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Research Article

Finite Difference Approach for Fourth-Order Impulsive Sturm-Liouville Boundary Value Problems

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Abstract

This paper presents a finite difference method to solve a novel type fourth-order boundary value problem with impulsive conditions. These differential equations, which model deflections in beams, provide insights into various applications in fields such as civil, mechanical, and aeronautical engineering. Analytical solutions to boundary value problems are often challenging to derive, highlighting the need for robust numerical methods. In this study, a formula for finite difference approximation is derived by using Taylor series expansions at selected grid points. By transforming differential equations into algebraic systems, the unknown solutions are determined based on the grid points. The proposed method is validated through a numerical example involving a fourth-order impulsive linear boundary value problem, and the results demonstrate its effectiveness.

Keywords: The finite difference method, fourth order boundary value problem, impulse conditions, approximate solutions

Dördüncü Mertebe İmpulsiv Sturm-Liouville Sınır Değer Problemleri için Sonlu Fark Yaklaşımı

Öz

Bu makale, impulsiv koşullara sahip yeni bir dördüncü dereceden sınır değer problemini çözmek için sonlu farklar yöntemini sunmaktadır. Kirişlerdeki sapmaları modelleyen bu diferansiyel denklemler, inşaat, makine ve havacılık mühendisliği gibi alanlardaki çeşitli uygulamaların aydınlatılmasını sağlar. Sınır değer problemlerine yönelik analitik çözümlerin elde edilmesi çoğu zaman zorlayıcıdır ve bu durum sağlam sayısal yöntemlere olan ihtiyacı vurgulamaktadır. Bu çalışmada, seçili grid noktalarında Taylor serisi açılımları kullanılarak sonlu farklar yaklaşımı için bir formül ortaya çıkarılmıştır. Diferansiyel denklemler cebirsel denklem sistemlere dönüştürülerek, bilinmeyen çözümler grid noktalarına göre belirlenmiştir. Önerilen yöntem, dördüncü dereceden impulsiv doğrusal sınır değer problemini içeren sayısal bir örnek üzerinden doğrulanmış ve sonuçlar yöntemin etkinliğini göstermiştir.

Anahtar Kelimeler: Sonlu fark yöntemi, dördüncü mertebe sınır değer problemi, impuls şartlar, yaklaşık çözümler

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1. Introduction

Fourth-order boundary value problems commonly arise in the mathematical modeling of beam and plate deformations, deflection theories, viscoelastic and inelastic flows, as well as numerous other applications in applied sciences. The theory of impulsive differential equations is significantly more complex than that of ordinary differential equations and has captivating applications in various fields. The interest of researchers in this domain is rapidly growing. Analytical solutions for such boundary value problems (BVP) are rarely attainable, which makes it necessary to study fundamental properties such as existence and uniqueness of solutions, linear dependence and independence, stability, and the existence of periodic solutions [1-17]

The theory of impulsive differential equations is essential for understanding real-life problems, where discontinuities in physical phenomena have considerable impacts on many applied problems in nature. For instance, in real processes, effects occurring over a short time and initially deemed negligible can induce sudden changes in the state of a system. Even minuscule, abrupt changes can lead to substantial differences in system behavior.

Examples of such phenomena include epidemic outbreaks that lead to rapid declines in the population of a species, sudden changes in the velocity of an oscillating object when an external force is applied, or sudden deflections in beams under various conditions. The mathematical models that incorporate these impulsive effects are expressed as impulsive differential equations. Solving these equations yields results that most closely align with real-world observations. Furthermore, realistic outcomes can be achieved through models applied to many fields, including theoretical physics, biotechnology, industrial robotics, medicine, and economics [18-27].

Over time, a variety of analytical and numerical methods have been developed to tackle these problems. Methods include spectral analysis [1-17], finite difference methods [18-21], variational approaches [22], Hermite and conforming finite elements [23, 24], compact finite difference methods [25], the differential transform method [26], and the Galerkin method [27], among others. The development of new methods continues to advance the field.

In this paper, we present the finite difference method as a means to solve fourth-order boundary value problems with impulsive conditions. This type of problem has not been previously addressed in the literature. We derive the finite difference formulations necessary for the solution, apply these formulations to specific problems, and illustrate the solutions using graphical representations.

2. Derivation of The Finite Difference Method for Solving of Fourth Order Impulsive BVP

Consider the fourth-order linear boundary value problem given by:

$$p(x)\eta^{(4)}(x) + q(x)\eta^{'''}(x) + r(x)\eta^{''}(x) + s(x)\eta^{'}(x) + t(x)\eta(x) = f(x), \quad x \in [a, m) \cup (m, b] \tag{1}$$

with the impulsive conditions at $x = m$:

$$\begin{aligned} \eta(m-0) = \alpha\eta(m+0), \quad \eta_x(m-0) = \beta\eta_x(m+0) \\ \eta_{xx}(m-0) = \gamma\eta_{xx}(m+0), \quad \eta_{xxx}(m-0) = \delta\eta_{xxx}(m+0) \end{aligned} \tag{2}$$

and boundary conditions:

$$\eta(a) = A_0, \quad \eta(b) = B_0, \quad \eta_x(a) = A_1, \quad \eta_x(b) = B_1 \tag{3}$$

where a, m, b, A_0, B_0, A_1, B_1, and α, β, γ, $\delta > 0$ are constants, and p, q, r, s, t, and f are smooth functions. To discretize problem (1), (3), the interval $[a, b]$ is divided into N equal sub-intervals $[x_0, x_1], [x_1, x_2], ..., [x_{N-1}, x_N]$ such that:

$$a = x_0 < x_1 < ... < x_N = b, \quad x_i = a + ih, \quad h = \frac{b-a}{N}$$

The values corresponding to η are denoted as $\eta(x_i) = \eta_i = \eta(x_0 + ih)$, $i \in \{0, 1, 2, ..., N\}$.

Using Taylor series expansion, we obtain the forward, backward, and central difference expressions for $\eta^{'}(x)$ at $x = x_i$:

$$\eta_i^{'} = \frac{\eta_{i+1} - \eta_i}{h}, \quad \eta_i^{'} = \frac{\eta_i - \eta_{i-1}}{h}$$
$$\eta_i^{'} = \frac{\eta_{i+1} - \eta_{i-1}}{2h} \tag{4}$$

Higher-order derivatives are derived by repeatedly applying first-order differences. The following approaches can be used:

$$D_+\eta(x) = \frac{\eta(x+h) - \eta(x)}{h}, \quad D_-\eta(x) = \frac{\eta(x) - \eta(x-h)}{h}$$

$$D_0\eta(x) = \frac{\eta(x+h) - \eta(x-h)}{2h}$$

where $D_+\eta(x)$, $D_-\eta(x)$, and $D_0\eta(x)$ denote the forward, backward, and centered finite differences of $\eta(x)$, respectively. The second-order derivative $D^2\eta(x)$ can be defined as the difference of first differences:

$$\begin{aligned} D^2\eta(x) = D_+(D_-\eta(x)) &= \frac{1}{h}[D_-\eta(x+h) - D_-\eta(x)] \\ &= \frac{1}{h^2}[\eta(x+h) - 2\eta(x) + \eta(x-h)] \end{aligned}$$

Hence, the forward, backward, and central differences for the second-order derivative are:

$$\eta_i^{''} = \frac{\eta_{i-1} - 2\eta_i + \eta_{i+1}}{h^2}, \quad \eta_i^{''} = \frac{\eta_{i-2} - 2\eta_{i-1} + \eta_i}{h^2}$$
$$\eta_i^{''} = \frac{\eta_i - 2\eta_{i+1} + \eta_{i+2}}{h^2} \tag{5}$$

The third-order derivative $D^3\eta(x)$ can be expressed as:

$$D^3\eta(x) = D_+D^2\eta(x) = \frac{1}{h^3}[\eta(x+2h) - 3\eta(x+h) + 3\eta(x) - \eta(x-h)]$$

which yields the forward, backward, and central differences for the third-order derivative:

$$\eta_i^{'''} = \frac{1}{h^3}[\eta_{i+3} - 3\eta_{i+2} + 3\eta_{i+1} - \eta_i], \quad \eta_i^{'''} = \frac{1}{h^3}[\eta_{i+1} - 3\eta_i + 3\eta_{i-1} - \eta_{i-2}] \tag{6}$$

$$\eta_i^{'''} = \frac{1}{h^3}[\eta_{i+2} - 3\eta_{i+1} + 3\eta_i - \eta_{i-1}]$$

The fourth-order derivative $D^4\eta(x)$ is defined by:

$$D^4\eta(x) = D_+D^3\eta(x) = \tfrac{1}{h^4}[\eta(x+3h) - 4\eta(x+2h) + 6\eta(x+h) - 4\eta(x) + \eta(x-h)]$$

and the corresponding finite difference expressions for the fourth-order derivative are:

$$\begin{aligned} \eta_i^{(4)} &= \frac{1}{h^4}[\eta_{i+4} - 4\eta_{i+3} + 6\eta_{i+2} - 4\eta_{i+1} + \eta_i] \\ \eta_i^{(4)} &= \frac{1}{h^4}[\eta_{i+2} - 4\eta_{i+1} + 6\eta_i - 4\eta_{i-1} + \eta_{i-2}] \\ \eta_i^{(4)} &= \frac{1}{h^4}[\eta_{i+3} - 4\eta_{i+2} + 6\eta_{i+1} - 4\eta_i + \eta_{i-1}] \end{aligned} \tag{7}$$

To solve the boundary value problem (1), (3) in the interval $[a, b]$, we substitute the finite difference expressions from (4), (5), (6), and (7) into the equations to obtain the following system of linear algebraic equations:

$$\begin{cases} 2p_i\eta_{i+2} + (-8p_i + 2hq_i + 2h^2r_i + h^3s_i)\eta_{i+1} + (12p_i - 6hq_i - 4h^2r_i + 2h^4t_i)\eta_i \\ +(-8p_i + 6hq_i + 2h^2r_i - h^3s_i)\eta_{i-1} + (2p_i - 2hq_i)\eta_{i-2} = 2h^4f_i, \quad i \in \{1, 2, \cdots N-1\} \end{cases} \tag{8}$$

with the boundary conditions:

$$\begin{aligned} &\eta(a) = A_0, \;\; \eta(b) = B_0 \\ &\eta_x(a) = \frac{\eta_{i+1} - \eta_{i-1}}{2h} = A_1, \quad \eta_x(b) = \frac{\eta_{i+1} - \eta_{i-1}}{2h} = B_1 \end{aligned} \tag{9}$$

where $\eta(x_i)$, $p(x_i)$, $q(x_i)$, $r(x_i)$, $s(x_i)$, $t(x_i)$, and $f(x_i)$ are represented as η_i, p_i, q_i, r_i, s_i, t_i, and f_i, respectively.

3. The Solution of Problem with Impulsive Conditions

To find the solution to the boundary value problem (1), (3) that satisfies the impulsive conditions:

$$\begin{aligned} &\eta(m-0) = \alpha\eta(m+0), \quad \eta_x(m-0) = \beta\eta_x(m+0) \\ &\eta_{xx}(m-0) = \gamma\eta_{xx}(m+0), \quad \eta_{xxx}(m-0) = \delta\eta_{xxx}(m+0) \end{aligned} \tag{10}$$

in the interval $[a, m) \cup (m, b]$, we divide the definition range $[a, b]$ into N equal sub-intervals using grid points:

$$x_i = a + ih, \quad i \in \{0, 1, ..., N\}$$

where $h = \frac{b-a}{N}$ is the mesh width, representing the distance between consecutive grid points. The values p_i, q_i, r_i, s_i, t_i, and f_i correspond to the function values of p, q, r, s, t, and f at the grid point $x = x_i$. The approximate solution at x_i is denoted by η_i [18, 19]. Thus, we can numerically solve the impulsive boundary value problem, which includes the values $\eta_0, \eta_1, ..., \eta_{N-1}, \eta_N$.

From the boundary conditions (2.3), we have:

$$\begin{aligned} &\eta(a) = A_0, \quad \eta(b) = B_0 \\ &\eta_x(a) = \frac{\eta_{i+1} - \eta_{i-1}}{2h} = A_1, \quad \eta_x(b) = \frac{\eta_{i+1} - \eta_{i-1}}{2h} = B_1 \end{aligned} \tag{11}$$

Approximate values at the impulsive conditions (10) can be determined as:

$$\eta(m-0) = \eta(x_k), \quad \eta(m+0) = \eta(x_{k+1}) \tag{12}$$

$$\eta_x(m-0) = \frac{1}{h}[\eta(x_k) - \eta(x_{k-1})], \quad \eta_x(m+0) = \frac{1}{h}[\eta(x_{k+2}) - \eta(x_{k+1})] \tag{13}$$

$$\begin{aligned} \eta_{xx}(m-0) &= \tfrac{1}{h^2}[\eta(x_{k-2}) - 2\eta(x_{k-1}) + \eta(x_k)] \\ \eta_{xx}(m+0) &= \tfrac{1}{h^2}[\eta(x_{k+1}) - 2\eta(x_{k+2} + \eta(x_{k+3})] \end{aligned} \tag{14}$$

$$\begin{aligned} \eta_{xxx}(m-0) &= \tfrac{1}{h^3}[\eta(x_{k+1}) - 3\eta(x_k) + 3\eta(x_{k-1}) - \eta(x_{k-2})] \\ \eta_{xxx}(m+0) &= \tfrac{1}{h^3}[\eta(x_{k+4}) - 3\eta(x_{k+3}) + 3\eta(x_{k+2}) - \eta(x_{k+1})] \end{aligned} \tag{15}$$

Using the finite difference approximations for a sufficiently large N and at points x_k and x_{k+1}, we derive the following from the impulse conditions (12)-(15):

$$\eta_k = \alpha\eta_{k+1} \tag{16}$$

$$\frac{1}{h}[\eta_k - \eta_{k-1}] = \frac{\beta}{h}[\eta_{k+2} - \eta_{k+1}] \tag{17}$$

$$\tfrac{1}{h^2}[\eta_{k-2} - 2\eta_{k-1} + \eta_k] = \tfrac{\gamma}{h^2}[\eta_{k+1} - 2\eta_{k+2} + \eta_{k+3}] \tag{18}$$

$$\tfrac{1}{h^3}[\eta_{k+1} - 3\eta_k + 3\eta_{k-1} - \eta_{k-2}] = \tfrac{\delta}{h^3}[\eta_{k+4} - 3\eta_{k+3} + 3\eta_{k+2} - \eta_{k+1}] \tag{19}$$

In the impulsive boundary value problem (1)-(3), considering (11) and the finite difference approximations (16)-(19), the resulting system of linear algebraic equations is:

$$\begin{cases} 2p_i\eta_{i+2} + (-8p_i + 2hq_i + 2h^2r_i + h^3s_i)\eta_{i+1} + (12p_i - 6hq_i - 4h^2r_i + 2h^4t_i)\eta_i \\ +(-8p_i + 6hq_i + 2h^2r_i - h^3s_i)\eta_{i-1} + (2p_i - 2hq_i)\eta_{i-2} = 2h^4f_i, \;\; i \in \{1, 2, \cdots, k-1\} \\ \eta_i - \alpha\eta_{i+1} = 0, \;\; i \in \{k\} \\ -\eta_{i-1} + \eta_i + \beta\eta_{i+1} - \beta\eta_{i+2} = 0, \;\; i \in \{k+1\} \\ \eta_{i-2} - 2\eta_{i-1} + \eta_i - \gamma\eta_{i+1} + 2\gamma\eta_{i+2} - \gamma\eta_{i+3} = 0, \;\; i \in \{k+2\} \\ -\eta_{i-2} + 3\eta_{i-1} - 3\eta_i + (1+\delta)\eta_{i+1} - 3\delta\eta_{i+2} + 3\delta\eta_{i+3} - \delta\eta_{i+4} = 0, \;\; i \in \{k+3\} \\ 2p_i\eta_{i+2} + (-8p_i + 2hq_i + 2h^2r_i + h^3s_i)\eta_{i+1} + (12p_i - 6hq_i - 4h^2r_i + 2h^4t_i)\eta_i \\ +(-8p_i + 6hq_i + 2h^2r_i - h^3s_i)\eta_{i-1} + (2p_i - 2hq_i\eta_{i-2} = 2h^4f_i, \;\; i \in \{k+4, k+5, \cdots, N-1\} \end{cases} \tag{20}$$

The solution of the system (20) can be obtained using computational software such as Maple, Matlab, or Mathematica.

In numerical methods, if the approximate solutions closely match the real solution, this property is known as convergence. However, convergence depends on specific conditions, and it may not always be guaranteed [18, 19, 21].

4. Numerical Example

In this section, the proposed method is applied to a linear boundary value problem with impulsive conditions. The approximate solutions obtained are compared with the exact solution.

We consider the following linear boundary value problem:

$$\eta^{(4)}(x) + \eta^{''}(x) = x, \quad x \in [0, 1) \cup (1, 2] \tag{21}$$

subject to the boundary conditions:

$$\eta(0) = 0, \quad \eta(2) = 0, \quad \eta_x(0) = 0, \quad \eta_x(2) = 0 \tag{22}$$

and with impulsive conditions at $x = 1$:

$$\begin{aligned} &\eta(1-0) = \eta(1+0), \quad \eta_x(1-0) = \eta_x(1+0), \\ &\eta_{xx}(1-0) = 2\eta_{xx}(1+0), \;\; \eta_{xxx}(1-0) = 2\eta_{xxx}(1+0) \end{aligned} \tag{23}$$

Such equations are based on the analysis of elastic beam deflections. If $\eta = \eta(x)$ represents the deflection of the beam at x, the differential equation characterizes the transverse displacement of the beam under buckling [8, 13, 21, 22].

First, we examine the solution of this problem without impulsive conditions in the interval $[0, 2]$ using the finite difference method (FDM). We show that the approximate solutions align closely with the analytical solution of the boundary value problem (21), (22):

$$\eta(x) = \frac{1}{6(\cos(2)+\sin(2)-1)}[(\cos(2) + \sin(2) - 1)x^3 + (6\cos(2) + 4\sin(2) - 6)\sin(x)$$
$$+(4\cos(2) - 6\sin(2) + 8)\cos(x) - (6\cos(2) + 4\sin(2) - 6)x - 4\cos(2) + 6\sin(2) - 8]$$

We take a uniform cartesian grid $x_i = 0 + ih$, where $i \in \{0, 1, ..., 20\}$, and $h = \frac{2-0}{20} = 0.1$. Here, $x_0 = 0$, $x_{20} = 0$, with boundary values $\eta_0 = 0$ and $\eta_{20} = 0$. Applying the finite difference method at a typical grid point x_i, we get:

$$\eta_{i+2} + (h^2 - 4)\eta_{i+1} + (6 - 2h^2)\eta_i + (h^2 - 4)\eta_{i-1} + \eta_{i-2} = h^4 x_i, \quad i \in \{1, ..., 19\} \tag{24}$$

Thus, the finite difference solutions $\eta_i \sim \eta(x_i)$ for the linear system (24) are obtained. This system is solved using software such as Maple, Matlab, or Mathematica. The results demonstrate a strong agreement, as illustrated in Figure 1.

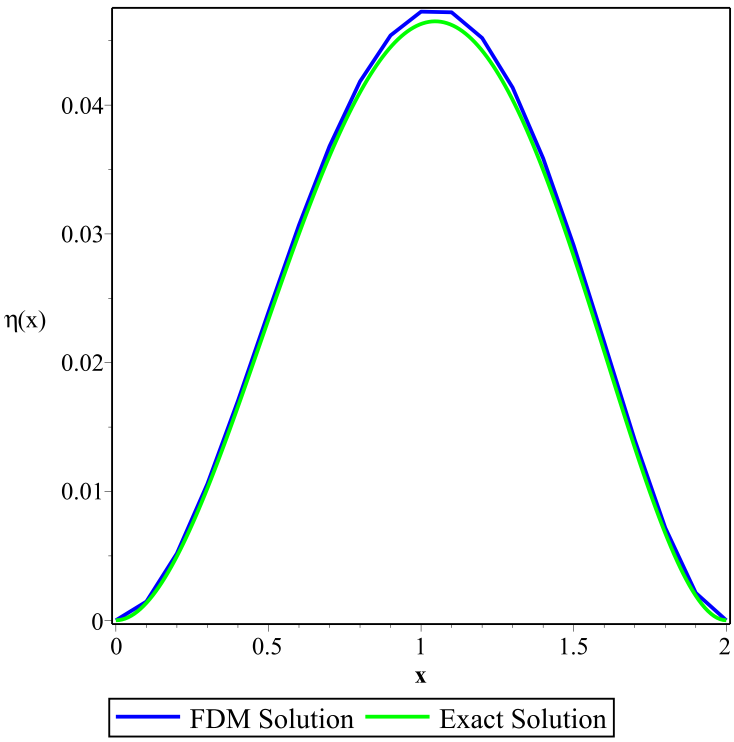

Figure 1: Graph of the FDM and Exact solution for $N = 20$

Then, we consider the boundary value problem (21), (22) with impulsive conditions (23) at the interface point $x = 1$. Selecting $N = 64$ and defining $\eta_{32} = \eta(1 - 0)$, $\eta_{33} = \eta(1 + 0)$, we apply the impulsive conditions (23) and obtain four additional algebraic equations:

$$\begin{cases} \eta_{32} - \eta_{33} = 0 \\ \eta_{30} - \eta_{32} - \eta_{33} + \eta_{35} = 0 \\ \eta_{28} - \eta_{32} - 2\eta_{33} + 2\eta_{37} = 0 \\ \eta_{26} - \eta_{32} - 2\eta_{33} + \eta_{39} = 0 \end{cases} \tag{25}$$

The algebraic system (24) combined with (25) can be solved using computational tools like Maple, Matlab, or Mathematica. The solution for the impulsive boundary value problem (21)-(23) is shown in Figure 2.

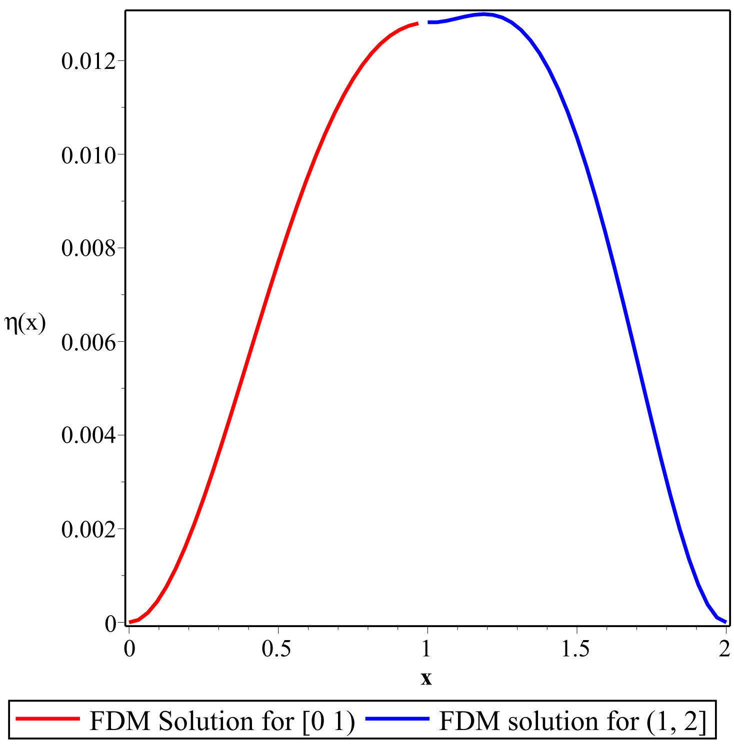

Figure 2: Graph of the FDM solution for impulsive BVP for $N = 64$

5. Conclusions

In this study, the finite difference approach was applied to obtain approximate solutions for a fourth-order boundary value problem with impulse conditions. These problems hold significant importance in practical applications and are commonly encountered in the analysis of elastic beam deflections. The problem examined in this study is distinct from those addressed in existing literature. The results obtained were illustrated through graphical representations, demonstrating the development and applicability of finite difference formulations for solving boundary value problems with impulse conditions. Mathematical models like these frequently appear in nonlinear oscillations, numerous physical phenomena, and real-life applications. Therefore, it is thought that the numerical method developed for the solution of the fourth-order impulsive differential equation would make valuable contributions to the existing literature. Future research could extend this method to nonlinear impulsive differential equations. It could also be adapted to fractional impulsive differential equations. Additionally, these problems could be studied under various initial and boundary conditions.

Ethics in Publishing:

There are no ethical issues regarding the publication of this study

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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