Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 26.03.2025
DOI : 10.15672/hujms.1597774

Research Article

Strongly completely monotonic functions on time scales

Jing-Feng Tian[1], Xiao-Yue Du[1], Zhong-Xuan Mao*[1], Jun-Yi Li[2]

[1] Hebei Key Laboratory of Physics and Energy Technology, Department of Mathematics and Physics, North China Electric Power University

[2] School of mathematics and statistic, Northwestern Polytechnical University

Abstract

In this paper, we introduce the concept of strongly completely monotonic functions on time scales and investigate several properties of such functions. Meanwhile, we present some key results considering three special cases including continuous, discrete, and quantum. As applications, we prove that certain functions involving the confluent and Gaussian hypergeometric functions are strongly completely monotonic.

Mathematics Subject Classification (2020). 26E70; 26A48; 33C15; 33C05

Keywords. Time scales; strongly completely monotonic function; completely monotonic function; confluent hypergeometric function

1. Introduction

Prior to the 1980s, continuity and discreteness were two separate lines of research considered by academics. Stefan Hilger introduced the time scale and its related concepts and properties in his PhD thesis [12] and academic paper [13], and succeeded in unifying the discrete and continuous analyses under the single framework of the time scale. Thus, the theory of time scale was established. After then, scholars provided some concepts on times scale, such as Taylor expansions [5], Laplace transforms [7], convolution [19] and some special functions [6]. In recent years, Mao and Tian established the concept of completely monotonic degree [16] and monotonicity rules [17] within the framework of time scales. Furthermore, time scales have found widespread applications in a variety of fields, including physics [22], chemical engineering [25], economics [2,3] and neural networks [15].

One of the key differences between real analysis and time scale analysis lies in the definition of derivatives. Here, we provide a brief introduction. In time scale analysis, there are three types of derivatives: the delta derivative (Δ), the nabla derivative (∇), and their linear combination, the diamond-alpha derivative ($\diamond_\alpha$). Let $\mathbb{T}$ is a time scale, which is an arbitrary nonempty closed subset of the real numbers. Define the forward

*Corresponding Author.
Email addresses: tianjf@ncepu.edu.cn (J.F. Tian), duxiaoyue@ncepu.edu.cn (X.Y. Du), maozhongxuan000@gmail.com (Z.X. Mao), ljy__ljy@mail.nwpu.edu.cn (J.Y. Li)
Received: 08.12.2024; Accepted: 28.02.2025

and backward jump operator by $\sigma(x) = \inf\{t \in \mathbb{T} : t > x\}$ and $\rho(x) = \sup\{t \in \mathbb{T} : t < x\}$, respectively. The delta derivative (see [15]) of a continuous function f on $\mathbb{T}^k$ is defined by

$$f^{\Delta}(x) = \begin{cases} \frac{f(\sigma(x))-f(x)}{\sigma(x)-x} & \text{if} \quad \sigma(x) > x, \\ \lim_{t\to x^+} \frac{f(t)-f(x)}{t-x} & \text{if} \quad \sigma(x) = x, \end{cases}$$

where

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup\mathbb{T}), \sup\mathbb{T}) & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

The nabla derivative (see [15]) of a continuous function f on $\mathbb{T}_k$ is defined by

$$f^{\nabla}(x) = \begin{cases} \frac{f(x)-f(\rho(x))}{x-\rho(x)} & \text{if} \quad \rho(x) < x, \\ \lim_{t\to x^-} \frac{f(x)-f(t)}{x-t} & \text{if} \quad \rho(x) = x, \end{cases}$$

where

$$\mathbb{T}_k = \begin{cases} \mathbb{T}\backslash\{m\} & \text{if } \mathbb{T} \text{ has a right-scattered minimum } m, \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

And the diamond-alpha derivative [21] is defined by

$$f^{\diamond_\alpha}(x) = \alpha f^{\Delta}(x) + (1-\alpha) f^{\nabla}(x), \quad \alpha \in [0,1],\ x \in \mathbb{T}^k \cap \mathbb{T}_k.$$

We recall the concepts of completely monotonic functions and absolutely monotonic functions [4,24]. Let $D^\infty(0,\infty)$ denote the set of all functions defined on $(0,\infty)$ that have derivatives of all orders.

Definition 1.1 ([4,24])**.** Let $f \in D^\infty(0,\infty)$ and

$$(-1)^n f^{(n)}(x) \geq 0, \quad x > 0$$

for all $n \in \mathbb{N}$. Then f is said to be a completely monotonic function on $(0,\infty)$.

Definition 1.2 ([24])**.** Let $f \in D^\infty(0,\infty)$ and

$$f^{(n)}(x) \geq 0, \quad x > 0$$

for all $n \in \mathbb{N}$. Then f is said to be an absolutely monotonic function on $(0,\infty)$.

In 1989, Trimble, Wells and Wright[23] introduced the concept of strongly completely monotonic functions, which is defined as follows.

Definition 1.3 ([23])**.** Let $f \in D^\infty(0,\infty)$ satisfy that the function

$$x \mapsto (-1)^n x^{n+1} f^{(n)}(x),$$

is non-negative and non-increasing on $(0,\infty)$ for all $n \in \mathbb{N}$. Then f is said to be a strongly completely monotonic function on $(0,\infty)$.

Clearly, if a function is strongly completely monotonic, it is also completely monotonic [14]. A function f is strongly completely monotonic on $(0,\infty)$ if and only if the function $xf(x)$ is completely monotonic on $(0,\infty)$ (see [26]) or there exists a non-negative and increasing function p such that $f(x) = \int_0^\infty e^{-xt} p(t)dt$ for all $x > 0$ (see [23]). In order to systematically research completely monotonic functions, Guo and Qi [11] established the concept of completely monotonic degree.

In 2023, the concept of complete monotonicity [18] has been further extended to the time scale domain, by using delta derivative.

Definition 1.4 ([18])**.** Suppose the function $f : \mathbb{T} \to \mathbb{R}$ has delta derivatives of all orders and satisfies

$$(-1)^n f^{\Delta^n}(x) = (-1)^n (f^{\Delta^{n-1}}(x))^{\Delta} \geq 0$$

for all $n \in \mathbb{N}$, then f is said to be a delta completely monotonic function.

This concept can be easily extended to results concerning nabla derivatives and diamond-alpha derivatives.

Definition 1.5. Suppose the function $f : \mathbb{T} \to \mathbb{R}$ has nabla derivatives of all orders and satisfies
$$(-1)^n f^{\nabla^n}(x) \geq 0$$
for all $n \in \mathbb{N}$, then f is said to be a nabla completely monotonic function.

Definition 1.6. Suppose the function $f : \mathbb{T} \to \mathbb{R}$ has delta derivatives and nabla derivatives of all orders and satisfies
$$(-1)^n f^{\diamond_{\alpha_1}\diamond_{\alpha_2}\cdots\diamond_{\alpha_n}}(x) \geq 0$$
for all $\alpha_1, \cdots, \alpha_n \in [0,1]$ and $n \in \mathbb{N}$, then f is said to be a diamond-alpha completely monotonic function.

Remark 1.7. If $\alpha_1 = \cdots = \alpha_n = 1$, then diamond-alpha completely monotonic function reduces to delta completely monotonic function, and it reduces to the nabla completely monotonic function if $\alpha_1 = \cdots = \alpha_n = 0$.

Inspired by the aforementioned studies, in this paper, we first aim to establish the concept of strongly completely monotonic functions within the framework of time scales, utilizing not only delta derivatives but also nabla derivatives and diamond-alpha derivatives. Subsequently, we explore various properties of these functions.

The paper is organized as follows. In Section 2, we introduce the concept of strongly completely monotonic functions on time scales. Based on these, we establish several theorems and corollaries on the three most commonly used time scales: continuous, discrete, and quantum. In Section 3, we investigate the applications of strongly completely monotonic functions in the field of special functions.

2. Strongly completely monotonic functions on time scales

This section is divided into two parts. In the first part, we will provide the definition of strongly completely monotonic functions on time scales. In the second part, we will explore the properties of these functions.

Firstly, we introduce the concepts of strongly completely monotonic functions on time scales, which are defined as follows.

Definition 2.1. Suppose the function $f : \mathbb{T} \to \mathbb{R}$ has delta derivatives of all orders and satisfies the function
$$(-1)^n x^{n+1} f^{\Delta^n}(x)$$
is non-negative and non-increasing for all $n = 0, 1, \cdots$, then the function f is said to be delta strongly completely monotonic, Δ-$SCM^{\mathbb{T}}$ for short.

Definition 2.2. Suppose the function $f : \mathbb{T} \to \mathbb{R}$ has nabla derivatives of all orders and satisfies the function
$$(-1)^n x^{n+1} f^{\nabla^n}(x)$$
is non-negative and non-increasing for all $n = 0, 1, \cdots$, then the function f is said to be nabla strongly completely monotonic, ∇-$SCM^{\mathbb{T}}$ for short.

Definition 2.3. Suppose the function $f : \mathbb{T} \to \mathbb{R}$ has delta derivatives and nabla derivatives of all orders and satisfies the function
$$(-1)^n x^{n+1} f^{\diamond_{\alpha_1}\diamond_{\alpha_2}\cdots\diamond_{\alpha_n}}(x)$$
is non-negative and non-increasing for all $\alpha_1, \cdots \alpha_n \in [0,1]$ and $n = 0, 1, \cdots$, then the function f is said to be diamond-alpha strongly completely monotonic, $\diamond_\alpha$-$SCM^{\mathbb{T}}$ for short.

Next, we present several theorems involving strongly completely monotonic functions on time scales, as follows.

Theorem 2.4. *If the function $f : \mathbb{T} \to \mathbb{R}$ has delta derivatives of all orders, and the inequalities*

$$(-1)^n x^{n+1} f^{\Delta^n}(x) \geq 0$$

and

$$(-1)^n \left(\sum_{k=0}^{n} x^k \sigma^{n-k}(x) f^{\Delta^n}(x) + \sigma^{n+1}(x) f^{\Delta^{n+1}}(x) \right) \leq 0$$

hold for all $n = 0, 1, \cdots$, then the function f is delta strongly completely monotonic.

Theorem 2.5. *If the function $f : \mathbb{T} \to \mathbb{R}$ has nabla derivatives of all orders, and the inequalities*

$$(-1)^n x^{n+1} f^{\nabla^n}(x) \geq 0$$

and

$$(-1)^n \left(\sum_{k=0}^{n} x^k \rho^{n-k}(x) f^{\nabla^n}(x) + \rho^{n+1}(x) f^{\nabla^{n+1}}(x) \right) \leq 0,$$

hold for all $n = 0, 1, \cdots$, then the function f is nabla strongly completely monotonic.

Theorem 2.6. *If the function $f : \mathbb{T} \to \mathbb{R}$ has delta derivatives and nabla derivatives of all orders, and the inequalities*

$$(-1)^n x^{n+1} f^{\diamond_{\alpha_1} \diamond_{\alpha_2} \cdots \diamond_{\alpha_n}}(x) \geq 0$$

and

$$(-1)^n \Big((\alpha_{n+1} g_n^{\Delta}(x) + (1-\alpha_{n+1}) g_n^{\nabla}(x)) h(x) + \alpha_{n+1} g_n^{\sigma}(x) h^{\Delta}(x) + (1-\alpha_{n+1}) g_n^{\rho}(x) h^{\nabla}(x) \Big) \leq 0$$

hold for $\alpha_1, \cdots \alpha_n \in [0,1]$ and $n = 0, 1, \cdots$, where $g_n(x) = x^{n+1}$ and $h(x) = f^{\diamond_{\alpha_1} \diamond_{\alpha_2} \cdots \diamond_{\alpha_n}}(x)$, then the function f is said to be diamond-alpha strongly completely monotonic.

Taking $\mathbb{T} = [0, \infty)$, we obtain the following theorem.

Theorem 2.7. *Let $f \in D^{\infty}(0, \infty)$ satisfying*

$$(-1)^n x^{n+1} f^{(n)}(x) \geq 0$$

and

$$(-1)^n \left((n+1) x^n f^{(n)}(x) + x^{n+1} f^{(n+1)}(x) \right) \leq 0$$

for all $x \in (0, \infty)$ and any $n \in \mathbb{N}$. Then the function f is strongly completely monotonic.

Taking $\mathbb{T} = \mathbb{N}$, we have the following theorem.

Theorem 2.8. *Suppose the derivative $\Delta^n f(x)$ exists for all $x \in \mathbb{N}$ and positive integer n, and f satisfies that*

$$(-1)^n x^{n+1} \Delta^n f(x) \geq 0$$

and

$$(-1)^n \left(\sum_{k=1}^{n+1} x^k (x+1)^{n-k} \Delta^n f(x) + (x+1)^{n+1} \Delta^{n+1} f(x) \right) \leq 0,$$

then the function f is delta strongly completely monotonic on $\mathbb{N}$, where $\Delta f(x) := f(x+1) - f(x)$ is the forward differnece.

Taking $\mathbb{T} = h\mathbb{N} (h > 1)$, we have the following theorem.

Theorem 2.9. *Suppose the n-order derivative $\Delta_h^n f(x)$ exists for all $x \in h\mathbb{N}$ and positive integer n, and f satisfies that*

$$(-1)^n x^{n+1} \Delta_h^n f(x) \geq 0$$

and

$$(-1)^n \left(\sum_{k=1}^{n+1} x^k (x+h)^{n-k} \Delta_h^n f(x) + (x+h)^{n+1} \Delta_h^{n+1} f(x) \right) \leq 0,$$

then the function f is delta strongly completely monotonic on $h\mathbb{N}$, where $\Delta_h f(x) := \frac{f(x+h)-f(x)}{h}$.

Let $\mathbb{T} = q^{\mathbb{N}} (q > 1)$, the following theorem holds.

Theorem 2.10. *Suppose the n-order derivative $\mathfrak{Q}_q^n f(x)$ exists for all $x \in q^{\mathbb{N}}$ and positive integer n, and f satisfies that*

$$(-1)^n x^{n+1} \mathfrak{Q}_q^n f(x) \geq 0$$

and

$$(-1)^n \left(\sum_{k=0}^{n} q^k x^n \mathfrak{Q}_q^n f(x) + q^{n+1} x^{n+1} \mathfrak{Q}_q^{n+1} f(x) \right) \leq 0,$$

then the function f is delta strongly completely monotonic on $q^{\mathbb{N}}$, where $\mathfrak{Q}_q f(x) := \frac{f(x)-f(qx)}{(1-q)x}$.

Clearly, if a function is strongly completely monotonic on time scales, then it is also completely monotonic on time scales. Specifically, we have the following property.

Property 2.11. *We have the following three conclusions.*

(i) *A delta strongly completely monotonic function is also a delta completely monotonic function.*
(ii) *A nabla strongly completely monotonic function is also a nabla completely monotonic function.*
(iii) *A diamond-alpha strongly completely monotonic function is also a diamond-alpha completely monotonic function.*

The linear relationships for strongly completely monotonic functions defined on time scales are easily obtainable.

Property 2.12. *Let $c_i, k_i \in \mathbb{R} (i = 1, 2, \cdots, n)$ be non-negative. If the functions $f_i, g_i : \mathbb{T} \to \mathbb{R}$ are Δ-$SCM^{\mathbb{T}}$, then the function $\sum_{i=1}^{n} (c_i f_i + k_i g_i)$ is Δ-$SCM^{\mathbb{T}}$.*

Property 2.13. *Let $c_i, k_i \in \mathbb{R} (i = 1, 2, \cdots, n)$ be non-negative. If the functions $f_i, g_i : \mathbb{T} \to \mathbb{R}$ are ∇-$SCM^{\mathbb{T}}$, then the function $\sum_{i=1}^{n} (c_i f_i + k_i g_i)$ is ∇-$SCM^{\mathbb{T}}$.*

Property 2.14. *Let $c_i, k_i \in \mathbb{R} (i = 1, 2, \cdots, n)$ be non-negative. If the functions $f_i, g_i : \mathbb{T} \to \mathbb{R}$ are $\diamond_\alpha$-$SCM^{\mathbb{T}}$, then the function $\sum_{i=1}^{n} (c_i f_i + k_i g_i)$ is $\diamond_\alpha$-$SCM^{\mathbb{T}}$.*

3. Applications

In this section, our main purpose is to present some applications of strongly completely monotonic functions. We provide a series of strongly completely monotonic functions, some of which are associated with the confluent hypergeometric function, and others with the Gaussian hypergeometric function.

The Gauss hypergeometric function [1] is defined by

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad a, b, c \in \mathbb{R}, \quad -c \notin \mathbb{N}, \quad x \in (0, 1),$$

where

$$(a)_0 = 1, \quad (a)_k := a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

is Pochhammer symbol, and

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

is the gamma function. The Gauss hypergeometric function not only plays a foundational role in the fields of mathematics and applied mathematics, but also is widely used in physics [20], signal processing [10], economy [8], communication technology [9], and so on. The confluent hypergeometric function of the first kind and that of the second kind are defined by

$$M(a,c;x) = \sum_{k=0}^\infty \frac{(a)_k}{(c)_k k!} x^k, \quad a, c \in \mathbb{R}, \ -c \notin \mathbb{N},$$

and

$$U(a,c;x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1} dt, \quad a > 0, x > 0,$$

respectively.

First, we prove that some functions, related with confluent hypergeometric function of the second kind are strongly completely monotonic.

Proposition 3.1. *Let $a \geq 1$ and $c \geq 2$. Then the confluent hypergeometric function of the second kind $U(a,c;x)$ is strongly completely monotonic on $(0,\infty)$.*

Proof. Define a function by

$$\phi_1(t) := t^{a-1}(1+t)^{c-a-1}, \quad a \geq 1, \quad c \geq 2.$$

This function is non-negative and non-decreasing since

$$\phi_1'(t) = t^{a-2}(t+1)^{-a+c-2}((c-2)t + a - 1) \geq 0.$$

Noticing that the function

$$\begin{aligned}(-1)^n x^{n+1} U^{(n)}(a,c;x) &= \frac{x^{n+1}}{\Gamma(a)} \int_0^\infty t^n e^{-xt} t^{a-1} (1+t)^{c-a-1} dt \\ &= \frac{x^{n+1}}{\Gamma(a)} \int_0^\infty e^{-xt} t^n \phi_1(t) dt \\ &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-s} s^n \phi_1\Big(\frac{s}{x}\Big) ds,\end{aligned}$$

is non-negative and non-increasing, by Definition 2.1, the function $U(a,c;x)$ is strongly completely monotonic. □

Since a strongly completely monotonic function must be completely monotonic, we have the following result.

Corollary 3.2. *Let $a \geq 1$ and $c \geq 2$. Then the function $U(a,c;x)$ is completely monotonic on $(0,\infty)$.*

Taking $c = a$ in Proposition 3.1, we have the following example.

Example 3.3. Let $c = a \geq 2$. Then the function

$$U(a,a;x) = e^x \Gamma(1-a,x)$$

is strongly completely monotonic and completely monotonic on $(0,\infty)$.

Taking $c = a+1$ in Proposition 3.1, we have the following example.

Example 3.4. Let $c = a + 1 \geq 2$. Then the function

$$U(a, a+1; x) = x^{-a},$$

is strongly completely monotonic and completely monotonic on $(0, \infty)$.

Taking $c = 2a$ in Proposition 3.1, we have the following corollary.

Corollary 3.5. *Let $c = 2a \geq 2$. Then the function*

$$U(a, 2a; x) = \frac{e^{x/2} x^{1/2-a} K_{a-1/2}\left(\frac{x}{2}\right)}{\sqrt{\pi}}$$

is strongly completely monotonic and completely monotonic on $(0, \infty)$, where

$$K_\nu(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(\nu t)\, dt$$

is the modified Bessel functions of the second kind.

Taking $a = 1$ in Proposition 3.1, we have the following example.

Example 3.6. Let $c \geq 2$. Then the function

$$U(1, c; x) = e^x x^{1-c} \Gamma(c-1, x)$$

is strongly completely monotonic on $(0, \infty)$.

Since the sum of multiple strongly completely monotonic functions is also strongly completely monotonic, it follows naturally that the function $\sum_{i=1}^m U(a, c_i; x)$ is strongly completely monotonic if each $c_i \geq 2$. Furthermore, we have the following proposition.

Proposition 3.7. *Let $m \in \mathbb{N}$, $a \geq 1$ and $\sum_{i=1}^m c_i \geq 2m$. Then the function $\sum_{i=1}^m U(a, c_i; x)$ is strongly completely monotonic on $(0, \infty)$.*

Proof. Define a function by

$$\phi_2(t) = t^{a-1} \sum_{i=1}^m (1+t)^{c_i - a - 1}.$$

This function is non-negative, and we assert that it is also non-decreasing. In fact, taking derivative leads to

$$\begin{aligned}
\phi_2'(t) &= (a-1)t^{a-2} \sum_{i=1}^m (1+t)^{c_i-a-1} + t^{a-1} \sum_{i=1}^m (c_i - a - 1)(1+t)^{c_i-a-2} \\
&= \sum_{i=1}^m t^{a-2}(1+t)^{c_i-a-2}\Big((a-1)(1+t) + (c_i - a - 1)t\Big) \\
&= \sum_{i=1}^m t^{a-2}(1+t)^{c_i-a-2}\Big((a-1) + (c_i - 2)t\Big) \\
&= (a-1)\sum_{i=1}^m t^{a-2}(1+t)^{c_i-a-2} + \sum_{i=1}^m t^{a-1}(1+t)^{c_i-a-2}(c_i - 2).
\end{aligned}$$

Since $a \geq 1$, we have $(a-1)\sum_{i=1}^m t^{a-2}(1+t)^{c_i-a-2} \geq 0$. Without loss of generality, we assume that $c_1 \geq \cdots \geq c_m$ and complete the proof of the assert via discussing two scenarios. If $c_1 \geq \cdots \geq c_m \geq 2$, then clearly we have

$$\sum_{i=1}^m t^{a-1}(1+t)^{c_i-a-2}(c_i - 2) \geq 0.$$

If there exist $k \in \{1, \cdots, m-1\}$ such that $c_i \geq 2$ for all $i \leq k$ and $c_i \leq 2$ for all $i \geq k+1$, we have

$$\begin{aligned}
&\sum_{i=1}^{m} t^{a-1}(1+t)^{c_i-a-2}(c_i-2) \\
= \; & \frac{t^{a-1}}{(1+t)^{a+2}}\Big(\sum_{i=1}^{k}(1+t)^{c_i}(c_i-2) + \sum_{i=k+1}^{m}(1+t)^{c_i}(c_i-2)\Big) \\
\geq \; & \frac{t^{a-1}}{(1+t)^{a+2}}\Big(\sum_{i=1}^{k}(1+t)^{c_{k+1}}(c_i-2) + \sum_{i=k+1}^{m}(1+t)^{c_{k+1}}(c_i-2)\Big) \\
= \; & \frac{t^{a-1}}{(1+t)^{a+2}}\sum_{i=1}^{m}(1+t)^{c_{k+1}}(c_i-2) \\
= \; & t^{a-1}(1+t)^{c_{k+1}-a-2}\sum_{i=1}^{m}(c_i-2) \geq 0.
\end{aligned}$$

Hence, we prove that the function $\phi_2(t)$ is non-decreasing.

Now, it can be seen that the function

$$\begin{aligned}
(-1)^n x^{n+1}\frac{d^n}{dx^n}\sum_{i=1}^{m}U(a,c_i;x) &= \frac{x^{n+1}}{\Gamma(a)}\int_0^\infty t^n e^{-xt}t^{a-1}\sum_{i=1}^{m}(1+t)^{c_i-a-1}dt \\
&= \frac{x^{n+1}}{\Gamma(a)}\int_0^\infty e^{-xt}t^n\phi_2(t)dt \\
&= \frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^n\phi_2\Big(\frac{s}{x}\Big)ds
\end{aligned}$$

is non-negative and non-increasing, by Definition 2.1, the function $\sum_{i=1}^{m}U(a,c_i;x)$ is strongly completely monotonic. □

Taking $a=1$ in Proposition 3.7, we have the following corollary.

Corollary 3.8. *Let $m \in \mathbb{N}$ and $\sum_{i=1}^{m} c_i \geq 2m$. Then the function*

$$\sum_{i=1}^{m}U(1,c_i;x) = e^x\sum_{i=1}^{m}x^{1-c_i}\Gamma(c_i-1,x)$$

is strongly completely monotonic on $(0,\infty)$.

Now, we present several strongly completely monotonic functions related to the confluent hypergeometric function of the first kind.

Proposition 3.9. *Let $a \geq 0$ and $c > 0$. Then the function $M(a,c;e^{-x})/x$ is strongly completely monotonic on $(0,\infty)$.*

Proof. It is easy to check that the function $M(a,c;e^{-x})$ is completely monotonic on $(0,\infty)$. By using the fact that a function f is strongly completely monotonic if and only if the function $xf(x)$ is completely monotonic (see [26]), we conclude that the function $M(a,c;e^{-x})/x$ is strongly completely monotonic on $(0,\infty)$. □

Remark 3.10. Let $a \geq 0$ and $c > 0$. Then the functions $M(a,c;x)$ and $M(a,c;e^x)$ are absolutely monotonic on $(0,\infty)$.

As a direct consequence of Proposition 3.9, we easily obtain the following example.

Example 3.11. The following functions

$$\begin{aligned}\frac{M(1,1;e^{-x})}{x} &= \frac{e^{e^{-x}}}{x},\\ \frac{M(1,2;e^{-x})}{x} &= \frac{e^{x}\left(e^{e^{-x}}-1\right)}{x},\\ \frac{M(1,3;e^{-x})}{x} &= \frac{2e^{x}\left(e^{x}\left(e^{e^{-x}}-1\right)-1\right)}{x},\\ \frac{M(2,3;e^{-x})}{x} &= \frac{2e^{x}\left(e^{e^{-x}}+e^{x}-e^{x+e^{-x}}\right)}{x},\end{aligned}$$

are all strongly completely monotonic on $(0,\infty)$.

Taking $c=2a$ in Proposition 3.9, we obtain the following corollary.

Corollary 3.12. *Let $a>0$. Then the function*

$$\frac{M(a,2a;e^{-x})}{x} = \frac{2^{2a-1}e^{\frac{e^{-x}}{2}}\left(e^{-x}\right)^{\frac{1}{2}-a}\Gamma\left(a+\frac{1}{2}\right)I_{a-\frac{1}{2}}\left(\frac{e^{-x}}{2}\right)}{x},$$

is strongly completely monotonic on $(0,\infty)$, where

$$I_{\nu}(x)=\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

is the modified Bessel function of the first kind with order ν.

Taking $a=1$ in Proposition 3.9, we obtain the following corollary.

Corollary 3.13. *Let $c>0$. Then the function*

$$\frac{M(1,c;e^{-x})}{x} = \frac{(c-1)e^{e^{-x}}\left(e^{-x}\right)^{1-c}\left(\Gamma(c-1)-\Gamma\left(c-1,e^{-x}\right)\right)}{x}$$

is strongly completely monotonic on $(0,\infty)$, where

$$\Gamma(\alpha,x)=\int_x^{\infty}t^{\alpha-1}e^{-t}dt$$

is the incomplete gamma function. Moreover, taking $c=3/2$, the function

$$\frac{M(1,3/2;e^{-x})}{x} = \frac{\sqrt{\pi}e^{\frac{x}{2}+e^{-x}}\operatorname{erf}\left(e^{-\frac{x}{2}}\right)}{2x}$$

is strongly completely monotonic on $(0,\infty)$, where

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^{x}e^{-t^2}dt$$

is the error function.

Using Property 2.12, we obtain the following proposition.

Proposition 3.14. *Let $a_i\ge 0$ and $c_i>0$ for $i=1,2,\cdots,m$. Then the function $\sum_{i=1}^{m}M(a_i,c_i;e^{-x})/x$ is strongly completely monotonic on $(0,\infty)$.*

Taking $c_i=2a_i$ $(i=1,2,\cdots,m)$ in Proposition 3.14, we obtain the following corollary.

Corollary 3.15. *Let $a_i>0$ for $i=1,2,\cdots,m$. Then the function*

$$\frac{\sum_{i=1}^{m}M(a_i,2a_i;e^{-x})}{x} = \sum_{i=1}^{m}\frac{2^{2a_i-1}e^{\frac{e^{-x}}{2}}\left(e^{-x}\right)^{\frac{1}{2}-a_i}\Gamma\left(a_i+\frac{1}{2}\right)I_{a_i-\frac{1}{2}}\left(\frac{e^{-x}}{2}\right)}{x},$$

is strongly completely monotonic on $(0,\infty)$.

Taking $a_i = 1$ $(i = 1, 2, \cdots, m)$ in Proposition 3.14, we obtain the following corollary.

Corollary 3.16. *Let $c_i > 0$ for $i = 1, 2, \cdots, m$. Then the function*

$$\frac{\sum_{i=1}^{m} M(1, c_i; e^{-x})}{x} = \sum_{i=1}^{m} \frac{(c_i - 1)e^{e^{-x}} \left(e^{-x}\right)^{1-c_i} \left(\Gamma(c_i - 1) - \Gamma\left(c_i - 1, e^{-x}\right)\right)}{x}$$

is strongly completely monotonic on $(0, \infty)$.

At last, we provide some strongly completely monotonic functions involving Gauss hypergeometric function.

Proposition 3.17. *Let $a, b, c > 0$. Then the function $F(a, b; c; e^{-x})/x$ is strongly completely monotonic on* $(0, \infty)$.

Proof. The proof can be completed by the fact that the function $F(a, b; c; e^{-x})$ is completely monotonic. $\square$

Remark 3.18. Let $a, b, c > 0$. Then the function $F(a, b; c; x)$ is absolutely monotonic on $(0, 1)$.

Remark 3.19. Let $a_i, b_i, c_i > 0$ for $i = 1, 2, \cdots, m$. Then the function $\sum_{i=1}^{m} F(a_i, b_i; c_i; x)$ is absolutely monotonic on $(0, 1)$.

As a direct consequence of Proposition 3.17, we easily obtain the following example.

Example 3.20. Let $a > 0$. Then the following functions

$$\begin{aligned}
\frac{F(1,1;2;e^{-x})}{x} &= -\frac{e^x \log(\sinh(x) - \cosh(x) + 1)}{x}, \\
\frac{F(1/2,1/2;3/2;e^{-x})}{x} &= \frac{e^{x/2} \sin^{-1}\left(e^{-\frac{x}{2}}\right)}{x}, \\
\frac{F(a,1;1;e^{-x})}{x} &= \frac{(\sinh(x) - \cosh(x) + 1)^{-a}}{x}, \\
\frac{F(1,1;3/2;e^{-x})}{x} &= \frac{e^{x/2} \sin^{-1}\left(e^{-\frac{x}{2}}\right)}{x\sqrt{\sinh(x) - \cosh(x) + 1}},
\end{aligned}$$

are all strongly completely monotonic on $(0, \infty)$.

Taking $a = b = 1/2, c = 1$ in Proposition 3.17, we obtain the following corollary.

Corollary 3.21. *The function*

$$\frac{F(\frac{1}{2}, \frac{1}{2}; 1; e^{-x})}{x} = \frac{2\mathcal{K}\left(e^{-x}\right)}{\pi x}$$

is strongly completely monotonic on $(0, \infty)$, *where*

$$\mathcal{K}(x) = \frac{\pi}{2} \sum_{k=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 x^{2k},$$

is the complete elliptic integrals function of the first kind.

Using Property 2.12, we obtain the following proposition.

Proposition 3.22. *Let $a_i, b_i, c_i > 0$ for $i = 1, 2, \cdots, m$. Then the function $\sum_{i=1}^{m} F(a_i, b_i; c_i; e^{-x})/x$ is strongly completely monotonic on* $(0, \infty)$.

References

[1] K. Aomoto and M. Kita, *Theory of hypergeometric functions*, Springer Monographs Math, Tokyo, 2011.

[2] F.M. Atici, D.C. Biles and A. Lebedinsky, *An application of time scales to economics*, Math. Comput. Model. **43**, 718–726, 2006.

[3] F.M. Atici and F. Uysal, *A production-inventory model of HMMS on time scales*, Appl. Math. Lett. **21**, 236–243, 2008.

[4] S. Bernstein, *Sur les fonctions absolument monotones(French)*, Acta Math. **52**, 1–66, 1929.

[5] M. Bohner, *Dynamic equations on time scales: An Introduction with Applications*, Birkhauser, 2001.

[6] M. Bohner and B. Karpuz, *The gamma function on time scales*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **20**(4), 507–522, 2013.

[7] M. Bohner and A. Peterson, *Laplace transform and Z-transform: unification and extension*, Meth. Appl. Anal. **9**(1), 151–158, 2002.

[8] R. Boucekkine and J. R. Ruiz-Tamarit, *Special functions for the study of economic dynamics: the case of the Lucas-Uzawa model*, J. Math. Econom. **44**(1), 33–54, 2008.

[9] Y.A. Brychkov and N.V. Savischenko, *Application of hypergeometric functions of two variables in wireless communication theory*, Lobachevskii. J. Math. **40**, 938–953, 2019.

[10] R. Chandramouli and N. Ranganathan, *Computing the bivariate gaussian probability integral*, IEEE Signal Proc. **6**(6), 129–131, 1999.

[11] B.-N. Guo and F. Qi, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218**(19), 9890–9897, 2012.

[12] S. Hilger, *Ein maßkettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten*, PhD thesis, 1988.

[13] S. Hilger, *Analysis on measure chains-A unified approach to continuous and discrete calculus*, Results. Math. **18**, 18–56, 1990.

[14] S. Koumandos and H.L. Pedersen, *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler's gamma function*, J. Math. Anal. Appl. **355**(1), 33–40, 2009.

[15] V. Lakshmilkantham and S. Sivasundaramn, *Stability of moving invariant sets and uncertain dynamic systems on time scales*, Comput. Math. Appl. **36**(10-12), 339–346, 1998.

[16] Z.-X. Mao and J.-F. Tian, *Delta complete monotonicity and completely monotonic degree on time scales*, Bull. Malays. Math. Sci. Soc. **46**(4), 142, 2023.

[17] Z.-X. Mao and J.-F. Tian, *Delta L'Hospital-, Laplace- and variable limit-type monotonicity rules on time scales*, Bull. Malays. Math. Sci. Soc. **47**(1), 1, 2024.

[18] Z.-X. Mao, J.-F. Tian and Y.-R. Zhu, *Psi, polygamma functions and Q-complete monotonicity on time scales*, J. Appl. Anal. Comput. **13**(3), 1137–1154, 2023.

[19] B. Martin and G.S. Guseinov, *The convolution on time scales*, Abstr. Appl. Anal. **2007**(3–4), 2007.

[20] N. Michel and M.V. Stoitsov, *Fast computation of the Gauss hypergeometric function with all its parameters complex with application to the Poschl-Teller-Ginocchio potential wave functions*, Comput. Physics. Commun. **178**(7), 535–551, 2008.

[21] Q. Sheng, M. Fadag, J. Henderson and J.M. Davis, *An exploration of combined dynamic derivatives on time scales and their applications*, Nonlinear Anal. Real World Appl. **7**(3), 395–413, 2006.

[22] X. Tian and Y. Zhang, *Fractional time-scales Noether theorem with Caputo Δ-derivatives for Hamiltonian systems*, Appl. Math. Comput. **393**, 125753, 2021.

[23] S.Y. Trimble, J. Wells and F.T. Wright, *Superadditive functions and a statistical application*, SIAM J. Math. Anal. **20**(5), 1255–1259, 1989.

[24] D.V. Widder, *The Laplace transform*, Princeton Mathematical Series, vol. 6. Princeton University Press, Princeton, 1941.

[25] J.-F. Xu, B. Pervaiz, A. Zada, et al., *Stability analysis of causal integral evolution impulsive systems on time scales*, Acta. Math. Sci. **41**, 781–800, 2021.

[26] L. Zhu, *A class of strongly completely monotonic functions related to gamma function*, J. Comput. Appl. Math. **367**, 112–469, 2020.