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FIDELITY ANALYSIS OF PATH ENTANGLED TWO-QUANTON SYSTEMS

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Abstract. Path (momentum) entanglement, arising from the spatial correlations of quantons, constitutes a cornerstone of quantum communication, metrology, and advanced interferometry. Despite its profound importance, the quantitative evaluation of path entanglement remains an intricate task, particularly under transformations imposed by interferometric setups. This study explores the fidelity of path entanglement in two interferometric configurations, P-BS and BS-P-BS, for spatially correlated two-quanton systems. Fidelity, which measures the preservation of quantum correlations, is analyzed alongside concurrence to capture the dynamics of entanglement under phase retarder manipulations. Our findings reveal contrasting behaviors between the two setups: while the P-BS configuration shows a decrease in fidelity with increasing concurrence, the BS-P-BS setup achieves maximum fidelity for maximally entangled states with carefully tuned retarder phases. These findings underscore the robustness of the BS-P-BS architecture in maintaining quantum correlations, rendering it a compelling candidate for quantum teleportation and high-fidelity quantum channel implementations. Furthermore, the interplay between retarder phases, concurrence, and fidelity offers novel insights for optimizing interferometric designs in advanced quantum information processing applications.

Keywords. Path-Entanglement, momentum correlation, fidelity.

1. Introduction

Path (momentum) entanglement is a fundamental concept in quantum mechanics, essential for understanding and demonstrating quantum phenomena [1]. Unlike spin or polarization entanglement, it arises from the spatial correlations of quantum mechanical particles (quantons) and offers a distinctive perspective on non-locality

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and non-contextuality. Path entanglement plays crucial roles in quantum communication [2-4], quantum metrology [5-7], and fundamental tests and applications of quantum mechanics [8-15]. Its practical applications also extend to quantum-enhanced interferometry, where path-entangled states enable measurements beyond classical precision limits, as in quantum lithography, and developing quantum technologies [16-23]. Given its importance in various quantum phenomena, it is essential to develop accurate measures for evaluating path entanglement, as these measurements are key to understanding the dynamics and preservation of entanglement in quantum systems.

Entanglement fidelity, which assesses how effectively entanglement is preserved during a quantum process, offers a unique perspective distinct from traditional entanglement measures as entanglement of formation, concurrence, and negativity [24-25]. These measures quantify various facets of quantum correlations, while entanglement fidelity captures a broader or complementary aspect, underscoring the multidimensional nature of the entanglement [26-32]. It is widely used to assess the performance of quantum operations, the robustness of quantum systems to noise, and the degree of preservation of quantum coherence [33-34]. Beyond its theoretical utility, fidelity is also indispensable in quantum technologies error correction, quantum state cloning, and the evaluation of quantum communication protocols [35-36].

In this work, we present a new investigation of the fidelity of path entanglement along two possible interferometer scenarios for two-quanton systems. Starting with spatially (momentum) correlated quantons produced from a source, we study the transformations induced by two possible interferometer configurations. By analyzing the final states, we assess their fidelity with the original state, offering insights into the preservation of quantum correlations under various conditions. Furthermore, our approach leverages the concurrence as a measure of entanglement to correlate the fidelity outcomes with the degree of quantum coherence retained in the system. This dual analysis of fidelity and concurrence not only strengthens our understanding of path entanglement but also demonstrates the tunability of quantum correlations using experimentally feasible setups.

The structure of this paper is as follows. In Section 2, we outline the theoretical foundations of path entanglement and fidelity, followed by a detailed description of the two interferometric scenarios and their mathematical formulation. Section 3 presents the results of our fidelity analysis, emphasizing the interplay between concurrence and entanglement preservation. Section 4 discusses the experimental implications and potential applications of our findings. Finally, Section 5 concludes with a summary of our contributions and suggestions for future work in the study of path entanglement.

2. Spatially Correlated Quanton Systems

Path (momentum) entanglement in quantum systems arises from the spatial correlations between particles, playing a fundamental role in quantum information processing and experimental implementations of quantum mechanics. In particular, two-quanton systems exhibit entanglement that depends on their relative motion and spatial configuration, making them ideal candidates for studying quantum coherence and non-locality. Understanding the mathematical formulation of such entangled states provides a foundation for analyzing their behavior under various transformations.

The state of two spatially (momentum) correlated quantons, due to the conservation of momentum, a quanton moving upward-left ($|u_\alpha\rangle_L$) necessitates its pair to move downward-right ($|d_\alpha\rangle_R$), while a downward-left ($|d_\alpha\rangle_L$) quanton corresponds to an ($|u_\alpha\rangle_R$) upward-right, $|\psi\rangle = |u_\alpha\rangle_R \otimes |d_\alpha\rangle_L + |d_\alpha\rangle_R \otimes |u_\alpha\rangle_L$, produced from a source in arbitrary α-directions is given by,

$$|\psi\rangle_{in} = \sqrt{\frac{1 - C(\alpha)}{2}}\,|\chi^+\rangle + \sqrt{\frac{1 + C(\alpha)}{2}}\,|\varphi^+\rangle \tag{1}$$

where $|\chi^\pm\rangle = \frac{1}{\sqrt{2}}[|00\rangle \pm |11\rangle]$ and $|\varphi^\pm\rangle = \frac{1}{\sqrt{2}}[|01\rangle \pm |10\rangle]$ are orthonormal Bell states, and $C(\alpha) = \frac{sin^2(2\alpha)}{1+\cos^2(2\alpha)}$ represents the concurrence of the system, which serves as a key measure of entanglement with the selection of $|u_\alpha\rangle = \left(\cos\left(\frac{\pi}{4} - \alpha\right), \sin\left(\frac{\pi}{4} - \alpha\right)\right)^T$, $|d_\alpha\rangle = \left(\sin\left(\frac{\pi}{4} - \alpha\right), \cos\left(\frac{\pi}{4} - \alpha\right)\right)^T$ as basis [14].

In such a system, a quanton moving upwards with angle α on the right is paired with a quanton moving downwards with the same angle on the left. Similarly, a quanton moving downwards on the right is paired with a quanton moving upwards on the left. Hence, the concurrence varies between 0 and 1, where $C = 0$ corresponds to a product state and $C = 1$ signifies maximal entanglement. This formulation allows for practical use in observing path-entanglement, performing entanglement measurements, and implementing quantum mechanical applications. Thus, by using spatially correlated quantons entanglement measurements can be performed in two possible scenarios in phase retarder (P) modified Mach-Zender-type interferometers, enabling the examination of different configurations.

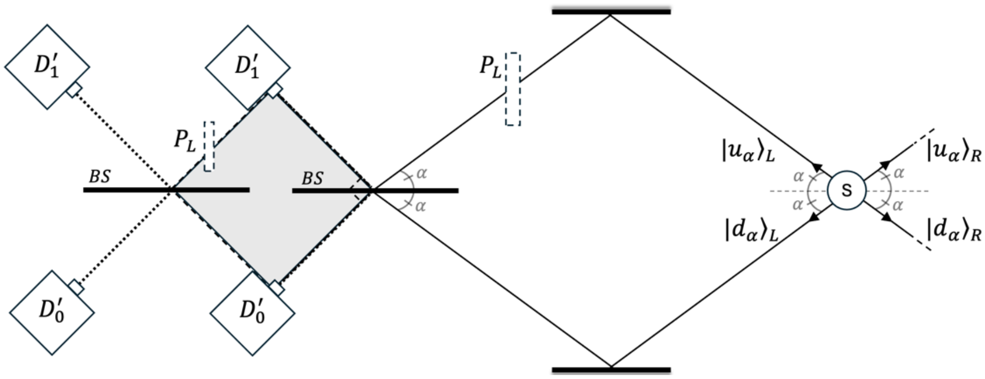

FIGURE 1. Schematic representations of the P-BS and BS-P-BS configurations. In the P-BS system, quantons are generated from the same source in arbitrary α directions and are detected after passing through a mutual retarder and BSs. In the BS-P-BS system, the grey section is incorporated into the circuit. Here, particles produced in α directions enter Mach-Zehnder (MZ) interferometers through the corresponding input ports of the first BSs and detected after following interaction with a retarder placed (wlog) in one of the arms and the second BS. The right side of the figure is entirely symmetric to the left side ($L \leftrightarrow R$).

First, the quantons are produced from the source S in reverse paths at an angle α, and they pass through retarders before reaching the symmetric and lossless BS with two ports. One can represent the operators BS, and P placed, wlog, on one of the trajectories in the matrix represantations,

$$BS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \; P = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

Thus, the state of the system initially given by (1), transformed into $|\psi_{out}\rangle_{P-BS} = (\mathrm{BS_R} \otimes \mathrm{BS_L})(\mathrm{P_R} \otimes \mathrm{P_L})|\psi\rangle$. With $\theta_\pm = \theta_R \pm \theta_L$, where θ_R and θ_L present the retarder phases for the right and left sides, the final state can be expressed as,

$$|\psi_{out}\rangle_{P-BS} = \sqrt{\frac{1-C}{2}}[\cos\theta_+|\varphi^+\rangle + \sin\theta_+|\chi^-\rangle] + \sqrt{\frac{1+C}{2}}[\sin\theta_-|\varphi^-\rangle + \cos\theta_-|\chi^+\rangle] \tag{3}$$

In the second scenario, one can expand the system to include two quantons emitted in arbitrary α directions from a single source, processed through modified Mach-Zehnder (MZ) interferometers with phase retarders. Initially, the quantons pass through the first beam splitters (BS_{R1} and BS_{L1}) and then traverse retarders placed in one arm of each interferometer. Afterward, they encounter the second beam splitters (BS_{R2} and BS_{L2}) before being detected by mutual detectors. Therefore, the system initially described by (1) evolves into the state $|\psi_{out}\rangle_{BS-P-BS} = (BS_{R2} \otimes BS_{L2})(P_R \otimes P_L)(BS_{R1} \otimes BS_{L1})|\psi\rangle$,

$$|\psi_{out}\rangle_{BS-P-BS} = \sqrt{\frac{1-C}{2}}\left[\cos\left(\frac{\theta_-}{2}\right)|\chi^+\rangle + \sin\left(\frac{\theta_-}{2}\right)|\varphi^-\rangle\right] + \sqrt{\frac{1+C}{2}}\left[\sin\left(\frac{\theta_+}{2}\right)|\chi^-\rangle + \cos\left(\frac{\theta_+}{2}\right)|\varphi^+\rangle\right] \quad (4)$$

The transformation between the initial and final states of the system reveals how various factors influencing the degree of entanglement can be effectively controlled. The concurrence parameter, C, plays a crucial role in controlling the system's entanglement, while the angles θ_R and θ_L provide insight into the impact of different interferometric configurations on entanglement. These findings lay a solid foundation for the development of quantum mechanical applications, enabling more precise and controllable entanglement measurements. Furthermore, they offer a deep understanding of how spatial correlations and interferometric setups can be utilized in entanglement experiments.

3. Fidelities

Fidelity is a function that measures the closeness between two quantum states. For two pure quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, the fidelity is defined as,

$$F(\psi_1, \psi_2) \equiv |\langle\psi_1|\psi_2\rangle|^2 \quad (5)$$

Here, $\langle\psi_1|\psi_2\rangle$ represents the inner product between the two states, and its square gives the transition probability between the initial and final states. The fidelity value ranges from 0 to 1, where $F = 1$ indicates the states are identical, and $F = 0$ represents completely orthogonal states. This measure plays a crucial role in understanding the preservation and evolution of quantum states during processes. Besides, the fidelity has broad applications in quantum information science, serving as a vital tool for evaluating the accuracy of quantum teleportation, comparing quantum states, and analyzing the performance of quantum channels. It helps determine whether a channel is suitable for teleportation by assessing how well quantum correlations are preserved. Fidelity also underpins other metrics, such as the Bures distance, which measures the closeness of quantum states in a more geometric sense. Moreover, it is essential for studying quantum memory, where the evolution of fidelity over time reveals the extent to which states are retained during storage and retrieval processes. With this foundation, we aim to quantify the extent to which spatially correlated quanton systems retain their initial quantum properties after undergoing transformations within interferometric setups. This is achieved by analyzing the fidelities between the initial and final quantum states derived in the preceding section. Specifically, to evaluate the fidelities of the P-BS and BS-P-BS configurations, we consider the initial state (1) and their respective final states, (3) and (4). This examination leads to the derivation of the following inner product expressions:

$$\langle\psi_{\text{in}}|\psi_{P-BS}\rangle = \frac{\sqrt{1-C^2}}{2}(\cos\theta_+ + \cos\theta_-) \tag{6}$$

$$\langle\psi_{\text{in}}|\psi_{BS-P-BS}\rangle = \left(\frac{1+C}{2}\right)\cos\frac{\theta_+}{2} + \left(\frac{1-C}{2}\right)\cos\frac{\theta_-}{2}$$

Considering (5) and (6), the fidelities for each system are then calculated as:

$$F(\psi_{in},\psi_{P-BS}) = |(1-C^2)\cos\theta_R\cos\theta_L| \tag{7}$$

$$F(\psi_{in},\psi_{BS-P-BS}) = \left|\cos\frac{\theta_R}{2}\cos\frac{\theta_L}{2} - C\sin\frac{\theta_R}{2}\sin\frac{\theta_L}{2}\right|^2$$

These expressions reveal the dependence of fidelity on the concurrence and the phase of the retarders θ_R and θ_L, providing a quantitative understanding of the system's evolution and the impact of entanglement. The fidelities in (7) are both constrained within the interval [0,1]. This result aligns with the fundamental properties of fidelity, where $F = 1$ corresponds to perfect overlap between the initial and final states, indicating no loss of quantum information, while $F = 0$ represents completely orthogonal states.

Accordingly, in the first scenario where $C = 1$ (or equivalently for $\alpha = \pi/4$) the fidelity becomes $F = 0$ for all maximally entangled states, regardless of the phase of the retarder. For $C = 1$ and $\theta_R = \theta_L = 0$ (no retarders are present) or $\theta = \frac{1}{2}\arctan(-\frac{\sqrt{1-C^2}}{C})$, it implies that while spatial correlations holds, fully orthogonal states emerge [14, 23-24]. In this setup, achieving a fidelity of $F = 1$ is possible only if $C = 0$ and simultaneously both retarder phases are either 0 or π. The dependence on $(1 - C^2)$ ensures that the fidelity decreases with increasing C, reflecting how the system's evolution impacts the preservation of its initial configuration.

In the second scenario, $F(\psi_{in}, \psi_{BS-P-BS})$ highlights the influence of the generating angle and phase adjustments in controlling the quantum system's evolution. The fidelity can reach $F = 1$ when $C = 1$, unlike the previous case. For the maximally entangled state, $F = \cos^2 \frac{\theta_+}{2}$ rendering it a function of the retarder (see Fig. 2).

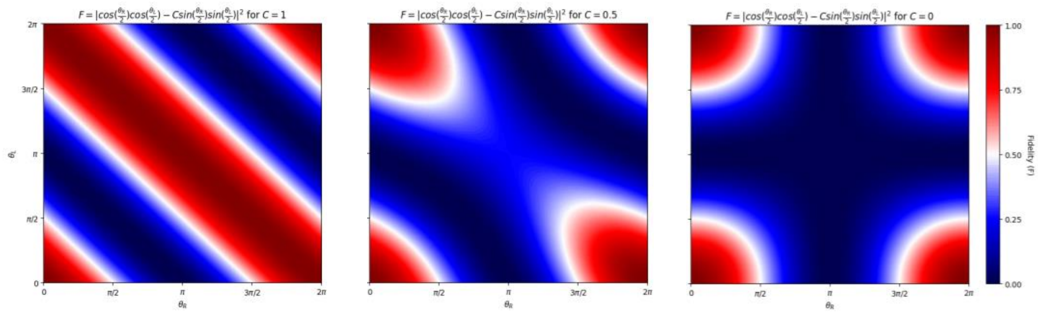

FIGURE 2. Graphs of fidelity as a function of retarder phases for various concurrence values in BS-P-BS configuration (animation).

In this configuration, for the values of $\theta = \frac{1}{2}\arctan(\frac{\sqrt{1-C^2}}{-C})$, which sustain the existing correlations, the fidelity will be preserved ensuring the retention of the state's original integrity [14]. Hence, these setups gain prominence for applications in teleportation fidelity and the construction of related quantum channels. As C approaches 0, the retarder phases diverge such that fidelity is preserved at 0 and 2π.

4. RESULTS

In this work, we investigate the fidelities of spatially correlated quanton systems processed through MZ-type P-BS and BS-P-BS configurations. By quantifying path entanglement through the concurrence $(C(\alpha))$, we analyze how these setups influence the fidelity under varying retarder phase conditions. Both setups provide insights into how spatial correlations and entanglement evolve under specific interferometric transformations, and the concurrence served as a critical parameter in quantifying path entanglement sourced by the generation angles.

The results reveal a fundamental distinction between the two configurations: for the P-BS system, maximal entanglement $(C = 1)$ invariably leads to $F = 0$, indicating a complete loss of overlap between initial and final states regardless of phase adjustments. As C decreases, the fidelity of the P-BS system increases. However, when $C = 0$, $F = 1$ can be achieved under specific conditions, preserving the initial quantum state without loss of information. Conversely, the BS-P-BS system demonstrates greater flexibility, achieving $F = 1$ even at $C = 1$ through precise tuning of the retarder phases θ_R and θ_L. These phases play a critical role in modulating the overlap between the initial and final states, allowing the system to sustain quantum correlations even at maximal entanglement. This behavior underscores the robustness of the BS-P-BS configuration, making it particularly advantageous for applications requiring the preservation of entangled states, such as quantum teleportation and the design of high-fidelity quantum channels. Furthermore, the presence of fully closed trajectories in the BS-P-BS system provides a suitable framework for investigating quantum mechanical phenomena that may share a common origin, and practical insights for applications.

In conclusion, the proposed setups, along with concurrence and phase retarder, offer a framework for optimizing interferometric configurations to preserve quantum fidelity. These findings deepen our understanding of entanglement dynamics and pave the way for more reliable quantum technologies, particularly through the BS-P-BS configuration's ability to maintain high fidelity at maximum entanglement.

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