it was shown in [17], the order of the damping caused by the radiation is of order $O(\eta^{3n})$ is negligible compared with the considered structural damping which is of $O(\eta)$ as predicted by the asymptotic formulae above. Figures 2 and 3 illustrate the resonant behaviour of a thin cylindrical shell immersed in a fluid at $n=2$ and $n=3$ for the vertical displacement normalised by Q^* (see, equation (57)). In all numerical calculations, the problem parameters are $\rho=2790$ kg/m^3, $\nu=0.3$, and $\rho_f=1000$ kg/m^3.

6. Concluding Remarks

An asymptotic procedure is developed for forced low-frequency vibrations of a thin viscoelastic cylindrical shell immersed in fluid. The effect of viscosity is accounted by adapting the concept of a complex Young's modulus.

Refined asymptotic formulae for the shell transverse displacement and the related complex resonance frequency are derived. They demonstrate that the incorporated effect of structural damping is much greater than the contribution of the damping due to the radiation of vibration energy into the fluid. As a result, the latter can be ignored in practical applications. This is also beneficial since its evaluation requires retaining extra higher order terms in the expansion (28), see also [17] for more details.

The proposed approach has a clear potential to be extended to more sophisticated models of viscoelastic behaviour as well as to a transversely inhomogeneous fluid-loaded shell, e.g., see [18]. The obtained results can also be readily generalized to scattering problems, including 3D ones.

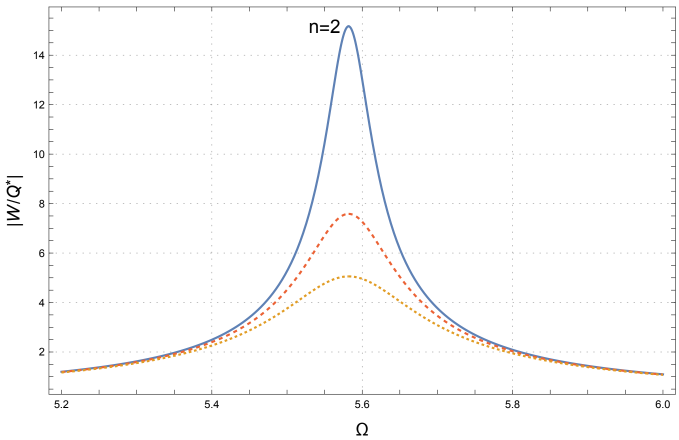

FIGURE 2. Displacement (57) for $n = 2$ with the Poisson ratio $\nu = 0.3$ and $\eta = 0.01$ with $\alpha_0 = 1$ (solid line), $\alpha_0 = 2$ (dashed red line) and $\alpha_0 = 3$ (dashed orange line).

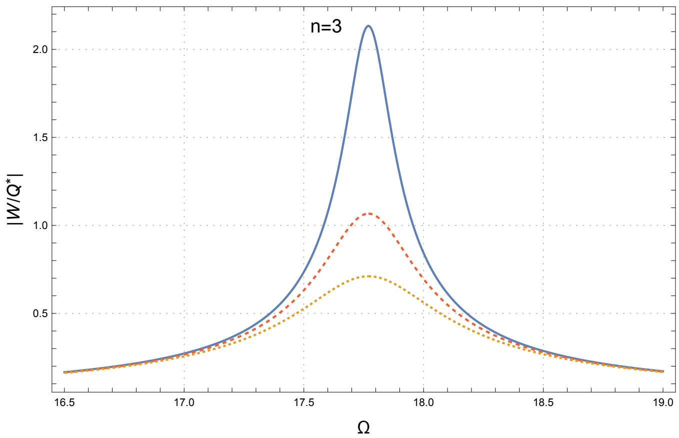

FIGURE 3. Displacement (57) for $n = 3$ with the Poisson ratio $\nu = 0.3$ and $\eta = 0.01$ with $\alpha_0 = 1$ (solid line), $\alpha_0 = 2$ (dashed red line) and $\alpha_0 = 3$ (dashed orange line).

Declaration of Competing Interests There are no competing financial interests or personal relationships that could influence the work on this paper.

Acknowledgements I wish to express my sincere gratitude to J. Kaplunov and B. Erbaş for their unwavering support to my studies and many encouraging discussions.

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