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Solving an inverse heat conduction problem by reduced differential transform method

Afshin Babaei[1] *and Alireza Mohammadpour*[2]

[1]Department of Mathematics, University of Mazandaran, Babolsar, P.O. Box: 47416-95447, Iran
[2]Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran.

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Abstract: In this paper, the inverse problem of determining an unknown boundary condition in an inverse heat conducton problem is considered. The reduced differential transform method is presented for recovering the unknown functions and obtaining a solution of the problem. This technique doesn't require any discretization, linearization or small perturbations. Finally, some examples are presented to illustrate the ability and efficiency of the present approach.

Keywords: Heat conduction problem; Inverse problem; Neumann boundary condition; Reduced differential transform method.

1 Introduction

Inverse heat conduction problems have many applications in various branches of science and engineering, mechanical and chemical engineering. Mathematicians and specialists in many other science branches are interested in inverse problems [1-5], each with different application in mind. Inverse solution consists not only in estimation of unknown information but also in finding full temperature field.

Mathematically, the inverse problems belong to the class of problems called the illposed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solution does not satisfy the general requirement of existence, uniqueness and stability under small changes to the input data. Today, various methods have been developed for the analysis of the inverse problems from measured temperatures inside the material [6-14].

In this paper, we apply the reduced differential transform method (RDTM) for solving the inverse heat conduction problem with the Neumann boundary condition. The problem consists in the calculation of temperature distribution in the domain, as well as in the reconstruction of functions describing the temperature and the heat flux on the boundary, in case when the temperature measurement in some points of the domain are known.

The structure of this paper is as follows. In section 2, formulating of an one dimensional inverse heat conduction problem is presented. In section 3, the basic idea of the reduced differential transform method will be introduced. Afterwards, in section 4 this method is employed to find the approximate analytical solution of the inverse heat conduction problem. Finally, in order to demonstrate the usefulness of method, some examples are presented in final section.

* Corresponding author e-mail: babaei@umz.ac.ir

2 Mathematical formulation of the problem

Let $D=\{(x,t): x\in[0,b], t\in[0,T]\}$ be the domain of the problem and T be end time of the process. On the boundary of this domain three component are distributed:

$$\Gamma_0=\{(x,0): x\in[0,b]\} \tag{1}$$

$$\Gamma_1=\{(0,t): t\in[0,T]\} \tag{2}$$

$$\Gamma_2=\{(b,t): t\in[0,T]\} \tag{3}$$

where the initial and boundary conditons are given. The heat conduction equation in the region D is :

$$\frac{\partial u}{\partial t}(x,t)=a\frac{\partial^2 u(x,t)}{\partial x^2}\quad (x,t)\in D \tag{4}$$

where a is the thermal diffusivity, u is temperature and t and x refer to time and special location respectively. On the boundary Γ_0 we have the initial condition as:

$$u(x,0)=\varphi(x),\quad x\in[0,b]. \tag{5}$$

The Dirichlet boundary conditions on Γ_1 and Γ_2 are as follows:

$$u(0,t)=\psi(t),\quad t\in[0,T]. \tag{6}$$

$$-k\frac{\partial u(b,t)}{\partial x}=q(t),\quad t\in[0,T] \tag{7}$$

where k ia a constant and $\varphi(x)$, $\psi(t)$ and $q(t)$ are piecewise continuous functions in their domains.

In the inverse problem, the temprature distribution $u(x,t)$ in region D and $q(t)$ in the Neumann boundary condition (7) are unknown. For determination of unknwons, we have the overspecified condition at the fixed point $x=x_p$, as:

$$u(x_p,t)=\psi_p(t),\quad t\in[0,T]. \tag{8}$$

where $x_p\in(0,b)$.

3 Reduced differential transform method

Consider a function of two variables $u(x,t)$ which is analytic and differentiated continuously in the domain of interest, and suppose that it can be represented as a product of two single-variables, i.e. $u(x,t)=f(x)g(t)$. The basic definitions and operations of RDTM [15-20] are reviewed.

Definition 3.1. If function $u(x,t)$ is analytic and differentiated continuosly with respect to time t and space x in the domain of interest, then let

$$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{9}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function which is called T-function in brief. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{10}$$

Combining (9) and (10) gives that:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0} t^k. \tag{11}$$

In real application, the function $u(x,t)$ by a finite series of equation (11) can be written as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{12}$$

By consideration of $U_0(x) = f(x)$ as transformaiton of initial condition

$$u(x,0) = f(x). \tag{13}$$

A straightforward iterative calculations gives the $U_k(x)$ values for $k = 1,2,...,n$. Then the inverse transformation of the $\{U_k(x)\}_{k=0}^{n}$ gives the approximation solutoin as:

$$\widetilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k, \tag{14}$$

where n is order of approximation solution. The exact solutoin is given by:

$$u(x,t) = \lim_{n \longrightarrow \infty} \widetilde{u}_n(x,t). \tag{15}$$

The fundamental operations of reduced differential transform that can be deduced from Eqs. (9) and (10) are listed in below [15-20].

Table 1. Some basic reduced differential transformations.

Function Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t) \right]_{t=0}$
$u(x,t) = v(x,t) + w(x,t)$	$U_k(x) = V_k(x) + W_k(x)$
$u(x,t) = cv(x,t)$	$U_k(x) = cV_k(x)$ $(c \text{ is a constant})$
$u(x,t) = x^m t^n$	$U_k(x) = x^m \delta(k-n) = x^m \begin{cases} 1 & k=n \\ 0 & k \neq n \end{cases}$
$u(x,t) = x^m t^n v(x,t)$	$U_k(x) = x^m V_{k-n}(x)$
$u(x,t) = \frac{\partial}{\partial t} v(x,t)$	$U_k(x) = (k+1) V_{k+1}(x)$
$u(x,t) = \frac{\partial^m}{\partial x^m} v(x,t)$	$U_k(x) = \frac{\partial^m}{\partial x^m} V_k(x).$

4 Application of reduced differential transform method

In this section we apply RDTM for solving the problem in section 3 and solving some examples. According to the RDTM and Table 1, we can construct the following iteration For the (4) as:

$$U_{k+1}(x) = \frac{a}{k+1}\frac{\partial^2}{\partial x^2}U_k(x), \quad k = 1,2,\cdots. \tag{16}$$

By starting from the $U_0(x)$ as transformation of initial condition (5), we can get the $U_k(x)$ values ($k = 1,2,\cdots$) by (16). Finally, the solution is determined by (14) and (15). After knowing the distribution $u(x,t)$ and the Neumann boundary condition (7) will be determined.

Example 1. Let us consider $b = 2$, $x_p = 1$, $a = 1$, $k = 1$ and $T = 2$. Also, suppose $\varphi(x) = \frac{1}{24}x^4$, $\psi(t) = \frac{1}{2}t^2$ and $\psi_p(t) = \frac{1}{24} + \frac{1}{2}t + \frac{1}{2}t^2$. With these assumptions, the exact solution of (4)-(6) is [11]

$$u(x,t) = \frac{t^2}{2} + \frac{t}{2}x^2 + \frac{1}{24}x^4.$$

By taking the differential transform of (4), we obtain

$$U_{k+1}(x) = \frac{1}{k+1}\frac{\partial^2}{\partial x^2}U_k(x), \; k = 0,1,2,... \tag{17}$$

where the t-dimensional spectrum function $U_k(x)$ are the transformed function. Also, the initial condition (5) gives

$$U_0(x) = \frac{1}{24}x^4. \tag{18}$$

Now, substituting (18) into (17), we obtain the following $U_k(x)$ values successively

$$U_1(x) = \frac{1}{2}x^2,$$

$$U_2(x) = \frac{1}{2},$$

and

$$U_k(x) = 0, \; k = 3,4,\cdots.$$

Finally the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = \frac{1}{24}x^4 + \frac{1}{2}x^2 t + \frac{1}{2}t^2.$$

From this solution, we get the boundary conditions

$$q(t) = -k\frac{\partial u(2,t)}{\partial x} = -\frac{4}{3} - 2t.$$

Example 2. Let us consider $b = 1.2$, $x_p = 1$, $a = 0.1$, $k = 2$ and $T = 2$. Also assume that $\varphi(x) = e^{1-x}$, $\psi(t) = e^{0.1t+1}$ and

$\psi_p(t) = e^{0.1t}$. With these assumptions, the problem (4)-(6) has the exact solution [11]

$$u(x,t) = e^{1-x-0.1t}. \tag{19}$$

Taking differential transform of (4) and $u(x,0) = \varphi(x)$ respectively, we obtain

$$U_{k+1}(x) = \frac{1}{k+1}\frac{\partial^2}{\partial x^2}, U_k(x),\ k = 0,1,2,\cdots. \tag{20}$$

Suppose

$$U_0(x) = e^{1-x}.$$

By substituting (4) into(20), we obtain the following values

$$U_1(x) = \frac{1}{10}e^{1-x},$$

$$U_2(x) = \frac{1}{2\times 10^2}e^{1-x},$$

$$U_3(x) = \frac{1}{3!\times 10^3}e^{1-x},$$

and, in general, by mathematical induction,

$$U_n(x) = \frac{1}{n!\times 10^n}e^{1-x}. \tag{21}$$

The differential inverse transform of $U_n(x)$ gives:

$$\begin{aligned}\widetilde{u}_n(x,t) &= e^{1-x} + \frac{1}{10}e^{1-x}t + \frac{1}{2!\times 10^2}e^{1-x}t^2 + \cdots + \frac{1}{n!\times 10^n}e^{1-x} \\ &= e^{1-x}[1 + 0.1t + \frac{(0.1t)^2}{2!} + \frac{(0.1t)^3}{3!} + \cdots + \frac{(0.1t)^n}{n!}.\end{aligned}$$

Therefore, the exact solution of problem is given by

$$u(x,t) = \lim_{n\longrightarrow\infty}\widetilde{u}_n(x,t) = e^{1-x}e^{0.1t} = e^{1-x+0.1t}, \tag{22}$$

which is the exact temprature distribution in whole region. The exact solution (22) gives the boundary conditions as:

$$q(t) = -k\frac{\partial u(1.2,t)}{\partial x} = 2e^{0.1t-0.2}.$$

5 Conclusion

In this paper, the reduced differential transform method was employed for solving the inverse problem of determining some unknown boundary conditions in an inverse heat conducton problem under a overdetermination condition. This method is a suitable method to obtain the approximate analytical solutions for these types of inverse problems, because, it gives rapidly converging series solutions. In the last section, the mentioned method was applied for two test examples. Exact closed form solution is obtained for these examples. The results show that the proposed method is a reliable technique for solving some inverse problems.

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