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Research Article

For a Depensatory Fishery System Hybrid Modeling and Optimal Control of Harvest Policies

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Abstract

Recent decades have brought an increasing concern for the sustainability of renewable resources, such as agricultural land, freshwater, forests, and fisheries. Management and control of them have been conducted through some institutions and governments, which mainly focus on efficiently managing those resources since they are affected by social and ecological uncertainties like climate change, difficulty in the application strategies, or uncertainties and noise in the data collection. Control engineering procedures represent a flexible and reasonable way to investigate and solve the difficulties, uncertainties, and noise listed above by formulating the problem mathematically.
In this work, we investigate fisheries and revenue optimization by using a hybrid model. The harvest of the fishery is done during some seasons of the year, which suggests that the model should include both discrete and continuous dynamics. To investigate the bio-economic system, the problem is formulated by two-hybrid dynamical fishery models. Those formulations are used to investigate optimal control and the stability of the sustainability of the system. In this respect, we investigate the optimal effort for the maximization of the revenue where the continuation of sustainability is preserved. Moreover, which parameters should be taken into account to check the stability in this case are determined. Whenever the system is unstable, the optimal effort for the sustainability of the system is determined.

1. Introduction

There are some critical concerns about the sufficiency and sustainability of the resources, as the population of the world increases. This forces foundations and governments to take action in finding effective policies to avoid the extinction of those resources. Sustainability, in simple terms, means the transfer of renewable resources to future generations without endangering them. Primarily considered renewable resources are agricultural lands, forests, water basins, and fisheries. Some of those resources face the risk of extinction even though a large amount of effort has been put into action for achieving sustainability. Therefore, methodologies from different research areas may give different insights into the problem. To achieve a more systematic view, there are some basic questions to be answered and considered. The amount and the sustainability of the resource, economic concerns due to the harvest, and the amount and the period of the harvesting are the first topics that come into mind.
There are three major aspects of the problem: economic, environmental, and social factors. They must be taken together to obtain a more realistic and optimum decision. To provide sustainability of the resource, the model should determine not only the amount of the resource but also the harvesting period and the harvesting amount. From this point of view, control system methodologies provide a better understanding of the dynamics of the system and also aid in the determination of the parameters listed above. The tools used by a control engineer help to gather robust methods for the complexities mentioned earlier and overcome the difficulties faced when applying the policy for this socioeconomic problem.

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The contentment of the public leads institutions and governments to pay more attention to the problem of sustainability of renewable resources. The essential resources in the literature that have been widely investigated are agricultural land [1,2], forests [3,4], freshwater basins [5,6] and fishery [7,8]. If fishing is considered, one can realize the crisis of an abrupt reduction in fish stocks. In the last half century, almost one of four fisheries collapsed [9]. Thus, this gives a powerful motivation to investigate the transfer of the resources to future generations. For effective management of these resources, the economic, ecological, and social aspects of the problem should be considered. Unexpected factors in the management of renewable resources or disregarding the economical aspects of market actors may result in ineligible resource management [7]. The tools and methods of control systems give a flexible and powerful way to overcome the difficulties in the implementation process to manage the fishery. Firstly, a mathematical model that represents the real-world problem most simply and approximately possible should be constructed. Then, the control system techniques are implemented to overcome the complexities mentioned above. Mathematical models of fishery date back to population dynamics [10, 11], and with the developments in the modeling techniques, they are also reconsidered in previous decades [12–15].

It is possible to find new approaches to the management of renewable sources from the perspective of sustainability control, and socioeconomic reflections on the management of those resources [16]. Fisheries' robust control [16, 17], harvest of the prey predator-prey system's feedback control [18], an optimal control problem to solve dynamic deterrence [19], and social-economical perspectives considering public policies [20, 21] are solid attempts to consider sustainability of the fishery with a different approach.

If one checks the literature on the resources that are sustainably managed, it is easy to realize that the models are mostly either continuous or discrete. Moreover, economic models, such as analysis of supply-demand curves or game-theoretical perspectives, are considered. However, in the real-world problem case, the system appears to be displaying continuous and also discrete dynamics if investigated carefully. Therefore, the use of a hybrid dynamical system modeling formulation is a beneficial choice. Hybrid dynamical systems are a challenging modeling scheme and they have taken the attention of various researchers from various disciplines in the last decades. Even though the idea of combining discrete and continuous dynamics may seem trivial to implement in a real-world problem, the mathematical formulation of the problem has been rather complex and differed among the researchers' needs. This type of modeling, despite its rather challenging arrangement, is giving us a way to obtain new aspects in the sense of stability [22–25]. In the literature of fishery models, there exist some hybrid models such as aquaculture models [26, 27] that consider the shrimp population using an impulsive model, the fishery models with on-off harvesting [12, 28], continuous biological, discrete economical variables [29], global dynamics with Poincare maps [30], or the version of the stochastic hybrid system of the fishery system [13].

In this work, to check the sustainable management of the fishery, a hybrid system modeling technique is used. For a better analysis and to overcome the difficulties in the management of the fishery, we consider the hybrid models in the sense of stability and optimal control. The reason for a hybrid model is that harvest of the fishery is applied only in some seasons and for the off seasons the population is left for reproduction. And therefore, the total sum of revenue will be positive only during those "on" seasons. For this purpose, a hybrid depensatory model is obtained and analyzed in terms of Lyapunov stability, and an investigation of optimal control is conducted. As a result, the optimal harvest rate and period are obtained dynamically. By checking the stability, new optimum values are obtained if the system is affected by a random factor.

The organization of the paper is as follows: in Sections 2, 3, and 4, a hybrid model is constructed, and optimal control and stability analysis are investigated, respectively. Finally, a summary and an outlook for new directions are introduced in Section 5.

2. Hybrid Model

Switching systems are one of the most investigated versions of hybrid systems. Therefore, firstly, we consider the switching system formalism to represent the hybrid nature of the problem. Consider the following system;

$$\dot{x} = f_{q(t)}(t, x(t), u(t)),$$

where for each discrete state $q \in Q$, a vector field $f_{q(t)}$ exists. In this section, we give two different hybrid models. The main reason for considering two models is that the solution of the optimal control problem will be an easy one when compared to the two-dimensional optimal control problem. Moreover, using a two-dimensional model, it becomes possible to conduct a stability analysis. Therefore, firstly, a one-dimensional hybrid system is constructed using Equation (2.1). Then a hybrid optimal control problem is solved. Secondly, a two-dimensional model for the same model is constructed and an analysis of the model considering the aspects of dynamical systems is given. Consider the fishery model in [16],

$$\dot{x} = rx^2(1 - \frac{x}{k}) - qux, \; x(0) = x_0. \tag{2.1}$$

Moreover, an optimal control law is obtained by solving an optimal control problem as in [16]

$$u(t) = \begin{cases} 0, & G(x,t) < 0, \\ u_{max}, & G(x,t) > 0, \end{cases}$$

where $G(x,t) = e^{-\delta t}(pqx - c) - \lambda qx$. By rearranging and calculating the Hamiltonian, this is accomplished. This is a problem that can be investigated as a hybrid dynamical system. For the one-dimensional case, we consider hybrid automata representation. Consider the states $Q = \{1, 2\}$ and the flows of those states are $f(1,x) = F(x)$ and $f(2,x) = F(x) - qu_{max}x$. The edges are $E = \{(1,2),(2,1)\}$. The guard conditions are $G(1,2) = (c, \infty)$ and $G(2,1) = (-\infty, c]$. Hybrid automata of this system can be seen in Figure 2.1. An optimal control is applied to this hybrid model. By applying a hybrid formulation to the system, we only find the trajectories that switch along the way. And find the optimum among them.

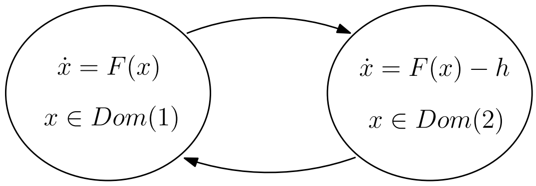

Figure 2.1: Fishery hybrid automata representation [31]

3. Optimal Control of the Model

Suppose that ψ_i represents the portion of fish stock eliminated from the ecosystem at ith harvest. This value is represented by Equation (2.1). Thus, the control variable u can be restated as $u_i = \frac{\psi_i}{q}$. Assume that τ_i represents the ith harvest time. Then the decision variables τ_i, $i = 1,...,m$ and ψ_i, $i = 1,...,m$ captures the conditions

$$0 \leq \tau_1 \leq \tau_2 \leq ... \leq \tau_m,$$

and

$$0 \leq \psi_i \leq 1, \ \ i = 1,...,m. \tag{3.1}$$

Now, the total revenue obtained within the time period $[0, \tau_m]$ is given by

$$R = \sum_{i=1}^{m} \{p\psi_i x(\tau_i^-) - H\}, \tag{3.2}$$

where p is the price and H (or cu) is the fixed cost of a single harvest. The problem is to find the optimal harvest times τ_i $i = 1,...,m$ and the optimal harvest rate ψ_i, $i = 1,...m$ to maximize the total revenue subject to the constraints (3.1) and (3.2), of the dynamical system

$$\begin{aligned} \dot{x} &= F(x) - qux, \\ x(0) &= x_0, \end{aligned}$$

where x_0 is a given constant and

$$x(\tau_i^+) = x(\tau_i^-) - \psi_i x(\tau_i^-), \ \ i = 1,...,p.$$

This problem is equivalent to the following optimal control problem as described in [27].

Problem 3.1. *Find a pair* $(\theta^*, \xi^*) \in \Theta \times W$ *such that*

$$\bar{J} = \inf_{(\theta,\xi) \in \Theta \times W} \bar{J}(\theta, \xi).$$

Moreover, if the procedure in [26] is used, the optimal time to harvest and the harvest rate can be calculated for the solution of the remaining part of the problem. In this situation, the Hamiltonian is

$$H = \langle p, F(x) - qu_i x \rangle + p_0.$$

For the given fishery problem, the optimal fish stock value is considered as $x_0 = 72.549$. It is taken due to the results of [16] by simplification of the functional with inflation rate equal to 0, also omitting measurement noise, etc. Firstly, considering $c = 45.23$ and initial fish stock $x_0 = 55$ where $x_0 < x_{opt}$, we obtain some results which can be seen in Table 3.1.

x_0	Periodicity (of harvesting	Total amount of revenue
55	1 day	1628.691
	3 days	1627.924
	12 days	1628.926
	1 month	1630.164
	2 months	1647.470
	6 months	1665.567

Table 3.1: Total amount of revenue for different harvesting periodicity if $x_0 = 55$

Revenue is calculated with the periodicities 1 day, 3 days, 12 days, 1 month, 2 months, and 6 months. Secondly, the initial fish stock is chosen as $x_0 = 90$, i.e. $x_0 > x_{opt}$, and the revenue is calculated using the same periodicities. One may see the results in the Table 3.2. If the results are observed, it is evident that the optimal periodicity for the case $x_0 > x_{opt}$ and $x_0 < x_{opt}$ are completely distinct from each other. It can be seen that for the values of x_0 and x_{opt} if $x_0 < x_{opt}$ then 6 months is the optimal periodicity to harvest. On the contrary, if $x_0 > x_{opt}$ then 3 days is the optimal harvest periodicity. This is a result of the fact that if $x_0 < x_{opt}$ the population has to increase first and then the harvest starts. And if $x_0 < x_{opt}$, harvest of the fishery should start without the need for an initial wait to increase the population. The figures Figure 3.1 and Figure 3.2 represent the most reasonable choices.

If we consider different initial fish stock values and different harvesting periods, as calculated and given in the tables, we realize that the initial fish stock plays a crucial role. Above or below values than the optimum x value, x_{opt} will result in totally different harvesting periodicity and rate. In that case, we aim to maximize the revenue as stated earlier.

x_0	**Periodicity (of harvesting)**	**Total amount of revenue**
	1 day	1911.800
	3 days	1913.436
90	12 days	1911.623
	1 month	1912.917
	2 months	1902.134
	6 months	1866.425

Table 3.2: Total amount of revenue for different harvesting periodicity if $x_0 = 90$.

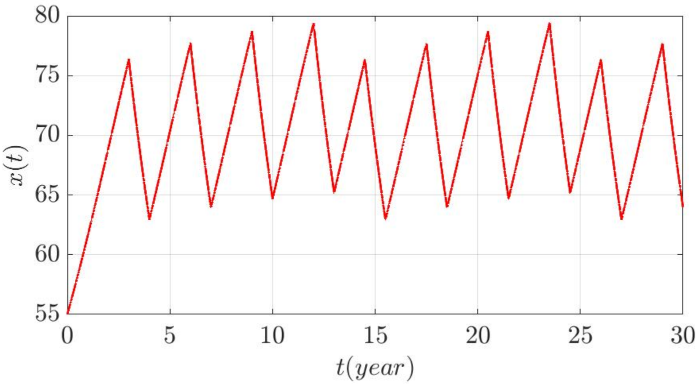

Figure 3.1: Biomass graph with initial fish stock $x_0 = 55$ and if harvesting period is 6 months.

4. Stability Analysis of the Model

For the hybrid model described by the hybrid automata in Figure 2.1, we have two different flows described by

$$\frac{dx}{dt} = F(x) - qu_i x,$$

where $i \in \{0,1\}$, and $u_0 = 0$ and $u_1 = u_{max}$. This indicates that the system switches between two flows

$$\begin{aligned} \frac{dx}{dt} &= F(x) - qu_{max}x, \quad \text{or} \\ \frac{dx}{dt} &= F(x). \end{aligned}$$

We choose Lyapunov function as $V(x) = \frac{x^2}{2}$. Then

$$\frac{\partial V}{\partial x} f_i(x) = x(F(x) - qu_i x) = xF(x) - qu_i x^2.$$

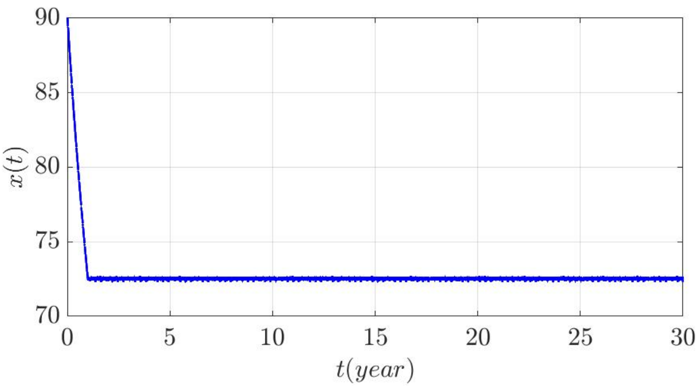

Figure 3.2: Biomass graph with initial fish stock $x_0 = 90$ and if harvesting period is 3 days.

For $F(x) = rx(1 - \frac{x}{K})$, we obtain,

$$\begin{aligned}\frac{\partial V}{\partial x} f_i(x) &= xrx(1 - \frac{x}{K}) - qu_i x^2 \\ &= rx^2 - \frac{rx^3}{K} - qu_i x^2 \\ &= x^2(r - \frac{rx}{K} - qu_i).\end{aligned}$$

Then the system is stable, if

$$r - \frac{rx}{K} - qu_i \leq -W(x),$$

where $W(x) : \mathbb{R}^n \to \mathbb{R}$ is a positive definite continuous function. This condition gives us some restrictions on the parameter values. Similar calculations imply

$$rx - \frac{rx^2}{K} - qu_i \leq -W(x),$$

if $F(x) = rx^2(1 - \frac{x}{K})$ is chosen as in [16]. Now, consider the following two-dimensional model [31]

$$\begin{aligned}\frac{dx}{dt} &= F(x) - qu_i x, \\ \frac{du}{dt} &= u_i(pqx - c),\end{aligned}$$

which is the same system of equations given by the equations in [32] except u in those equations is replaced by u_i. This effect creates a switching system that will be examined in terms of the global stability analysis. Choose $F(x) = rx(\frac{x}{k_c} - 1)(1 - \frac{x}{k})$ and calculate the binomic equilibrium ($\dot{x} = \dot{u}_i = 0$). A straightforward calculation gives the equilibrium points

$$\begin{aligned}x^* &= \frac{c}{pq} \\ u_i^* &= \frac{r(xk - kk_c - x^2 + xk_c)}{qkk_c},\end{aligned}$$

which is a unique equilibrium point [31]. Since $F(x)$ is a depensation model, the equilibrium point is unstable. Similarly, for a choice of $F(x) = rx^2(1 - \frac{x}{K})$, equilibrium points are

$$\begin{aligned}x^* &= \frac{c}{pq} \\ u_i^* &= \frac{rc}{pq^2}(1 - \frac{c}{kpq}).\end{aligned}$$

The other equilibrium point is $(x^*, u_i^*) = (\frac{c}{pq}, 0)$.

5. Conclusion and Discussion

The fishery problem is considered by using a hybrid system formulation. To obtain optimal control problem and stability analysis one-dimensional and two-dimensional models are considered. Firstly, an optimal control of the one-dimensional case is solved. Secondly, a two-dimensional version of the model is constructed and the stability analysis is explored. In this respect, optimal values are obtained, stability analysis is performed and important parameters in the model are determined. Thus, whenever the system is unstable, a new optimal control policy may be obtained and the stability analysis may be conducted.
Since the endangered fishery is gaining attention, this research area is very open to new applications. For example, considering the stochastic versions of the methods discussed in this work would produce more realistic results since the system is open to randomness. Moreover, since only a portion of the population can reproduce while the other part is still juvenile, consideration of age-structured models would be meaningful and this suggests us to use delay hybrid systems, for example [33]. Again, stability and optimal control problems may be investigated as in [34]. In addition to those different mathematical modeling aspects, more complex systems such as multi-species models can be considered as in the works of [35], and [36]. Of course, in that case, real-world measurements will allow us to calculate the system parameters reasonably. At this point, highly in fashion and proven to become effective machine learning techniques can be considered such as [37].

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