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Research Article

Tessarine Number Sequences and Quantum Calculus Approach

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Abstract: This paper presents a detailed study of a new generation of tessarine number sequences with components including quantum integers Also, several fundamental identities are defined such as Binet formulas, Catalan, Cassini, D'ocagnes. After that, the q-Fibonacci tessarine and q-Lucas tessarine polynomials and function sequences are definedand obtained several properties for these sequences.

Keywords: q-Fibonacci tessarine number sequences, q-Lucas tessarine number sequences, Tessarine numbers

Tessarine Sayı Dizileri ve Kuantum Kalkulus Yaklaşımı

Öz: Bu makalede, kuantum tam sayıları içeren bileşenlere sahip yeni nesil tessarin sayı dizilerinin ayrıntılı bir çalışması sunulmaktadır. Ayrıca, Binet formülleri, Catalan, Cassini, D'ocagnes gibi çeşitli temel özdeşlikler tanımlanmıştır. Daha sonra, q-Fibonacci tessarine ve q-Lucas tessarine polinomları ve fonksiyon dizileri tanımlanmış ve bu diziler için çeşitli özellikler elde edilmiştir.

Anahtar Kelimeler: q- Fibanacci sayı dizileri, q- Lucas sayı dizileri, Tessarine sayıları

1. Introduction

Complex numbers were discovered by the Italian mathematician G. Cardano while he tries to solve a simpler state of the cubic equation and using the notation $i=\sqrt{-1}$. The complex numbers as points with rectangular coordinates were represented by Euler. After that, Cockle, (1849) proposed the tessarine numbers, an algebraic successor to complex numbers and quaternionic algebra, employing more modern notation. In the exponential series, he employed tessarine numbers to separate the hyperbolic sine and cosine series (Cockle, 1849; Cockle, 1850).

A tessarine number is a hypercomplex number of the form

$$\gamma = \gamma_0+\gamma_1 i_1+\gamma_2 i_2+\gamma_3 i_3$$

where γ_0, γ_1, γ_2 and γ_3 are real numbers and i_1, i_2 and i_3 are the imaginary units which satisfy the following rules:

$$i_1^2=-i_2^2=i_3^2=-1 \ ; i_1 i_2=i_2 i_1=i_3. \tag{1}$$

The addition and multiplication of tessarine numbers $\gamma=\gamma_0+\gamma_1 i_1+\gamma_2 i_2+\gamma_3 i_3$ and $\delta=\delta_0+\delta_1 i_1+\delta_2 i_2+\delta_3 i_3$ are defined, respectively, as:

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$$\gamma + \delta = (\gamma_0 + \delta_0) + (\gamma_1 + \delta_1)i_1 + (\gamma_2 + \delta_2)i_2 + (\gamma_3 + \delta_3)i_3$$

and

$$\gamma\delta = \gamma_0\delta_0 - \gamma_1\delta_1 + \gamma_2\delta_2 - \gamma_3\delta_3 + (\gamma_0\delta_1 + \gamma_1\delta_0 - \gamma_2\delta_3 - \gamma_3\delta_2)i_1$$
$$+(\gamma_0\delta_2 + \gamma_2\delta_0 - \gamma_3\delta_1 - \gamma_1\delta_3)i_2 + (\gamma_0\delta_3 + \gamma_3\delta_0 + \gamma_1\delta_2 + \gamma_2\delta_1)i_3.$$

It is easy to see that the multiplication of tessarine numbers is commutative In mathematics, the Fibonacci numbers and Lucas numbers are an infinite sequences of integers in which each number is the sum of the two preceding ones. These numbers have been researched extensively because of their complex characteristics and deep connections to several fields of mathematics and their related numbers are of essential importance due to their various applications in biology, physics, statistics, and computer science (Horadam, 1961; 1963; Nalli & Haukkanen, 2009; Koshy, 2018; 2019; Oduol & Okoth, 2020).

For n ≥ 2, the second order linear sequences F_n and L_n are defined by:

$$\begin{cases} F_n = F_{n-1} + F_{n-2} \\ L_n = L_{n-1} + L_{n-2.} \end{cases}$$

Here, the initial conditions are $F_0 = 0, F_1 = 1, L_0 = 2$ and $L_1 = 1$, respectively. The Binet formulas of these numbers are

$$\begin{cases} F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta} \\ L_n = \alpha^n + \beta^n \end{cases}$$

where α and β are roots of characteristic equation $\varphi^2 - \varphi - 1 = 0$.

Now, we give definitions and facts from the quantum calculus necessary for understanding of this paper (Kac & Cheung, 2002; Kome et. al., 2022; Babadağ, 2023).

For any integers n and m, we define the function

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1} \tag{2}$$

and

$$\begin{cases} [m + n]_q = [m]_q + q^m[n]_q \\ [mn]_q = [m]_q + [n]_{q^m}\,. \end{cases} \tag{3}$$

During the last few years, few researchers have studied the tessarine numbers with Fibonacci numbers, Lucas numbers , Homothetic Motions, Surfaces and Neural Networks (Babadağ, 2017; Babadağ & Uslu, 2021; Senna & Valle, 2021). For example, Babadağ & Uslu, (2021) defined the Fibonacci tessarines with Fibonacci and Lucas numbers, and they examined the identities related to the tessarines, Fibonacci numbers and Lucas numbers. Babadağ, (2017) defined the homothetic motions and homothetic exponential motions. Senna & Valle, (2021) give tessarine and quaternion-valued deep neural networks for image classification.

In this paper, using different perspective, we define the tessarine number sequences with components including quantum integers and get several new results for these number sequences. Section 2. we give the q-Fibonacci tessarine number sequences and q-Lucas tessarine number sequences. Section 3. we explore some identities used in various areas of mathematics, including

Binet's formula, the exponential generating function, and Catalan, Cassini, and D'Ocagne's identities and we define quantum tessarine polynomial sequences or, briefly q-Fibonacci tessarine polynomial sequences $\gamma_{q,n}(t)$ and q-Lucas tessarine polynomial sequences $\delta_{q,n}(t)$ and then derive the Binet formula for these type of polynomial sequence. In addition some results of q-tessarine polynomial sequences are given. Then, we define quantum tessarine function sequences or briefly q-Fibonacci and q-Lucas tessarine function sequences $\Gamma_{q,n}(t)$ and $\Delta_{q,n}(t)$.

2. Material and Methods

To reach our goal, we obtain relation between the q-Fibonacci tessarine numbers and q-Lucas tessarine numbers. After that using these numbers, we introduce some identities used in various areas of mathematics, including Binet's formula, the exponential generating function, and Catalan, Cassini, and D'Ocagne's identities and we give quantum tessarine polynomial sequences and then derive the Binet formula for these type of polynomial sequence.

3. Resultants

Resultants are plays an important role in physics, combinatorics, number theory and other fields of the mathematics. Since quantum calculus may be viewed as generalization of ordinary calculus, there is a relationship between quantum calculus and number sequences

Definition 3.1. Tessarine number sequences of the form

$$\gamma_n = \alpha^{n-1}[n]_q + \alpha^n[n+1]_q i_1 + \alpha^{n+1}[n+2]_q i_2 + \alpha^{n+2}[n+3]_q i_3 \tag{4}$$

are called the n^{th} q-Fibonacci tessarine number sequences and

$$\delta_n = \alpha^n \frac{[2n]_q}{[n]_q} + \alpha^{n+1} \frac{[2n+2]_q}{[n+1]_q} i_1 + \alpha^{n+2} \frac{[2n+4]_q}{[n+2]_q} i_2 + \alpha^{n+3} \frac{[2n+6]_q}{[n+3]_q} i_3$$

are called as the n^{th} q-Lucas tessarine number sequences. Furthermore, we can rewrite these number sequences in the following forms:

$$\gamma_n = \frac{\alpha^n(1-q^n)}{\alpha-\alpha q} + \frac{\alpha^{n+1}(1-q^{n+1})}{\alpha-\alpha q} i_1 + \frac{\alpha^{n+2}(1-q^{n+2})}{\alpha-\alpha q} i_2 + \frac{\alpha^{n+3}(1-q^{n+3})}{\alpha-\alpha q} i_3$$

and

$$\delta_n = \alpha^n \frac{1-q^{2n}}{1-q^n} + \alpha^{n+1} \frac{1-q^{2n+2}}{1-q^{n+1}} i_1 + \alpha^{n+2} \frac{1-q^{2n+4}}{1-q^{n+2}} i_2 + \alpha^{n+3} \frac{1-q^{2n+6}}{1-q^{n+3}} i_3 \tag{5}$$

where $i_1,\ i_2$ and i_3 are the imaginary units which satisfy the rules in (1).

Theorem 3.1. (Binet formulas) For any integer *n*,

$$\begin{cases} \gamma_n = \alpha^{n-1}[n]_q \underline{\alpha} + (\alpha q)^n \underline{\beta} \\ \delta_n = \alpha^n \dfrac{[2n]_q}{[n]_q} \underline{\gamma} + \alpha^{n+1}(1-q)\underline{\beta}. \end{cases} \tag{6}$$

Furthermore, these formulas have another expression of the form

$$\begin{cases} \gamma_n = \dfrac{\alpha^n \underline{\alpha} - (\alpha q)^n \underline{\varphi}}{\alpha - \alpha q} \\ \delta_n = \alpha^n \underline{\alpha} + (\alpha q)^n \underline{\varphi} \end{cases} \tag{7}$$

where

$$\begin{cases} \underline{\alpha} = 1 + \alpha i_1 + \alpha^2 i_2 + \alpha^3 i_3 \\ \underline{\beta} = i_1 + \alpha[2]_q i_2 + \alpha^2 [3]_q i_3 \\ \underline{\varphi} = 1 + (\alpha q) i_1 + (\alpha q)^2 i_2 + (\alpha q)^3 i_3 \end{cases} \tag{8}$$

Proof. By using (2), (3), (4) and (8), we can write

$$\begin{aligned} \gamma_n &= \alpha^{n-1}[n]_q + \alpha^n [n+1]_q i_1 + \alpha^{n+1}[n+2]_q i_2 + \alpha^{n+2}[n+3]_q i_3 \\ &= \alpha^{n-1}[n]_q + \alpha^n\big([n]_q + q_n\big) i_1 + \alpha^{n+1}([n]_q + q^n[2]_q) i_2 + \alpha^{n+2}([n]_q + q^n[3]_q) i_3 \\ &= \alpha^{n-1}[n]_q (1 + \alpha i_1 + \alpha^2 i_2 + \alpha^3 i_3) + \alpha^n q^n \big(i_1 + \alpha[2]_q i_2 + \alpha^2[3]_q i_3\big) \\ &= \alpha^{n-1}[n]_q \underline{\alpha} + (\alpha q)^n \underline{\beta}. \end{aligned}$$

For q-Lucas tessarine number sequences by using (2), (5) and (8):

$$\begin{aligned} \delta_n &= \alpha^n \frac{1-q^{2n}}{1-q^n} + \alpha^{n+1} \frac{1-q^{2n+2}}{1-q^{n+1}} i_1 + \alpha^{n+2} \frac{1-q^{2n+4}}{1-q^{n+2}} i_2 + \alpha^{n+3} \frac{1-q^{2n+6}}{1-q^{n+3}} i_3 \\ &= \alpha^n (1 + \alpha i_1 + \alpha^2 i_2 + \alpha^3 i_3) + (\alpha q)^n (1 + (\alpha q) i_1 + (\alpha q)^2 i_2 + (\alpha q)^3 i_3) \\ &= \alpha^n \underline{\alpha} + (\alpha q)^n \underline{\varphi}. \end{aligned}$$

Theorem 3.2. (Exponential generating functions) The exponential generating functions for q-Fibonacci and q-Lucas tessarine number sequences are:

$$f(x) = \frac{e^{\alpha x} \underline{\alpha} - e^{(\alpha q) x} \underline{\varphi}}{\alpha - \alpha q}$$

$$g(x) = e^{\alpha x} \underline{\alpha} + e^{(\alpha q) x} \underline{\varphi}$$

respectively.

Proof. Using Binet-like formula for q-Fibonacci tessarine numbers and $e^{\alpha x} = \sum_{n=0}^{\infty} \alpha^n \frac{x^n}{n!}$ we get:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \gamma_n \frac{x^n}{n!} \\ &= \frac{1}{\alpha - \alpha q} \sum_{n=0}^{\infty} \left(e^{\alpha x} \underline{\alpha} - e^{(\alpha q) x} \underline{\varphi} \right) \frac{x^n}{n!} \\ &= \frac{e^{\alpha x} \underline{\alpha} - e^{(\alpha q) x} \underline{\varphi}}{\alpha - \alpha q}. \end{aligned}$$

The proof for q-Lucas number sequences can be done similarly.

Theorem 3.3. (Catalan identities) For positive integers n, r such that $n > r$, then we have

$$\gamma_{n+r}\gamma_{n-r} - \gamma_n^2 = \alpha^{2n-1}q^n([-r]_q + [r]_q)\underline{\alpha}\ \underline{\varphi}$$

$$\delta_{n+r}\delta_{n-r} - \delta_n^2 = \alpha^{2n}q^n(q-1)([-r]_q + [r]_q)\underline{\alpha}\ \underline{\varphi}$$

where

$$\underline{\alpha}\ \underline{\varphi} = (1-\alpha^2 q)(1+(\alpha^2 q)^2) + (1+(\alpha^2 q)^2)\alpha(1+q)i_1 + (1-\alpha^2 q)\alpha^2(1+q^2)i_2 + \alpha^3(1+q^2)i_3.$$

Proof. By using (2), (3) and (7), we obtain

$$\begin{aligned}\gamma_{n+r}\gamma_{n-r} - \gamma_n^2 &= \left(\frac{\alpha^{n+r}\underline{\alpha} - (\alpha q)^{n+r}\underline{\varphi}}{\alpha - \alpha q}\right)\left(\frac{\alpha^{n-r}\underline{\alpha} - (\alpha q)^{n-r}\underline{\varphi}}{\alpha - \alpha q}\right) - \left(\frac{\alpha^{n}\underline{\alpha} - (\alpha q)^{n}\underline{\varphi}}{\alpha - \alpha q}\right)^2 \\ &= \frac{\alpha^{2n}q^n(2 - q^{-r} - q^r)\underline{\alpha}\underline{\varphi}}{\alpha - \alpha q} \\ &= \alpha^{2n-1}q^n([-r]_q + [r]_q)\underline{\alpha}\ \underline{\varphi}.\end{aligned}$$

Similarly we can get the result for q-Lucas number sequences.

Theorem 3.4. (Cassini identities) For $n \geq 1$, we have

$$\gamma_{n+1}\gamma_{n-1} - \gamma_n^2 = \alpha^{2n-1}q^{n-1}(q-1)\underline{\alpha}\underline{\varphi}$$

$$\delta_{n+1}\delta_{n-1} - \delta_n^2 = \alpha^{2n}q^{n-1}(q-1)^2\underline{\alpha}\ \underline{\varphi}.$$

Proof. By writing $r = 1$ in Theorem (3.4), we can get the results.

Theorem 3.5. (d'Ocagne identities) For positive integer n and m, we have

$$\gamma_m\gamma_{n+1} - \gamma_n\gamma_{m+1} = \alpha^{m+n-1}\big([m]_q - [n]_q\big)\left(\underline{\alpha}^2 + \alpha(q-1)\underline{\alpha}\ \underline{\beta}\right)$$

$$\delta_m\delta_{n+1} - \delta_n\delta_{m+1} = \alpha^{m+n+1}(q-1)^2\big([m]_q - [n]_q\big)\underline{\alpha}\ \underline{\varphi}.$$

Proof. Using (2), (3) and (6), we obtain

$$\begin{aligned}\gamma_m\gamma_{n+1} - \gamma_n\gamma_{m+1} &= \left(\alpha^{m-1}[m]_q\underline{\alpha} + (\alpha q)^m\underline{\beta}\right)\left(\alpha^n[n+1]_q\underline{\alpha} + (\alpha q)^{n+1}\underline{\beta}\right) \\ &\quad - \left(\alpha^{n-1}[n]_q\underline{\alpha} + (\alpha q)^n\underline{\beta}\right)\left(\alpha^m[m+1]_q\underline{\alpha} + (\alpha q)^{m+1}\underline{\beta}\right) \\ &= \alpha^{m+n-1}\big([m]_q[n+1]_q - [n]_q[m+1]_q\big)\underline{\alpha}^2 \\ &\quad + \alpha^{m+n}\left(q^n\big(q[m]_q - [m+1]_q\big) + q^m\big([n+1]_q - q[n]_q\big)\right)\underline{\alpha}\ \underline{\beta}\end{aligned}$$

$$= \alpha^{m+n-1}\frac{q^n - q^m}{1-q}\underline{\alpha}^2 + \alpha^{m+n}(q^m - q^n)\underline{\alpha\beta}$$
$$= \alpha^{m+n-1}\left([m]_q - [n]_q\right)\underline{\alpha}^2 + \alpha^{m+n}\left([m]_q - [n]_q\right)(q-1)\underline{\alpha\beta}$$
$$= \alpha^{m+n-1}\left([m]_q - [n]_q\right)\left(\underline{\alpha}^2 + \alpha(q-1)\underline{\alpha}\,\underline{\beta}\right).$$

For the proof of q-Lucas tessarine number sequences, we use (2), (3) and (7) and get the result easily.

Theorem 3.6. For positive integers, n, r and s, the q-Fibonacci and q-Lucas tessarine number sequences satisfy the following identity

$$\delta_{n+r}\gamma_{n+s} - \delta_{n+s}\gamma_{n+r} = 2q^n\alpha^{2n+r+s-1}\left([s]_q - [r]_q\right)\underline{\alpha}\,\underline{\varphi}$$

Proof. By using (2) and (7), we get

$$\delta_{n+r}\gamma_{n+s} - \delta_{n+s}\gamma_{n+r} = \left(\alpha^{n+r}\underline{\alpha} + (\alpha q)^{n+r}\underline{\varphi}\right)\left(\frac{\alpha^{n+s}\underline{\alpha} - (\alpha q)^{n+s}\underline{\varphi}}{\alpha - \alpha q}\right)$$
$$-\left(\alpha^{n+s}\underline{\alpha} + (\alpha q)^{n+s}\underline{\varphi}\right)\left(\frac{\alpha^{n+r}\underline{\alpha} - (\alpha q)^{n+r}\underline{\varphi}}{\alpha - \alpha q}\right)$$
$$= 2q^n\alpha^{2n+r+s-1}\frac{q^r - q^s}{1-q}\underline{\alpha}\,\underline{\varphi}$$
$$= 2q^n\alpha^{2n+r+s-1}\left([s]_q - [r]_q\right)\underline{\alpha}\,\underline{\varphi}.$$

Definition 3.2. For complex polynomials $h(t)$ and $g(t)$, the q-Fibonacci $F_{q,n}(t)$ and q-Lucas $L_{q,n}(t)$ polynomials are defined as follows:

$$\begin{cases} F_{q,n}(t) = h(\mathrm{t})\, F_{q,n-1}(t) - g(t)F_{q,n-2}(t) \\ L_{q,n}(t) = h(\mathrm{t})L_{q,n-1}(t) - g(t)L_{q,n-2}(t). \end{cases} \tag{9}$$

Here, $F_{q,0}(t) = 0, F_{q,1}(t) = 1, L_{q,0}(t) = 2$ and $L_{q,1}(t) = h(t)$, respectively. Classify the q-polynomials $F_{q,n}(t)$ and $L_{q,n}(t)$ according to the $h(t)$ and $g(t)$ values, respectively.
i. Assume that $h(t) = aq + 1$ and $g(t) = a^2q$ are constant polynomials. In this case, we can write as follows:

$$\begin{cases} F_{q,n}(t) = (aq+1)F_{q,n-1}(t) - a^2qF_{q,n-2}(t) \\ L_{q,n}(t) = (aq+1)L_{q,n-1}(t) - a^2qL_{q,n-2}(t). \end{cases}$$

ii. Assume that $h(t) = \lambda(s)$ and $g(t) = -1$ are not constant polynomials. For this case, we obtain the following equality:

$$\begin{cases} F_{q,n}(t) = \lambda(s)F_{q,n-1}(t) + F_{q,n-2}(t) \\ L_{q,n}(t) = \lambda(s)L_{q,n-1}(t) + L_{q,n-2}(t). \end{cases}$$

Roots of $r^2 - h(t)r - 1 = 0$ in (9) are

$$\alpha(r) = \frac{h(t) + \sqrt{h^2(t) + 4}}{2}$$

$$\beta(r) = \frac{h(t) - \sqrt{h^2(t) + 4}}{2}.$$

Then, the Binet formulas for q-polynomials $F_{q,n}(t)$ and $L_{q,n}(t)$ are

$$F_{q,n}(t) = \frac{\alpha(r)^n - \beta(r)^n}{\alpha(r) - \beta(r)}$$

and

$$L_{q,n}(t) = \alpha(r)^n + \beta(r)^n.$$

Definition 3.3. The q-Fibonacci tessarine polynomial sequences $\gamma_{q,n}(t)$ and the q-Lucas tessarine polynomial sequences $\delta_{q,n}(t)$ are defined by the recurrence relation

$$\gamma_{q,n}(t) = F_{q,n}(t) + F_{q,n+1}(t)i_1 + F_{q,n+2}(t)i_2 + F_{q,n+3}(t)i_3$$

$$\delta_{q,n}(t) = L_{q,n}(t) + L_{q,n+1}(t)i_1 + L_{q,n+2}(t)i_2 + L_{q,n+3}(t)i_3.$$

The initial conditions of the $\gamma_{q,n}(t)$ and $\delta_{q,n}(t)$ are

$$\begin{aligned}\gamma_{q,0}(t) &= F_{q,0}(t) + F_{q,1}(t)i_1 + F_{q,2}(t)i_2 + F_{q,3}(t)i_3 \\ &= i_1 + h(t)i_2 + \left(h(t)^2 - g(t)\right)i_3,\end{aligned}$$

$$\begin{aligned}\gamma_{q,1}(t) &= F_{q,1}(t) + F_{q,2}(t)i_1 + F_{q,3}(t)i_2 + F_{q,4}(t)i_3 \\ &= 1 + h(t)i_1 + \left(h(t)^2 - g(t)\right)i_2 + \left(h(t)^3 - 2h(t)g(t)\right),\end{aligned}$$

$$\begin{aligned}\delta_{q,0}(t) &= L_{q,0}(t) + L_{q,1}(t)i_1 + L_{q,2}(t)i_2 + L_{q,3}(t)i_3 \\ &= 2 + h(t)i_1 + \left(h(t)^2 - 2g(t)\right)i_2 + \left(h(t)^3 - 3h(t)g(t)\right)i_3\end{aligned}$$

and

$$\begin{aligned}\delta_{q,1}(t) &= L_{q,1}(t) + L_{q,2}(t)i_1 + L_{q,3}(t)i_2 + L_{q,4}(t)i_3 \\ &= h(t) + \left(h(t)^2 - 2g(t)\right)i_1 + \left(h(t)^3 - 3h(t)g(t)\right)i_2 \\ &\quad + (h(t)^4 - 4h(t)^2 g(t) - 2g(t)^2)i_3\end{aligned}$$

where i_1, i_2 and i_3 are the imaginary units satisfies the multiplication rule in (1).

Theorem 3.7. The Binet-like formulas of the q-tessarine polynomials $\gamma_{q,n}(t)$ and $\delta_{q,n}(t)$ are

$$\gamma_{q,n}(t)=\frac{\alpha(t)^n\underline{\alpha}(t)-\beta(t)^n\underline{\beta}(t)}{\alpha(t)-\beta(t)}$$

and

$$\delta_{q,n}(t)=\alpha(t)^n\underline{\alpha}(t)+\beta(t)^n\underline{\beta}(t).$$

Here,

$$\underline{\alpha}(t)=1+\alpha(t)i_1+\alpha(t)^2i_2+\alpha(t)^3i_3$$

$$\underline{\beta}(t)=1+\beta(t)i_1+\beta(t)^2i_2+\beta(t)^3i_3.$$

Proof. The proof can be done easily by using Binet-like formulas of q-Fibonacci and q-Lucas tessarine polynomial sequences.

Definition 3.4. (Kac & Cheung, 2002; Stum & Quiros, 2013) Suppose that $p(t)$ is an arbitrary function. Its q-derivative operator is given by

$$d_qp(t)=p(qt)-p(t).$$

Note that in particular $d_q(t)=(q-1)t$,

$$\lim_{q\to1}D_qp(t)=\lim_{q\to1}\frac{p(qt)-p(t)}{(q-1)t}=\frac{dp(t)}{dt} \tag{10}$$

where $q\neq1$.

Definition 3.5. The $n^{th}$$q$-Fibonacci and q-Lucas tessarine function sequences are defined as follows:

$$\Gamma_{q,n}(t)=\mathcal{F}_{q,n}(t)+\mathcal{F}_{q,n+1}(t)i_1\ +\mathcal{F}_{q,n+2}(t)i_3+\mathcal{F}_{q,n+3}(t)i_4$$

and

$$\Delta_{q,n}(t)=\mathcal{L}_{q,n}(t)+\mathcal{L}_{q,n+1}(t)i_1+\mathcal{L}_{q,n+2}(t)i_2+\mathcal{L}_{q,n+3}(t)i_3,$$

respectively. Where $\mathcal{F}_{q,n}(t)$ and $\mathcal{L}_{q,n}(t)$ are the n^{th} q-Fibonacci and q-Lucas functions and i_1, i_2 and i_3 are the imaginary units satisfy rules in (1).
The q-derivative of $\Gamma_{q,n}(t)$ and $\Delta_{q,n}(t)$ is defined as:

$$D_q\Gamma_{q,n}(t)=D_q\mathcal{F}_{q,n}(t)+D_q\mathcal{F}_{q,n+1}(t)i_1\ +D_q\mathcal{F}_{q,n+2}(t)i_3+D_q\mathcal{F}_{q,n+3}(t)i_4$$

and

$$D_q\Delta_{q,n}(t)=D_q\mathcal{L}_{q,n}(t)+D_q\mathcal{L}_{q,n+1}(t)i_1+D_q\mathcal{L}_{q,n+2}(t)i_2+D_q\mathcal{L}_{q,n+3}(t)i_3$$

where $D_q\mathcal{F}_{q,n}(t)$ means the derivative of $\mathcal{F}_{q,n}(t)$.

Example 3.1. For any integer n, if $\mathcal{F}_{q,n}(t)=(t-a)^n$,then

$$D_q\mathcal{F}_{q,n}(t)=[n]_q\mathcal{F}_{q,n-1}(t)$$

Proof. From (10), compute q-derivative of the function sequences $\mathcal{F}_{q,n}(t)$,

$$\begin{aligned}D_q\mathcal{F}_{q,n}(t)&=\frac{\left(q(t-a)\right)^n-(t-a)^n}{(q-1)(t-a)}\\&=\frac{q^n-1}{q-1}(t-a)^{n-1}\end{aligned}$$

$$= [n]_q \mathcal{F}_{q,n-1}(t)$$

and from above example the derivative of the n^{th} q-Fibonacci tessarine function sequences is

$$D_q \Gamma_{q,n}(t) = [n]_q \mathcal{F}_{q,n-1}(t) + [n+1]_q \mathcal{F}_{q,n}(t) i_1 + [n+2]_q \mathcal{F}_{q,n+1}(t) i_2 + [n+3]_q \mathcal{F}_{q,n+2}(t) i_3$$

4. Discussion and Conclusion

In this study, tessarine number sequences are defined using notations from quantum calculus. We derive several fundamental identities, including Binet-like formulas, exponential generating functions, and Catalan-like, Cassini-like, and d'Ocagne-like identities for these numbers. Additionally, we introduce new tessarine polynomial and function sequences, namely the q-Fibonacci tessarine and q-Lucas tessarine polynomials and function sequences. We then present various properties and identities for these polynomials and function sequences. In the future, researchers may explore additional identities of tessarine number sequences within the framework of quantum calculus.

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