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Numerical Solution of Nonlinear Advection Equation Using Reproducing Kernel Method

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Abstract

In this study, an iterative approximation is proposed by using the reproducing kernel method (RKM) for the nonlinear advection equation. To apply the iterative RKM, specific reproducing kernel spaces are defined and their kernel functions are presented. The proposed method requires homogenising the initial or boundary conditions of the problem under consideration. After homogenising the initial condition of the advection equation, a linear operator selection is made, and then the approximate solution is constructed using orthonormal basis functions in serial form. Convergence analysis of the approximate solution is demonstrated through the lemma and theorem. Numerical outcomes are provided in the form of graphics and tables to show the efficiency and accuracy of the presented method.

1. Introduction

In this paper, an iterative reproducing kernel approximation is presented for obtaining a serial solution of the nonlinear advection equation as follows [1]:

$$y_\kappa(\zeta,\kappa)+y(\zeta,\kappa)y_\zeta(\zeta,\kappa)=f(\zeta,\kappa), \tag{1.1}$$

$$0\le\zeta\le 1, 0\le\kappa\le 1,$$

$$y(\zeta,0)=h(\zeta). \tag{1.2}$$

Here, $f(\zeta,\kappa)$ is a continuous function.

In environmental sciences, advection is transporting chemical or biological material by bulk motion. The advection equation has significant importance in meteorology and oceanography [2]. Various analytical and numerical methods have been proposed in the literature to obtain solutions to the advection equation. For instance, Khan and Wu proposed the homotopy perturbation transform method for the advection equation in [3], the Fourier series method is applied by Sanugi and Evans in [4], Wazwaz employed the Adomian decomposition method for the advection equation in [5], the finite difference method is presented by Molenkamp in [6], the Laplace decomposition method is employed in [7]. Nisar et al. [8] suggested a numerical technique for the nonlinear advection equation using the Pad*é* approximation. The explicit finite difference scheme is used to obtain a numerical solution of the advection diffusion equation by Ara et al. [9]. Cosgun and Sari [10] employed the reversed fixed point iteration for advection-diffusion processes. The homotopy analaysis method is implemented for the fractional advection equation by Alkan [11]. Mirza et al. [12] proposed an analytical solution to the fractional advection diffusion equation. Mirzaee et al. [13] suggested the finite difference and spline approximation for stochastic the advection-diffusion equation with fractional order.
The origin of the reproducing kernel method goes back to Zaremba's researches at the beginning of last century. He focused on boundary value problems with Dirichlet conditions in [14]. This concept is improved as theoretically in [15] and [16]. Also, some specific reproducing kernel spaces that have trigonometric and polynomial kernels are presented in [17]. The reproducing kernel method is applied to many model problems. For instance, Bagley-Torvik and Painlevé equations [18], fractional order systems [19], Fredholm integro-differential

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equations [20], integro-differential equations with Fredholm operator [21], eighth order boundary value problems [22], fractional Riccati differential equations [23], sine-Gordon equation [24], nonlinear system of PDEs [25], fractional advection-dispersion equation [26], time fractional telegraph equation [27], nonlinear hyperbolic telegraph equation [28], reaction-diffusion equations [29], time fractional partial integro-differential equations [30], class of fractional partial differential equation [31], time fractional Tricomi and Keldysh equations [32], and so on [33]-[38].
This paper is arranged as follows: Section 2 presents some specific reproducing kernel spaces and basic definitions. Section 3 provides a detailed explanation of the linear operator selection and the construction of the approximate solution for the nonlinear advection equation. In Section 4, a theorem and lemma show the convergence of the constructed approximate solution. In Section 5, the proposed method is tested on two equations, and the numerical outcomes are presented with tables and graphs to demonstrate the effectiveness of the method. Section 6 gives a brief conclusion.

Symbols and nomenclature

Notation	Meaning
κ	Time variable
ζ	Space variable
$W_2^{(2,2)}$	Special Hilbert space
Δ	$[0,1]\times[0,1]$
$T_{(t,x)}(\zeta,\kappa)$	Reproducing kernel function
AC	Absolutely continuous
L	Linear operator
CC	Completely continuous
$\omega(\zeta,\kappa)$	Exact solution
$\omega_n(\zeta,\kappa)$	Approximate solution
$\mathbb{C}$	Complex numbers
$L^2[0,1]$	Squared integrable Lebesgue space in $[0,1]$

2. Preliminaries

This section introduces the special one- and two-variable Hilbert spaces used in the construction of the approximate solution and the reproducing kernel functions of these spaces.

Definition 2.1. *Let $\Theta\neq\emptyset$ an abstract set, H be a Hilbert space and B is defined as $B:\Theta\times\Theta\to\mathbb{C}$.*

$$i. B(.,r)\in H,\quad \forall r\in\Theta,$$
$$ii. \langle\mu(.),B(.,r)\rangle=\mu(r)\quad \forall r\in\Theta,\ \forall\mu\in H.$$

If the above conditions are satisfied, then B and H are called reproducing kernel function and reproducing kernel Hilbert space, respectively.

Before the construction of the representation solution, some specific reproducing kernel spaces and their kernel functions will be given to solve the advection equation. The procedure for obtaining the reproducing kernels can be found in [36].

$\mathbf{W_2^1[0,1]}$ Hilbert space

$$W_2^1[0,1]=\{\tau(\zeta)\mid\tau \text{ is AC function},\ \tau'\in L^2[0,1]\}.$$

The inner product, norm and kernel function for the space $W_2^1[0,1]$ are given as follows.
1. The inner product:

$$\langle\tau(\zeta),\omega(\zeta)\rangle_{W_2^1}=\tau(0)\,\omega(0)+\int_0^1\tau'(\zeta)\,\omega'(\zeta)\,d\zeta.$$

2. The norm:

$$\|\tau\|^2_{W_2^1}=\langle\tau,\tau\rangle_{W_2^1},\quad \tau,\omega\in W_2^1[0,1].$$

3. The kernel function:

$$R_t^{\{1\}}(\zeta)=\begin{cases}1+\zeta, & \zeta\le t,\\ 1+t, & t>\zeta.\end{cases}$$

$\mathbf{W_2^2[0,1]}$ Hilbert space

$$W_2^2[0,1]=\{\tau(\zeta)|\tau,\ \tau'\text{are AC functions},\tau''\in L^2[0,1]\}$$

The inner product, norm and kernel function for the space $W_2^2[0,1]$ are given as follows.
1. The inner product:

$$\langle\tau(\zeta),\omega(\zeta)\rangle_{W_2^2}=\tau(0)\,\omega(0)+\tau'(0)\,\omega'(0)+\int_0^1\tau''(\zeta)\,\omega''(\zeta)\,d\zeta.$$

2. The norm:

$$\|\tau\|^2_{W_2^2} = \langle \tau, \tau \rangle_{W_2^2}, \ \omega, \tau \in W_2^2\,[0,\,1]\,.$$

3. The kernel function:

$$R_t^{\{2\}}(\zeta) = \begin{cases} 1+\zeta t+\frac{1}{2}t\zeta^2-\frac{1}{6}\zeta^3, & \zeta \le t, \\ 1-\frac{1}{6}t^3+\frac{1}{2}\zeta t^2+t\zeta, & \zeta > t. \end{cases}$$

In a similar manner to the above, namely under same inner product and norm, the following closed subspace of $W_2^2\,[0,\,1]$ can be defined as

$$W_2^2[0,1] = \{\tau(\zeta)|\tau,\tau^{'} \text{are AC functions}, \tau^{''} \in L^2[0,1], \tau(0)=0\},$$

and its kernel function is

$$R_x^{\{2\}}(\kappa) = \begin{cases} \kappa x+\frac{1}{2}x\kappa^2-\frac{1}{6}\kappa^3, & \kappa \le x, \\ -\frac{1}{6}x^3+\frac{1}{2}\kappa x^2+x\kappa, & \kappa > x. \end{cases}$$

$\mathbf{W_2^{(2,2)}}(\Delta)$ Hilbert space

Let be $\Delta = [0,1]\times[0,1]$. $W_2^{(2,2)}(\Delta)$ should be defined for obtain representation solution of model problem (1.1) subject to initial condition (1.2).

$$W_2^{(2,2)}(\Delta) = \{\omega(\zeta,\kappa)|\frac{\partial^2\omega}{\partial\zeta\partial\kappa} \text{ is completely continuous in } \Delta, \frac{\partial^4\omega}{\partial\zeta^2\partial\kappa^2} \in L^2(\Delta), \omega(\zeta,0)=0\}.$$

The inner product and norm for the space $W_2^{(2,2)}(\Delta)$ are given as follows.
1. The inner product :

$$\begin{aligned}\langle \omega(\zeta,\kappa),u(\zeta,\kappa)\rangle_{W_2^{(2,2)}} &= \sum_{i=0}^{1}\int_0^1 [\frac{\partial^2}{\partial\kappa^2}\frac{\partial^i}{\partial\zeta^i}\omega(0,\kappa)\frac{\partial^2}{\partial\kappa^2}\frac{\partial^i}{\partial\zeta^i}u(0,\kappa)]d\kappa + \sum_{j=0}^{1}\langle\frac{\partial^j}{\partial\kappa^j}\omega(\zeta,0),\frac{\partial^j}{\partial\kappa^j}u(\zeta,0)\rangle_{W_2^2} \\ &+ \int_0^1\int_0^1 [\frac{\partial^2}{\partial\zeta^2}\frac{\partial^2}{\partial\kappa^2}\omega(\zeta,\kappa)\frac{\partial^2}{\partial\zeta^2}\frac{\partial^2}{\partial\kappa^2}u(\zeta,\kappa)]d\zeta d\kappa, \ \omega,u \in W_2^{(2,2)}(\Delta).\end{aligned}$$

2. The norm:

$$\|\omega\|^2_{W_2^{(2,2)}} = \langle \omega,\omega\rangle_{W_2^{(2,2)}}, \ \omega \in W_2^{(2,2)}(\Delta).$$

The following basic theorem of reproducing kernel theory shows that the kernel function of $W_2^{(2,2)}(\Delta)$ is derived as multiplying of kernel functions of $W_2^2[0,1]$ for ζ and κ variables.

Theorem 2.2. *[36] Let $T_{(t,x)}(\zeta,\kappa)$ be a kernel function of $W_2^{(2,2)}(\Delta)$. So, it can be written that*

$$T_{(t,x)}(\zeta,\kappa) = R_t^{\{2\}}(\zeta)R_x^{\{2\}}(\kappa),$$

where $R_t^{\{2\}}(\zeta)$ and $R_x^{\{2\}}(\kappa)$ are reproducing kernel functions of $W_2^2[0,1]$. For any $\omega(\zeta,\kappa)\in W_2^{(2,2)}(\Delta)$

$$\omega(t,x) = \langle \omega(\zeta,\kappa), T_{(t,x)}(\zeta,\kappa)\rangle_{W_2^{(2,2)}}$$

and

$$T_{(\zeta,\kappa)}(t,x) = T_{(t,x)}(\zeta,\kappa).$$

$\mathbf{W_2^{(1,1)}}(\Delta)$ Hilbert space

$$W_2^{(1,1)}(\Delta) = \{\omega(\zeta,\kappa)|\ \omega \text{ is CC function in } \Delta, \frac{\partial^2\omega}{\partial\zeta\partial\kappa} \in L^2(\Delta)\}.$$

The inner product, norm and kernel function for the space $W_2^{(1,1)}(\Delta)$ are given as follows.
1. The inner product:

$$\begin{aligned}\langle \omega(\zeta,\kappa),u(\zeta,\kappa)\rangle_{W_2^{(1,1)}} &= \int_0^1 [\frac{\partial}{\partial\kappa}\omega(0,\kappa)\frac{\partial}{\partial\kappa}u(0,\kappa)]d\kappa + \langle\omega(\zeta,0),u(\zeta,0)\rangle_{W_2^1} \\ &+ \int_0^1\int_0^1 [\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\kappa}\omega(\zeta,\kappa)\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\kappa}u(\zeta,\kappa)]d\zeta d\kappa, \ \omega,u \in W_2^{(1,1)}(\Delta).\end{aligned}$$

2. The norm:

$$\|\omega\|^2_{W_2^{(1,1)}} = \langle \omega,\omega\rangle_{W_2^{(1,1)}}, \ \omega \in W_2^{(1,1)}(\Delta).$$

3. The kernel function:

$$\tilde{T}_{(t,x)}(\zeta,\kappa) = R_t^{\{1\}}(\zeta)R_x^{\{1\}}(\kappa).$$

3. Iterative Solution for Eqs. (1)-(2) in Space $W_2^{(2,2)}(\Delta)$

This section will explain how to construct an iterative solution for the nonlinear advection equation and provide the necessary theoretical information. First, the initial condition of Eq. (1.1) is homogenised, and then the linear operator selection is made. After the homogenisation process, the selection of the linear operator L is as follows:

$$L : W_2^{(2,2)}(\Delta) \to W_2^{(1,1)}(\Delta),$$

$$L\omega(\zeta,\kappa) = \omega_\kappa(\zeta,\kappa) + h(\zeta)\omega_\zeta(\zeta,\kappa) + h^{'}(\zeta)\omega(\zeta,\kappa). \tag{3.1}$$

The Eq. (3.1) can be expressed as:

$$\begin{cases} L\omega(\zeta,\kappa) = F(\zeta,\kappa,\omega(\zeta,\kappa),\omega_\zeta(\zeta,\kappa)), \quad \zeta,\kappa \in [0,1], \\ \omega(\zeta,0) = 0. \end{cases} \tag{3.2}$$

Here, $F(\zeta,\kappa,\omega(\zeta,\kappa),\omega_\zeta(\zeta,\kappa)) = f(\zeta,\kappa) - h^{'}(\zeta)h(\zeta) - \omega(\zeta,\kappa)\omega_\zeta(\zeta,\kappa)$.

If $\{(\zeta_i,\kappa_i)\}_{i=1}^{\infty}$ is a countable dense subset in Δ, then $\Psi_i(\zeta,\kappa)$ is defined as:

$$\begin{aligned} \Psi_i(\zeta,\kappa) &= L_{(t,x)}T_{(t,x)}(\zeta,\kappa)|_{(t,x)=(\zeta_i,\kappa_i)} \\ &= \{\frac{\partial}{\partial x}T_{(t,x)}(\zeta,\kappa) + h(t)\frac{\partial}{\partial t}T_{(t,x)}(\zeta,\kappa) + h^{'}(t)T_{(t,x)}(\zeta,\kappa)\}|_{(t,x)=(\zeta_i,\kappa_i)} \\ &= \frac{\partial}{\partial \kappa}T_{(\zeta_i,\kappa_i)}(\zeta,\kappa) + h(\zeta_i)\frac{\partial}{\partial t}T_{(\zeta_i,\kappa_i)}(\zeta,\kappa) + h^{'}(\zeta_i)T_{(\zeta_i,\kappa_i)}(\zeta,\kappa). \end{aligned} \tag{3.3}$$

The following theorem shows that $\Psi_i(\zeta,\kappa)$ is completely continuous and linear operator L is bounded.

Theorem 3.1. $\Psi_i(\zeta,\kappa) \in W_2^{(2,2)}(\Delta), i = 1,2,...$

Proof. The following conditions should be provide to prove this theorem.
1. $\frac{\partial^4\Psi_i(\zeta,\kappa)}{\partial\zeta^2\partial\kappa^2} \in L^2(\Delta)$
2. $\frac{\partial^2\Psi_i(\zeta,\kappa)}{\partial\zeta\partial\kappa}$ is completely continuous function
3. $\Psi_i(\zeta,\kappa)$ satisfies the initial condition.
One can show that any elements of $W_2^{(2,2)}(\Delta)$ satisfies the above conditions 1-3.
Now, from the kernel function property, the following equation can be written

$$\partial^5_{t\zeta^2\kappa^2}T_{(t,x)}(\zeta,\kappa) = \partial^3_{t\zeta^2}R_t^{\{2\}}(\zeta)\partial^2_{\kappa^2}R_x^{\{2\}}(\kappa).$$

The $\partial^3_{t\zeta^2}R_t^{\{2\}}(\zeta)$ and $\partial^2_{\kappa^2}R_x^{\{2\}}(\kappa)$ functions are bounded in $[0,1]$ due to their continuity in $[0,1]$. Therefore, the following inequality can be expressed:

$$|\partial^5_{t\zeta^2\kappa^2}T_{(t,x)}(\zeta,\kappa)| \le M_1.$$

The following inequalities can be written by the same way of above:

$$|\partial^5_{x\zeta^2\kappa^2}T_{(t,x)}(\zeta,\kappa)| \le M_2,$$
$$|\partial^4_{\zeta^2\kappa^2}T_{(t,x)}(\zeta,\kappa)| \le M_3.$$

Here, M_1, M_2 and M_3 are positive constants. From (3.3),

$$\begin{aligned} |\frac{\partial^4\Psi_i(\zeta,\kappa)}{\partial\zeta^2\partial\kappa^2}| &\le |M_2 + h(\zeta_i)M_1 + h^{'}(\zeta_i)M_3| \\ &\le M_2 + |h(\zeta_i)|M_1 + |h^{'}(\zeta_i)|M_3. \end{aligned}$$

Therefore, $\frac{\partial^4\Psi_i(\zeta,\kappa)}{\partial\zeta^2\partial\kappa^2} \in L^2(\Delta)$. Noting that Δ is closed, thus, $\frac{\partial^2\Psi_i(\zeta,\kappa)}{\partial\zeta\partial\kappa}$ is completely continuous in Δ. And also, $\Psi_i(\zeta,\kappa)$ satisfies the initial condition because $T_{(t,x)}(\zeta,0) = 0$. Thus $\Psi_i(\zeta,\kappa) \in W_2^{(2,2)}(\Delta)$.

Theorem 3.2. $\{\Psi_i(\zeta,\kappa)\}_{i=1}^{\infty}$ *is a complete system of* $W_2^{(2,2)}(\Delta)$, *for* $i = 1,2,...$.

Proof. We have

$$\begin{aligned} \Psi_i(\zeta,\kappa) &= (L^*\Phi_i)(\zeta,\kappa) = \langle (L^*\Phi_i)(t,x), T_{(\zeta,\kappa)}(t,x)\rangle_{W_2^{(2,2)}} \\ &= \langle \Phi_i(t,x), L_{(t,x)}T_{(\zeta,\kappa)}(t,x)\rangle_{W_2^{(1,1)}} = L_{(t,x)}T_{(\zeta,\kappa)}(t,x)|_{(t,x)=(\zeta_i,\kappa_i)} \\ &= L_{(t,x)}T_{(t,x)}(\zeta,\kappa)|_{(t,x)=(\zeta_i,\kappa_i)}. \end{aligned}$$

Clearly, $\Psi_i(\zeta,\kappa)\in W_2^{(2,2)}(\Delta)$, for each fixed $\omega(\zeta,\kappa)\in W_2^{(2,2)}(\Delta)$, if $\langle\omega(\zeta,\kappa),\Psi_i(\zeta,\kappa)\rangle_{W_2^{(2,2)}}=0$.
Namely,

$$\langle\omega(\zeta,\kappa),(L^*\Phi_i)(\zeta,\kappa)\rangle_{W_2^{(2,2)}}=\langle L\omega(\zeta,\kappa),\Phi_i(\zeta,\kappa)\rangle_{W_2^{(1,1)}}=(L\omega)(\zeta_i,\kappa_i)=0,\ \ i=1,2,... \tag{3.4}$$

$(L\omega)(\zeta,\kappa)=0$ since $\{(\zeta_i,\kappa_i)\}_{i=1}^{\infty}$ is dense in Δ. When the inverse operator L^{-1} is used in Eq.(3.4), it can be clearly seen that $\omega=0$.

The orthonormal system $\{\overline{\Psi_i}(\zeta,\kappa)\}_{i=1}^{\infty}$ can be attained by the Gram-Schmidt orthogonalization of $\{\Psi_i(\zeta,\kappa)\}_{i=1}^{\infty}$ as

$$\overline{\Psi_i}(\zeta,\kappa)=\sum_{k=1}^{i}\beta_{ik}\Psi_k(\zeta,\kappa).$$

The orthogonalization process is given by formula as follow:

$$\beta_{11}=\frac{1}{\|\Psi_1\|},\ \ \beta_{ik}=\frac{1}{d_{ik}},\ \ \beta_{ij}=-\frac{1}{d_{ik}}\sum_{k=j}^{i-1}c_{ik}\beta_{kj}\ \text{ for }\ j<i,$$

and also

$$d_{ik}=\sqrt{\|\Psi_i\|^2-\sum_{k=1}^{i-1}c_{ik}^2},\ \ c_{ik}=\langle\Psi_i,\overline{\Psi_k}\rangle_{W_2^{(2,2)}}.$$

Theorem 3.3. *Let $\{(\zeta_i,\kappa_i)\}_{i=1}^{\infty}$ be dense in Δ, then the iterative solution of Eq. (3.2) is*

$$\omega(\zeta,\kappa)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(\zeta_k,\kappa_k,\omega(\zeta_k,\kappa_k),\partial_\zeta\omega(\zeta_k,\kappa_k))\overline{\Psi}_i(\zeta,\kappa). \tag{3.5}$$

Proof.$\{\Psi_i(\zeta,\kappa)\}_{i=1}^{\infty}$ is a complete system of $W_2^{(2,2)}(\Delta)$. Therefore, it can be written

$$\begin{aligned}\omega(\zeta,\kappa) &= \sum_{i=1}^{\infty}\langle\omega(\zeta,\kappa),\overline{\Psi}_i(\zeta,\kappa\rangle_{W_2^{(2,2)}}\overline{\Psi}_i(\zeta,\kappa)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle\omega(\zeta,\kappa),\Psi_k(\zeta,\kappa)\rangle_{W_2^{(2,2)}}\overline{\Psi}_i(\zeta,\kappa)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle\omega(\zeta,\kappa),L^*\Phi_k(\zeta,\kappa)\rangle_{W_2^{(2,2)}}\overline{\Psi}_i(\zeta,\kappa)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle L\omega(\zeta,\kappa),\Phi_k(\zeta,\kappa)\rangle_{W_2^{(1,1)}}\overline{\Psi}_i(\zeta,\kappa)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle L\omega(\zeta,\kappa),\tilde{T}_{(\zeta_k,\kappa_k)}(\zeta,\kappa)\rangle_{W_2^{(1,1)}}\overline{\Psi}_i(\zeta,\kappa)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}L\omega(\zeta_k,\kappa_k)\overline{\Psi}_i(\zeta,\kappa)\\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(\zeta_k,\kappa_k,\omega(\zeta_k,\kappa_k),\partial_\zeta\omega(\zeta_k,\kappa_k))\overline{\Psi}_i(\zeta,\kappa).\end{aligned} \tag{3.6}$$

The proof is completed.
When finite n-terms are taken in Eq.(3.6), the approximate solution $\omega_n(\zeta,\kappa)$ is expressed as follows:

$$\omega_n(\zeta,\kappa)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(\zeta_k,\kappa_k,\omega(\zeta_k,\kappa_k),\partial_\zeta\omega(\zeta_k,\kappa_k))\overline{\Psi}_i(\zeta,\kappa).$$

The convergence of approximate solution will be presented in the next section.

4. Convergence Analysis

Here, it will be shown that the iterative approximate solution is uniformly convergent. Taking A_i as:

$$A_i=\sum_{k=1}^{i}\beta_{ik}F(\zeta_k,\kappa_k,\omega(\zeta_k,\kappa_k),\partial_\zeta\omega(\zeta_k,\kappa_k)),$$

then Eq.(3.5) can be written as

$$(\zeta,\kappa)=\sum_{i=1}^{\infty}A_i\overline{\Psi}_i(\zeta,\kappa).$$

Now from the initial conditions of Eq. (3.2), if taking $(\zeta_1,\kappa_1)=0$, $\omega(\zeta_1,\kappa_1)$ can be calculated. When $\omega_0(\zeta_1,\kappa_1)=\omega(\zeta_1,\kappa_1)$ is taked, then the n-term approximation of $\omega(\zeta,\kappa)$ can be given as follow:

$$\omega_n(\zeta,\kappa)=\sum_{i=1}^{n}B_i\overline{\Psi}_i(\zeta,\kappa), \tag{4.1}$$

here

$$B_i=\sum_{k=1}^{i}\beta_{ik}F(\zeta_k,\kappa_k,\omega_{k-1}(\zeta_k,\kappa_k),\partial_\zeta\omega_{k-1}(\zeta_k,\kappa_k)). \tag{4.2}$$

Now, the uniform convergence of the approximate solution $\omega_n(\zeta,\kappa)$ will be shown. Therefore the following lemma should be given.

Lemma 4.1. *If $F(\zeta,\kappa,\omega(\zeta,\kappa),\omega_{\zeta}(\zeta,\kappa))$ is continuous and $\omega_n \to \hat{\omega}$ for $(\zeta_n,\kappa_n)\to(t,x)$, then*

$$F(\zeta_n,\kappa_n,\omega_{n-1}(\zeta_n,\kappa_n),\partial_{\zeta}\omega_{n-1}(\zeta_n,\kappa_n))\to F(t,x,\hat{\omega}(t,x),\partial_{\zeta}\hat{\omega}(t,x)).$$

Proof. Since

$$\begin{aligned}|\omega_{n-1}(\zeta_n,\kappa_n)-\hat{\omega}(t,x)| &= |\omega_{n-1}(\zeta_n,\kappa_n)-\omega_{n-1}(t,x)+\omega_{n-1}(t,x)-\hat{\omega}(t,x)|\\ &\le |\omega_{n-1}(\zeta_n,\kappa_n)-\omega_{n-1}(t,x)|+|\omega_{n-1}(t,x)-\hat{\omega}(t,x)|.\end{aligned}$$

By using the reproducing kernel feature, it can be said that

$$\omega_{n-1}(\zeta_n,\kappa_n)=\langle\omega_{n-1}(\zeta,\kappa),T_{(\zeta_n,\kappa_n)}(\zeta,\kappa)\rangle_{W_2^{(2,2)}},\ \ \omega_{n-1}(t,x)=\langle\omega_{n-1}(\zeta,\kappa),T_{(t,x)}(\zeta,\kappa)\rangle_{W_2^{(2,2)}}.$$

It follows that

$$|\omega_{n-1}(\zeta_n,\kappa_n)-\omega_{n-1}(t,x)|=|\langle\omega_{n-1}(\zeta,\kappa),T_{(\zeta_n,\kappa_n)}(\zeta,\kappa)-T_{(t,x)}(\zeta,\kappa)\rangle|.$$

It is known that there exists a constant M from the convergence of $\omega_{n-1}(\zeta,\kappa)$, such that

$$\|\omega_{n-1}(\zeta,\kappa)\|_{W_2^{(2,2)}}\le M\|\hat{\omega}(t,x)\|_{W_2^{(2,2)}},\ \text{ as } n\ge M.$$

Also, it can be proven that

$$\|T_{(\zeta_n,\kappa_n)}(\zeta,\kappa)-T_{(t,x)}(\zeta,\kappa)\|_{W_2^{(2,2)}}\to 0,\ \text{ for } n\to\infty$$

by using Theorem 2.2. So,

$$\omega_{n-1}(\zeta_n,\kappa_n)\to\hat{\omega}(t,x),\ \text{ as } (\zeta_n,\kappa_n)\to(t,x).$$

Similarly, the following expression can be written

$$\partial_{\zeta}\omega_{n-1}(\zeta_n,\kappa_n)\to\partial_{\zeta}\hat{\omega}(t,x),\ \text{ as } (\zeta_n,\kappa_n)\to(t,x).$$

Therefore,

$$F(\zeta_n,\kappa_n,\omega_{n-1}(\zeta_n,\kappa_n),\partial_{\zeta}\omega_{n-1}(\zeta_n,\kappa_n))\to F(t,x,\hat{\omega}(t,x),\partial_{\zeta}\hat{\omega}(t,x)).$$

So, the proof is completed.

Theorem 4.2. *Let $\{(\zeta_i,\kappa_i)\}_{i=1}^{\infty}$ be dense in Δ. Assume that $\|\omega_n\|$ is a bounded, and the Eq. (4.1) has a unique solution. Then, $\omega_n(\zeta,\kappa)\to\omega(\zeta,\kappa)$ and*

$$\omega(\zeta,\kappa)=\sum_{i=1}^{\infty}B_i\overline{\Psi}_i(\zeta,\kappa).$$

Proof. It will be shown that the convergence of $\omega_n(\zeta,\kappa)$. From the Eq. (4.1), it can be easily seen that

$$\omega_{n+1}(\zeta,\kappa)=\omega_n(\zeta,\kappa)+B_{n+1}\overline{\Psi}_{n+1}(\zeta,\kappa).$$

By using of $\{\overline{\Psi}_i\}_{i=1}^{\infty}$, the following equation can be written:

$$\|\omega_{n+1}\|^2=\|\omega_n\|^2+B_{n+1}^2=\sum_{i=1}^{n+1}B_i^2. \tag{4.3}$$

Therefore, from Eq. (4.3), it can be seen that $\|\omega_{n+1}\|>\|\omega_n\|$. By the using boundedness of $\|\omega_n\|$, it can be easily seen that $\|\omega_n\|$ is convergent. And also there exists a constant c such that

$$\sum_{i=1}^{\infty}B_i^2=c. \tag{4.4}$$

So, Eq. (4.4) shows that $\{B_i\}_{i=1}^{\infty}\in l^2$. If $m>n$, then

$$\begin{aligned}\|\omega_m-\omega_n\|^2 &= \|\omega_m-\omega_{m-1}+\omega_{m-1}-\omega_{m-2}+\cdots+\omega_{n+1}-\omega_n\|^2\\ &= \|\omega_m-\omega_{m-1}\|^2+\|\omega_{m-1}-\omega_{m-2}\|^2+\cdots+\|\omega_{n+1}-\omega_n\|^2.\end{aligned}$$

On account of

$$\|\omega_m-\omega_{m-1}\|^2=B_m^2,$$

consequently

$$\|\omega_m-\omega_n\|^2=\sum_{l=n+1}^{m}B_l^2\to 0,\ \text{ as } n\to\infty.$$

From the completeness of $W_2^{(2,2)}(\Delta)$, it can be expressed that $\omega_n \to \hat{\omega}$ as $n \to \infty$. Now, it will be shown that $\hat{\omega}$ is the solution of Eq. (3.2). Taking limits in Eq. (4.1) we get

$$\hat{\omega}(\zeta,\kappa) = \sum_{i=1}^{\infty} B_i \overline{\Psi}_i(\zeta,\kappa).$$

Note that

$$(L\hat{\omega})(\zeta,\kappa) = \sum_{i=1}^{\infty} B_i L\overline{\Psi}_i(\zeta,\kappa),$$

$$\begin{aligned}(L\hat{\omega})(\zeta_l,\kappa_l) &= \sum_{i=1}^{\infty} B_i L\overline{\Psi}_i(\zeta_l,\kappa_l) = \sum_{i=1}^{\infty} B_i \langle L\overline{\Psi}_i(\zeta,\kappa), \Phi_l(\zeta,\kappa)\rangle_{W_2^{(1,1)}} \\ &= \sum_{i=1}^{\infty} B_i \langle \overline{\Psi}_i(\zeta,\kappa), L^*\Phi_l(\zeta,\kappa)\rangle_{W_2^{(2,2)}} = \sum_{i=1}^{\infty} B_i \langle \overline{\Psi}_i(\zeta,\kappa), \Psi_l(\zeta,\kappa)\rangle_{W_2^{(2,2)}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{l=1}^{i} \beta_{il}(L\hat{\omega})(\zeta_l,\kappa_l) &= \sum_{i=1}^{\infty} B_i \langle \overline{\Psi}_i(\zeta,\kappa), \sum_{l=1}^{i} \beta_{il}\Psi_l(\zeta,\kappa)\rangle_{W_2^{(2,2)}} \\ &= \sum_{i=1}^{\infty} B_i \langle \overline{\Psi}_i(\zeta,\kappa), \overline{\Psi}_l(\zeta,\kappa)\rangle_{W_2^{(2,2)}} = B_l.\end{aligned}$$

From Eq. (4.2), we have

$$L\hat{\omega}(\zeta_l,\kappa_l) = F(\zeta_l,\kappa_l,\omega_{l-1}(\zeta_l,\kappa_l),\partial_\zeta \omega_{l-1}(\zeta_l,\kappa_l)).$$

Since $\{(\zeta_i,\kappa_i)\}_{i=1}^{\infty}$ is dense in Δ, there exists a subsequence $\{(\zeta_{n_j},\kappa_{n_j})\}_{j=1}^{\infty}$ such that $(\zeta_{n_j},\kappa_{n_j}) \to (t,x)$, for each $(t,x) \in \Delta$, $(j \to \infty)$. It can be expressed that

$$L\hat{\omega}(\zeta_{n_j},\kappa_{n_j}) = F(\zeta_{n_j},\kappa_{n_j},\omega_{n_{j-1}}(\zeta_{n_j},\kappa_{n_j}),\partial_\zeta \omega_{n_{j-1}}(\zeta_{n_j},\kappa_{n_j})).$$

Using Lemma 4.1 and the continuity of F, it can be written that

$$(L\hat{\omega})(t,x) = F(t,x,\hat{\omega}(t,x),\partial_\zeta \hat{\omega}(t,x)), \text{ for } j \to \infty. \tag{4.5}$$

The Eq. (4.5) demonstrates that $\hat{\omega}(\zeta,\kappa)$ provides Eq. (3.2). The proof is completed.

5. Numerical Outcomes

In this section, the iterative reproducing kernel method is tested on two nonlinear advection equations. When calculating numerical results, $\zeta_i = \frac{i}{q}$, $i = 0,1,\ldots,q$, $\kappa_i = \frac{i}{p}$, $i = 0,1,\ldots,p$ and $n = q \times p$ are selected. The numerical results obtained for different values of p and q are shown in tables and graphs. Also, the algorithm process of the method is presented as follows.

5.1. Algorithm of method

The iterative RKM process is presented as follow:

Step 1. Choose iteration number as $n = q \times p$ discrete point in the $[0,1] \times [0,1]$.
Step 2. Enter $\Psi_i(\zeta,\kappa) = L_{(t,x)}T_{(t,x)}(\zeta,\kappa)|_{(t,x)=(\zeta_i,\kappa_i)}$.
Step 3. Attain β_{ik} orthogonalization coefficients.
Step 4. For $i = 1,2,\ldots,n$, set $\overline{\Psi}_i(\zeta,\kappa) = \sum_{k=1}^{i} \beta_{ik}\Psi_k(\zeta,\kappa)$.
Step 5. Enter initial approximation $\omega_0(\zeta_i,\kappa_i)$.
Step 6. For $i = 1,2,\ldots,n$, evaluate $B_i = \sum_{k=1}^{i} \beta_{ik} F(\zeta_k,\kappa_k,\omega_{k-1}(\zeta_k,\kappa_k),\partial_\zeta \omega_{k-1}(\zeta_k,\kappa_k))$.
Step 7. For $i = 1,2,\ldots,n$, evaluate $\omega_i(\zeta,\kappa) = \sum_{k=1}^{i} B_k \overline{\Psi}_k(\zeta_k,\kappa_k)$.

5.2. Examples

Example 5.1. *The following nonlinear advection equation is considered:*

$$y_\kappa(\zeta,\kappa) + y(\zeta,\kappa)y_\zeta(\zeta,\kappa) = f(\zeta,\kappa), \ \ \zeta,\kappa \in [0,1]. \tag{5.1}$$

The exact solution of Eq. (5.1) is

$$y(\zeta,\kappa) = \zeta^2(\kappa+2),$$

and the initial condition of problem is

$$y(\zeta,0)=2\zeta^2.$$

After the homogenisation of initial condition, Eq.(5.1) turns into the following form:

$$\omega_\kappa(\zeta,\kappa)+2\zeta^2\omega_\zeta(\zeta,\kappa)+4\zeta\omega(\zeta,\kappa)+\omega(\zeta,\kappa)\omega_\zeta(\zeta,\kappa)+8\zeta^3=f(\zeta,\kappa). \quad (5.2)$$

The initial condition of Eq.(5.2) is

$$\omega(\zeta,0)=0,$$

and the exact solution of Eq.(5.2) is

$$\omega(\zeta,\kappa)=\zeta^2\kappa.$$

In Eq.(5.2),

$$f(\zeta,\kappa)=2\kappa^2\zeta^3+8\kappa\zeta^3+8\zeta^3+\zeta^2.$$

The absolute error values are computed for $n=225$ in Table 5.1 and $n=400$ in Table 5.2. The graphics of approximate solution, absolute error and exact solution are presented in Figure 1 for $n=400$ $(q=p=20)$.

ζ/κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	6.96×10^{-6}	4.26×10^{-6}	6.39×10^{-6}	8.48×10^{-6}	1.02×10^{-5}	1.14×10^{-5}	1.21×10^{-5}	1.71×10^{-5}	5.42×10^{-5}
0.2	1.30×10^{-5}	6.77×10^{-6}	9.18×10^{-6}	1.09×10^{-5}	1.19×10^{-5}	1.05×10^{-5}	7.08×10^{-6}	1.48×10^{-5}	1.03×10^{-4}
0.3	1.90×10^{-5}	7.81×10^{-6}	1.03×10^{-5}	1.08×10^{-5}	1.05×10^{-5}	5.78×10^{-6}	3.51×10^{-6}	4.59×10^{-6}	1.42×10^{-4}
0.4	2.48×10^{-5}	8.25×10^{-6}	1.19×10^{-5}	1.15×10^{-5}	1.03×10^{-5}	2.90×10^{-6}	1.16×10^{-5}	2.95×10^{-6}	1.84×10^{-4}
0.5	3.05×10^{-5}	7.68×10^{-6}	1.32×10^{-5}	1.23×10^{-5}	1.03×10^{-5}	4.60×10^{-7}	1.91×10^{-5}	9.59×10^{-6}	2.26×10^{-4}
0.6	3.60×10^{-5}	6.03×10^{-6}	1.42×10^{-5}	1.30×10^{-5}	1.02×10^{-5}	2.08×10^{-6}	2.65×10^{-5}	1.61×10^{-5}	2.69×10^{-4}
0.7	4.13×10^{-5}	3.18×10^{-6}	1.45×10^{-5}	1.33×10^{-5}	9.83×10^{-6}	4.95×10^{-6}	3.43×10^{-5}	2.29×10^{-5}	3.12×10^{-4}
0.8	4.62×10^{-5}	6.95×10^{-7}	1.42×10^{-5}	1.34×10^{-5}	9.36×10^{-6}	7.82×10^{-6}	4.20×10^{-5}	2.97×10^{-5}	3.54×10^{-4}
0.9	5.07×10^{-5}	5.55×10^{-6}	1.32×10^{-5}	1.32×10^{-5}	8.87×10^{-6}	1.06×10^{-5}	4.96×10^{-5}	3.63×10^{-5}	3.97×10^{-4}

Table 5.1: The absolute error values of Example 5.1 for $p=15$ and $q=15$.

ζ/κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	7.56×10^{-7}	1.72×10^{-6}	2.45×10^{-6}	3.23×10^{-6}	4.03×10^{-6}	4.51×10^{-6}	4.07×10^{-6}	4.88×10^{-6}	1.85×10^{-5}
0.2	1.59×10^{-6}	3.06×10^{-6}	3.52×10^{-6}	4.13×10^{-6}	4.80×10^{-6}	4.55×10^{-6}	1.44×10^{-6}	6.04×10^{-7}	3.15×10^{-5}
0.3	1.93×10^{-6}	4.09×10^{-6}	4.04×10^{-6}	4.37×10^{-6}	4.90×10^{-6}	3.98×10^{-6}	1.95×10^{-6}	5.13×10^{-6}	4.18×10^{-5}
0.4	1.85×10^{-6}	5.02×10^{-6}	4.39×10^{-6}	4.47×10^{-6}	4.99×10^{-6}	3.56×10^{-6}	5.03×10^{-6}	1.05×10^{-5}	5.23×10^{-5}
0.5	1.38×10^{-6}	5.83×10^{-6}	4.65×10^{-6}	4.46×10^{-6}	5.04×10^{-6}	3.19×10^{-6}	7.95×10^{-6}	1.56×10^{-5}	6.32×10^{-5}
0.6	5.51×10^{-7}	6.48×10^{-6}	4.82×10^{-6}	4.33×10^{-6}	5.00×10^{-6}	2.79×10^{-6}	1.08×10^{-5}	2.06×10^{-5}	7.42×10^{-5}
0.7	6.02×10^{-7}	6.90×10^{-6}	4.93×10^{-6}	4.10×10^{-6}	4.87×10^{-6}	2.33×10^{-6}	1.37×10^{-5}	2.55×10^{-5}	8.52×10^{-5}
0.8	2.04×10^{-6}	7.06×10^{-6}	4.99×10^{-6}	3.78×10^{-6}	4.64×10^{-6}	1.78×10^{-6}	1.67×10^{-5}	3.06×10^{-5}	9.62×10^{-5}
0.9	3.71×10^{-6}	6.89×10^{-6}	4.99×10^{-6}	3.40×10^{-6}	4.32×10^{-6}	1.16×10^{-6}	1.97×10^{-5}	3.56×10^{-5}	1.07×10^{-4}

Table 5.2: The absolute error values of Example 5.1 for $p=20$ and $q=20$.

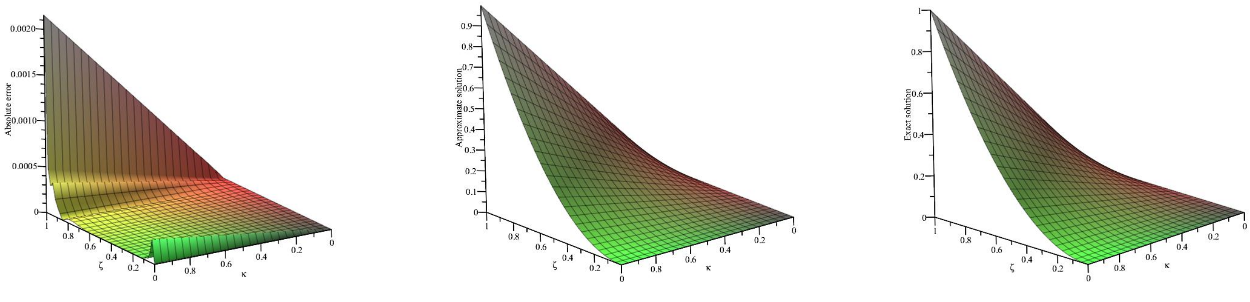

Figure 5.1: The graphs of the absolute error, approximate solution and exact solution for $p=20$ and $q=20$ in Example 5.1.

Example 5.2. *The following nonlinear advection equation is considered:*

$$y_\kappa(\zeta,\kappa)+y(\zeta,\kappa)y_\zeta(\zeta,\kappa)=f(\zeta,\kappa),\ 0\le\zeta,\kappa\le 1. \tag{5.3}$$

The exact solution of problem is

$$y(\zeta,\kappa)=\zeta(\frac{\kappa^2}{2}+1),$$

and the initial condition of problem is

$$y(\zeta,0)=\zeta.$$

After the homogenisation of initial condition, Eq.(5.3) turns into the following form:

$$\omega_\kappa(\zeta,\kappa)+\zeta\omega_\zeta(\zeta,\kappa)+\omega(\zeta,\kappa)+\omega_\zeta(\zeta,\kappa)\omega(\zeta,\kappa)+\zeta=f(\zeta,\kappa). \tag{5.4}$$

The initial condition of Eq. (5.4) is

$$\omega(\zeta,0)=0,$$

and the exact solution of Eq. (5.4) is

$$\omega(\zeta,\kappa)=\zeta\frac{\kappa^2}{2}.$$

In Eq. (5.4),

$$f(\zeta,\kappa)=\kappa\zeta+\kappa^2\zeta+\frac{1}{4}\zeta\kappa^4+\zeta.$$

The absolute error values are computed for $n=225$ in Table 5.3 and $n=400$ in Table 5.4. The graphics of approximate solution, absolute error and exact solution are presented in Figure 2 for $n=400$ $(q=p=20)$.

ζ/κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	6.03×10^{-6}	2.73×10^{-6}	3.36×10^{-6}	4.54×10^{-6}	5.75×10^{-6}	6.91×10^{-6}	8.06×10^{-6}	9.21×10^{-6}	1.03×10^{-5}
0.2	1.16×10^{-5}	6.49×10^{-6}	6.72×10^{-6}	8.93×10^{-6}	1.14×10^{-5}	1.38×10^{-5}	1.61×10^{-5}	1.84×10^{-5}	2.07×10^{-5}
0.3	1.65×10^{-5}	1.15×10^{-5}	1.05×10^{-5}	1.35×10^{-5}	1.74×10^{-5}	2.12×10^{-5}	2.48×10^{-5}	2.83×10^{-5}	3.18×10^{-5}
0.4	2.01×10^{-5}	1.79×10^{-5}	1.49×10^{-5}	1.81×10^{-5}	2.34×10^{-5}	2.87×10^{-5}	3.37×10^{-5}	3.85×10^{-5}	4.32×10^{-5}
0.5	2.20×10^{-5}	2.57×10^{-5}	2.06×10^{-5}	2.35×10^{-5}	3.02×10^{-5}	3.71×10^{-5}	4.37×10^{-5}	5.00×10^{-5}	5.62×10^{-5}
0.6	2.16×10^{-5}	3.45×10^{-5}	2.75×10^{-5}	2.96×10^{-5}	3.73×10^{-5}	4.59×10^{-5}	5.43×10^{-5}	6.22×10^{-5}	7.00×10^{-5}
0.7	1.84×10^{-5}	4.47×10^{-5}	3.66×10^{-5}	3.76×10^{-5}	4.61×10^{-5}	5.66×10^{-5}	6.70×10^{-5}	7.70×10^{-5}	8.67×10^{-5}
0.8	1.14×10^{-5}	5.54×10^{-5}	4.76×10^{-5}	4.75×10^{-5}	5.65×10^{-5}	6.87×10^{-5}	8.13×10^{-5}	9.35×10^{-5}	1.05×10^{-4}
0.9	2.59×10^{-7}	6.69×10^{-5}	6.15×10^{-5}	6.08×10^{-5}	7.06×10^{-5}	8.46×10^{-5}	9.97×10^{-5}	1.14×10^{-4}	1.29×10^{-4}

Table 5.3: The absolute error values of Example 5.2 for $p=15$ and $q=15$.

ζ/κ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	9.78×10^{-7}	1.26×10^{-6}	1.82×10^{-6}	2.47×10^{-6}	3.10×10^{-6}	3.72×10^{-6}	4.33×10^{-6}	4.95×10^{-6}	5.57×10^{-6}
0.2	2.52×10^{-6}	2.71×10^{-6}	3.61×10^{-6}	4.99×10^{-6}	6.29×10^{-6}	7.54×10^{-6}	8.78×10^{-6}	1.00×10^{-5}	1.12×10^{-5}
0.3	4.42×10^{-6}	4.51×10^{-6}	5.41×10^{-6}	7.57×10^{-6}	9.60×10^{-6}	1.15×10^{-5}	1.34×10^{-5}	1.53×10^{-5}	1.72×10^{-5}
0.4	6.41×10^{-6}	6.83×10^{-6}	7.27×10^{-6}	1.02×10^{-5}	1.30×10^{-5}	1.57×10^{-5}	1.83×10^{-5}	2.09×10^{-5}	2.35×10^{-5}
0.5	8.24×10^{-6}	9.85×10^{-6}	9.36×10^{-6}	1.30×10^{-5}	1.68×10^{-5}	2.03×10^{-5}	2.36×10^{-5}	2.70×10^{-5}	3.03×10^{-5}
0.6	9.60×10^{-6}	1.37×10^{-5}	1.18×10^{-5}	1.61×10^{-5}	2.09×10^{-5}	2.53×10^{-5}	2.95×10^{-5}	3.37×10^{-5}	3.79×10^{-5}
0.7	1.01×10^{-5}	1.84×10^{-5}	1.51×10^{-5}	1.97×10^{-5}	2.56×10^{-5}	3.11×10^{-5}	3.63×10^{-5}	4.41×10^{-5}	4.67×10^{-5}
0.8	9.45×10^{-6}	2.41×10^{-5}	1.94×10^{-5}	2.41×10^{-5}	3.11×10^{-5}	3.79×10^{-5}	4.44×10^{-5}	5.07×10^{-5}	5.71×10^{-5}
0.9	7.12×10^{-6}	3.08×10^{-5}	2.52×10^{-5}	2.99×10^{-5}	3.78×10^{-5}	4.60×10^{-5}	5.41×10^{-5}	6.19×10^{-5}	6.97×10^{-5}

Table 5.4: The absolute error values of Example 5.2 for $p=20$ and $q=20$.

6. Conclusion

In this study, a numerical approach is proposed for the nonlinear advection equation. This approach is based on the reproducing kernel function obtained from special Hilbert spaces and the selection of a linear operator. The approximate solution is constructed by the basis function obtained by applying the reproducing kernel function to the selected linear operator. The convergence analysis of the proposed approach is given in detail. To demonstrate the validity of the method, the RKM is applied to two different nonlinear advection equations. The obtained results verify the effectiveness of the method. It is thought that the proposed method will contribute to the literature. The proposed method can be applied to integral differential equations with nonhomogeneous initial or boundary conditions by improving it.

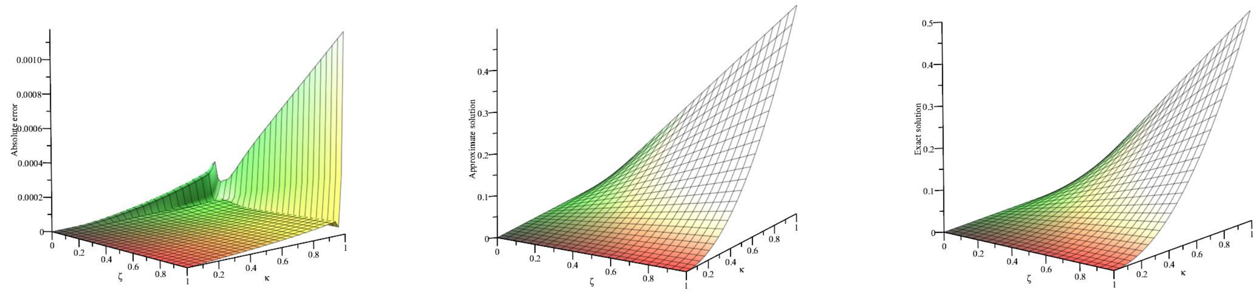

Figure 5.2: The graphs of the absolute error, approximate solution and exact solution for $p = 20$ and $q = 20$ in Example 5.2.

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