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ON WEAK SYMMETRIES OF *(k, μ)*– CONTACT METRIC MANIFOLDS

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ABSTRACT

In this study, we consider weakly symmetric and weakly Ricci-symmetric *(k, μ)*-contact metric manifolds. We find necessary conditions in order that a *(k, μ)*-contact metric manifold be weakly symmetric and weakly Ricci symmetric.

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Key Words: *Weakly symmetric , weakly Ricci-symmetric, (k, μ)-contact metric manifolds.*

(k, μ)–DEĞME METRİK MANİFOLDLARIN ZAYIF SİMETRİLERİ ÜZERİNE

ÖZET

Bu çalışmada, zayıf simetrik ve zayıf Ricci-simetrik *(k, μ)*-değme metrik manifoldları göz önüne aldık. *(k, μ)*-değme metrik manifoldların zayıf simetrik ve zayıf Ricci-simetrik olması için gerekli şartları bulduk.

Anahtar Kelimeler: *Zayıf simetrik, zayıf Ricci-simetrik, (k, μ)-değme metrik manifoldlar.*

1. INTRODUCTION

Let *(M, g)* be an n-dimensional, n≥2, semi-Riemannian manifold of class C^∞. We denote by ∇ the Levi-Civita connection. Then we have

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

The Riemannian-Christoffel tensor and the Ricci tensor of (M, g) are defined by $R(X,Y,Z,W)=g(R(X,Y)Z,W)$ and

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i) \tag{1}$$

respectively, where X, Y, Z, W $\in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on M and $\{e_1, e_2, ..., e_n\}$ is a local orthonormal basis for the vector fields on M.

A non-flat differentiable manifold (M^n, g), (n>3), is called pseudosymmetric if there exists a 1-form α on M such that

$$(\nabla_X R)(Y,Z,W) = 2\alpha(X)R(Y,Z)W+\alpha(Y)R(X,Z)W$$
$$+\alpha(Z)R(Y,X)W+\alpha(W)R(Y,Z)X+g(R(Y,Z)W,X)A,$$

where $X, Y, Z, W \in \chi(M)$ are arbitrary vector fields and $A \in \chi(M)$ is the vector field corresponding through g to the 1-form α which is given by $g(X, A)=\alpha(A)$ ([4]).

A non-flat differentiable manifold (M^n, g), $(n>3)$, is called weakly symmetric if there exists a vector field P and 1-forms $\alpha, \beta, \gamma, \delta$ on M such that

$$(\nabla_X R)(Y,Z,W) = \alpha(X)R(Y,Z)W+\beta(Y)R(X,Z)W \quad (2)$$
$$+\gamma(Z)R(Y,X)W+\delta(W)R(Y,Z)X+g(R(Y,Z)W,X)P,$$

holds for all vector fields $X, Y, Z, W\in\chi(M)$ ([10] and [11]). A weakly symmetric manifold (M, g) is pseudosymmetric if $\beta=\gamma=\delta=\frac{1}{2}\alpha$ and $P=A$, locally symmetric if $\alpha=\beta=\gamma=\delta=0$ and $P=0$. A weakly symmetric is said to be proper if at least one of the 1-forms $\alpha, \beta, \gamma, \delta$ is not zero or $P\neq 0$.

A differentiable manifold (M^n, g), $(n>3)$, is called weakly Ricci-symmetric if there exists 1-forms $\varepsilon, \sigma, \rho$ such that the condition

$$(\nabla_X S)(Y,Z) = \varepsilon(X)S(Y,Z)+\sigma(Y)S(X,Z)+\rho(Z)S(X,Y), \quad (3)$$

holds for all vector fields $X, Y, Z\in\chi(M)$ ([10] and [11]). If $\varepsilon=\sigma=\rho$ then M is called pseudo Ricci-symmetric ([5]).

From (2), an easy calculation shows that if M is weakly symmetric then we have

$$(\nabla_X S)(Z,W) = \alpha(X)S(Z,W)+\beta(R(X,Z)W) \quad (4)$$
$$+\gamma(Z)S(X,W)+\delta(W)S(X,Z)+p(R(X,W)Z),$$

where P is defined by $p(X)=g(X, P)$ for all $X\in\chi(M)$ ([11]).

In [11], the authors considered weakly symmetric and weakly Ricci-symmetric Einstein and Sasakian manifolds. In [5], the authors studied weakly symmetric and weakly Ricci-symmetric K-contact manifolds. Also, in [1], the authors studied pseudosymmetric contact metric manifolds of Chaki type. In this study we consider weakly symmetric and weakly Ricci-symmetric (k, μ)-contact metric manifolds.

2. PRELIMINARIES

Let M be a $(2n+1)$-dimensional contact metric manifold with structure tensors (φ, ξ, η, g). Then the structure tensors satisfy are following equations

$$\varphi^2 = -I+\eta\otimes\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(X) = g(X,\xi) \quad (5)$$
$$g(\varphi X, \varphi Y) = g(X,Y)-\eta(X)\eta(Y), \quad g(\varphi X,Y) = d\eta(X,Y), \quad (6)$$

for any vector field X and Y on M [2]. The (1,1)-tensor field h defined by $h=-\frac{1}{2}L_\xi\varphi$, where L denotes Lie differentiation. Then the vector field ξ is Killing if and only if h vanishes. It is well known that h and φh are symmetric operators, h anti-commutes with φ (i.e., $\varphi h + h\varphi=0$), $h\xi=0$, $\eta o h=0$, $trh=0$ and $tr\varphi h=0$, where trh denotes the trace of h. Since h anti-commutes with φ, if X is an eigenvector of h corresponding to the eigenvalue

λ then φX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. Moreover, for any contact metric manifold M, the following is satisfied

$$\nabla_X \xi = -\varphi X - \varphi h X \tag{7}$$

here ∇ is the Riemannian connection of g. If ξ is Killing on a contact metric manifold M, then M is said to be a K-contact Riemannian manifold. We also recall that on a K-contact Riemannian manifold it is valid $R(X,\xi)\xi = X - \eta(X)\xi$.

The *(k, μ)-nullity distribution* of a Riemannian manifold (M, g) for a real numbers k, μ is a distribution

$$N(k,\mu): p \rightarrow N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]$$
$$+\mu[g(Y,Z)hX - g(X,Z)hY]\}$$

for any $X,\ Y \in Tp(M)$. We consider that M is a contact metric manifold with belonging ξ to the (k, μ)-nullity distribution i.e.[3],

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{8}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{9}$$

$$S(X,\xi) = 2nk\eta(X), \tag{10}$$

$$Q\xi = 2nk\xi. \tag{11}$$

In particular, on a contact metric manifold, M is Sasakian if and only if $k=1$ and $\mu=0$.

3. MAIN RESULTS

In this chapter we investigate weakly symmetric and weakly Ricci-symmetric (k, μ)-contact metric manifolds. Firstly we have:

Theorem 1 There exists no weakly symmetric (k, μ)-contact metric manifold M^{2n+1}, $(k\neq 0)$, $n>1$, if $\alpha+\gamma+\delta$ is not everywhere zero.

Proof. Assume that M^{2n+1} is a weakly symmetric (k, μ)-contact metric manifold. Putting $W=\xi$ in (4) we get

$$(\nabla_X S)(Z,\xi) = \alpha(X)S(Z,\xi) + \beta(R(X,Z)\xi)$$
$$+\gamma(Z)S(X,\xi) + \delta(\xi)S(X,Z) + p(R(X,\xi)Z). \tag{12}$$

So using (8), (9) and (10) we have

$$(\nabla_X S)(Z,\xi) = 2nk\alpha(X)\eta(Z) + k\beta(X)\eta(Z) - k\beta(Z)\eta(X)$$
$$+\mu\beta(hX)\eta(Z) - \mu\beta(hZ)\eta(X) + 2nk\gamma(Z)\eta(X)$$
$$+\delta(\xi)S(X,Z) + k\eta(Z)p(X) - kg(X,Z)p(\xi)$$
$$+\mu\eta(Z)p(hX). \tag{13}$$

By the covariant differentiation of the Ricci tensor S, the left side can be written as

$$(\nabla_X S)(Z,\xi) = \nabla_X S(Z,\xi)-S(\nabla_X Z,\xi)-S(Z,\nabla_X \xi).$$

By the use of (7), (10) and the parallelity of the metric tensor g we have

$$(\nabla_X S)(Z,\xi) =-2nkg(\varphi X,Z)-2nkg(\varphi hX,Z)+S(\varphi X,Z)+S(\varphi hX,Z). \tag{14}$$

Comparing the right hand sides of (13) and (14), we obtain

$$\begin{aligned}-2nkg(\varphi X,Z)-2nkg(\varphi hX,Z) &+S(\varphi X,Z)+S(\varphi hX,Z) \\ &= 2nk\alpha(X)\eta(Z)+k\beta(X)\eta(Z) \\ &\quad -k\beta(Z)\eta(X)+\mu\beta(hX)\eta(Z)-\mu\beta(hZ)\eta(X) \\ &\quad +2nk\gamma(Z)\eta(X)+\delta(\xi)S(X,Z) \\ &\quad +k\eta(Z)p(X)-kg(X,Z)p(\xi)+\mu\eta(Z)p(hX).\end{aligned} \tag{15}$$

Putting $X=Z=\xi$ in (15) and using (5), (6) and (10) we get

$$2nk[\alpha(\xi)+\gamma(\xi)+\delta(\xi)] = 0.$$

Since $n>1$ and $k\neq0$, we obtain

$$\alpha(\xi)+\gamma(\xi)+\delta(\xi) = 0. \tag{16}$$

So vanishing of the 1-form $\alpha+\gamma+\delta$ over the vector field ξ necessary in order that M be a (k, μ)-contact metric manifold.

Now we will show that $\alpha+\gamma+\delta=0$ holds for all vector fields on M.

In (4), taking $Z=\xi$, similar to the previous calculations it follows that

$$\begin{aligned}-2nkg(\varphi X,W)-2nkg(\varphi hX,Z) &+S(\varphi X,W)+S(\varphi hX,Z) \\ &= 2nk\alpha(X)\eta(W)+k\beta(X)-k\beta(\xi)\eta(X) \\ &\quad +\mu\beta(hX)+2nk\gamma(\xi)\eta(X)+2nk\delta(\xi)\eta(X) \\ &\quad +kp(X)-k\eta(X)p(\xi)+\mu p(hX).\end{aligned} \tag{17}$$

Replacing W with ξ in (17) and by making use of (5), (8) and (10) we have

$$\begin{aligned}&2nk\alpha(X)+k\beta(X)-k\beta(\xi)\eta(X) \\ &\quad +\mu\beta(hX)+2nk\gamma(\xi)\eta(X)+2nk\delta(\xi)\eta(X) \\ &\quad +kp(X)-k\eta(X)p(\xi)+\mu p(hX) = 0.\end{aligned} \tag{18}$$

Putting $X=\xi$ in (17) and by virtue of (5), (8) and (10) we find

$$2nk\alpha(\xi)\eta(W)+2nk\gamma(\xi)\eta(W)+2nk\delta(W) \tag{19}$$

$$+k\eta(W)p(\xi)-kp(W)-\mu p(hW) = 0.$$

Replacing W with X in (19) and taking the summation with (18), in view of (16), we obtain

$$2nk\alpha(X)+k\beta(X)-k\beta(\xi)\eta(X) \tag{20}$$

$$+\mu\beta(hX)+2nk\delta(X)+2nk\gamma(\xi)\eta(X) = 0.$$

Now putting $X=\xi$ in (15) we have

$$2nk\alpha(\xi)\eta(Z)+k\beta(\xi)\eta(Z)-k\beta(Z) \tag{21}$$

$$-\mu\beta(hZ)+2nk\gamma(Z)+2nk\delta(\xi)\eta(Z) = 0.$$

So replacing Z with X in (21) and taking the summation with (20), in view of (16), we find

$$2nk[\alpha(X)+\gamma(X)+\delta(X)]= 0.$$

Since $n>1$ and $k\neq 0$, we get

$$\alpha(X)+\gamma(X)+\delta(X) = 0,$$

for all X. This implies $\alpha+\gamma+\delta=0$, which completes the proof of the theorem.

Theorem 2 There exists no weakly Ricci-symmetric (k, μ)-contact metric manifold M^{2n+1}, $(k\neq 0)$, $n>1$, if $\varepsilon+\sigma+\rho$ is not everywhere zero.

Proof. Assume that M^{2n+1} is a weakly Ricci-symmetric (k, μ)-contact metric manifold. Replacing Z with ξ in (3) and using (10) we have

$$(\nabla_X S)(Y,\xi) = 2nk\varepsilon(X)\eta(Y)+2nk\sigma(Y)\eta(X)+\rho(\xi)S(X,Y). \tag{22}$$

Replacing Z with Y in (14) and comparing the right hand sides of the equations (22) and (14) we obtain

$$-2nkg(\varphi X,Y)-2nkg(\varphi hX,Z)+S(\varphi X,Y)+S(\varphi hX,Z) \tag{23}$$

$$= 2nk\varepsilon(X)\eta(Y)+2nk\sigma(Y)\eta(X)+\rho(\xi)S(X,Y).$$

Taking $X=Y=\xi$ in (23) and by making use of (5), (6) and (10) we get

$$2nk[\varepsilon(\xi)+\sigma(\xi)+\rho(\xi)] = 0,$$

which gives, (since $n>1$ and $k\neq 0$),

$$\varepsilon(\xi)+\sigma(\xi)+\rho(\xi) = 0. \tag{24}$$

Putting $X=\xi$ in (23) we have

$$2nk\eta(Y)[\varepsilon(\xi)+\rho(\xi)]+2nk\sigma(Y) = 0.$$

So by virtue of (24) this yields $2nk[\eta(Y)\sigma(\xi)+\sigma(Y)]=0$,which gives us (since $n>1$ and $k\neq 0$)

$$\sigma(Y) = \sigma(\xi)\eta(Y). \tag{25}$$

Similarly taking $Y=\xi$ in (23) we also have

$$\varepsilon(X)+\eta(X)[\sigma(\xi)+\rho(\xi)] = 0.$$

Applying (24) into the last equation we get

$$\varepsilon(X) = \varepsilon(\xi)\eta(X). \tag{26}$$

Since $(\nabla_\xi S)(\xi,X)=0$, then from (3) we obtain

$$2nk\eta(X)[\varepsilon(\xi)+\sigma(\xi)]+2nk\rho(X) = 0. \tag{27}$$

So by making use of (24), the equation (27) reduces to

$$\rho(X) = \rho(\xi)\eta(X). \tag{28}$$

Therefore the summation of the equations (25), (26) and (28) give us

$$\varepsilon(X)+\sigma(X)+\rho(X) = (\varepsilon(\xi)+\sigma(\xi)+\rho(\xi))\eta(X),$$

and then, from (24), it follows that

$$\varepsilon(X)+\sigma(X)+\rho(X) = 0,$$

for all X. Thus $\varepsilon+\sigma+\rho=0$. Our theorem is proved.

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