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Research Article

Solitons of mean curvature flow in certain warped products: nonexistence, rigidity, and Moser-Bernstein type results

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ABSTRACT. We apply suitable maximum principles to obtain nonexistence and rigidity results for complete mean curvature flow solitons in certain warped product spaces. We also provide applications to self-shrinkers in Euclidean space, as well as to mean curvature flow solitons in real projective, pseudo-hyperbolic, Schwarzschild, and Reissner-Nordström spaces. Furthermore, we establish new Moser-Bernstein type results for entire graphs constructed over the fiber of the ambient space that are mean curvature flow solitons.

Keywords: Warped products, Euclidean space, real projective space, pseudo-hyperbolic spaces, Schwarzschild and Reissner-Nordström spaces, mean curvature flow solitons, self-shrinkers.

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1. INTRODUCTION

Let $\psi : \Sigma^n \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the position vector ψ evolves in the direction of the mean curvature vector $\vec{H}$, then it gives rise to a solution to mean curvature flow:

$$\Psi : [0, T) \times \Sigma^n \to \mathbb{R}^{n+1}$$

satisfying $\Psi(0, \cdot) = \psi(\cdot)$ and

$$\frac{\partial \Psi}{\partial t}(t, p) = \vec{H}(t, p),$$

where $\vec{H}(t, p)$ stands for the (non-normalized) mean curvature vector of the hypersurface $\Sigma_t^n = \Psi(t, \Sigma^n)$ at a point $\Psi(t, p)$. This equation is called the *mean curvature flow equation*. The study of the mean curvature flow from the perspective of partial differential equations was started with Huisken [24] on the flow of convex hypersurfaces. One of the most important problems in the mean curvature flow is to understand the possible singularities that the flow goes through. A key starting point for singularity analysis is Huisken's monotonicity formula [24] because the monotonicity implies that the flow is asymptotically self-similar near a given type I singularity and thus, is modeled by self-shrinking solutions of the flow.

An n-dimensional two-sided hypersurface $\psi : \Sigma^n \to \mathbb{R}^{n+1}$ is called a *self-shrinker* if it satisfies

$$H = -\langle \psi, N \rangle,$$

where H and N denote the (non-normalized) mean curvature function and the unit normal vector field of the hypersurface, respectively. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a

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given singularity of the mean curvature flow and, as it was pointed out by Colding and Minicozzi in [14], self-shrinkers are critical hypersurfaces for the entropy functional. The subject experienced an increasing activity after the seminal paper by Colding and Minicozzi [14] that inspired an impressive amount of work on existence and classification problems, rigidity and gap results, stability and spectral properties, see for instance [7, 8, 10, 11, 12, 13, 15, 18, 19, 23, 26, 27, 31] and the references therein.

More recently, Alías, de Lira and Rigoli [4] extended these investigations introducing the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a vector field K and establishing the corresponding notion of mean curvature soliton. In particular, when $\overline{M}^{n+1}$ is a warped product of the type $I \times_f M^n$ and $K = f(t)\partial_t$, they applied weak maximum principles to guarantee that a complete n-dimensional mean curvature flow soliton is a slice of $\overline{M}^{n+1}$. In [16], Colombo, Mari and Rigoli also studied some properties of mean curvature flow solitons in general Riemannian manifolds and in warped products, with emphasis on constant curvature and Schwarzschild type spaces. They focused on splitting and rigidity results under various geometric conditions, ranging from the stability of the soliton to the fact that the image of its Gauss map be contained in suitable regions of the sphere. Moreover, they also investigated the case of entire mean curvature flow graphs.

Proceeding with this picture, our purpose in this paper is to apply suitable maximum principles in order to obtain nonexistence and rigidity results concerning complete n-dimensional mean curvature flow solitons with respect to the conformal vector field $K = f(t)\partial_t$ of a warped product space of the type $I \times_f M^n$ (see Sections 3 and 4). Applications to self-shrinkers in the Euclidean space, as well as to mean curvature flow solitons in the real projective, pseudo-hyperbolic, Schwarzschild and Reissner-Nordström spaces are also given. Furthermore, we study entire graphs constructed over the fiber M^n and which are mean curvature flow solitons with respect to K, obtaining new Moser-Bernstein type results (see Section 5).

2. Preliminaries

2.1. **Two-sided hypersurfaces in a warped product.** Let (M^n, g_M) be an n-dimensional $(n \geq 2)$ connected Riemannian manifold and let $I \subset \mathbb{R}$ be an open interval in $\mathbb{R}$ endowed with the metric dt^2. The product manifold $\overline{M}^{n+1} = I \times M^n$ endowed with the Riemannian metric

$$\overline{g} = \pi_I^*(dt^2) + f(\pi_I)^2 \pi_M^*(g_M), \tag{2.1}$$

where f is a positive smooth function on I, the maps π_I and π_M denote the projections onto I and M^n, respectively, is called a warped product with fiber M^n, base I and warping function f. Along this work, we will simply write $\overline{M}^{n+1} = I \times_f M^n$.

In this setting, we will consider the conformal closed vector field $K = f(t)\partial_t$ globally defined on $\overline{M}$, where $\partial_t = \frac{\partial}{\partial t}$ stands for the unit coordinate vector field tangent to I. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber (see [30, Proposition 7.35]) , it follows that

$$\overline{\nabla}_X K = f'(t) X \tag{2.2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$.

Along this work, we will deal with connected two-sided hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in $\overline{M}^{n+1} = I \times_f M^n$, which means that its normal bundle is trivial, that is, there is on it a globally defined unit normal vector field $N \in T\Sigma^{\perp}$. In this setting, we will denote by g the induced metric of Σ^n and we will consider its shape operator (or Weingarten endomorphism),

$A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which is given by $A(X) = -\overline{\nabla}_X N$. So, the (non-normalized) mean curvature function of Σ^n is defined as been $H = \mathrm{tr}(A)$.

In the warped product $\overline{M}^{n+1} = I \times_f M^n$ there exists a remarkable family of two-sided hypersurfaces: its slices $M_{t_*} = \{t_*\} \times M$, with $t_* \in I$. The shape operator and the mean curvature of M_{t_*} with respect to $N = \partial_t$ are, respectively, $A_{t_*} = -\dfrac{f'(t_*)}{f(t_*)} I$, where I denotes the identity operator, and $H_{t_*} = -n\dfrac{f'(t_*)}{f(t_*)}$.

We will deal with two particular functions naturally attached to a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, the (vertical) height function $h = \pi_I \circ \psi$ and the angle function $\Theta = \overline{g}(N, \partial_t)$. Let us denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics $\overline{g}$ and g, respectively. With a straightforward computation we show that the gradient of π_I on M^n is given by

$$\overline{\nabla}\pi_I = \overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = \partial_t$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_I)^\top = \partial_t^\top, \tag{2.3}$$

where $\partial_t^\top = \partial_t - \Theta N$ is the tangential component of ∂_t along Σ^n. From (2.3) we deduce that

$$|\nabla h|^2 + \Theta^2 = 1, \tag{2.4}$$

where ∇h is the gradient of h in the metric g and $|X|^2 = g(X, X)$ for any $X \in \mathfrak{X}(\Sigma)$. Moreover, from (2.2) and (2.3) we deduce that the Hessian of h in the metric g is given by

$$\begin{aligned} \nabla^2 h(X, X) &= g(\nabla_X \partial_t^\top, X) \\ &= \overline{g}(\overline{\nabla}_X(\partial_t - \Theta N), X) \\ &= \frac{f'(h)}{f(h)}(|X|^2 - g(\nabla h, X)^2) + g(AX, X)\Theta \end{aligned} \tag{2.5}$$

for any $X \in \mathfrak{X}(\Sigma)$. Hence, from (2.5) we obtain that the Laplacian of h in the metric g is

$$\Delta h = \frac{f'(h)}{f(h)}\left(n - |\nabla h|^2\right) + H\Theta. \tag{2.6}$$

2.2. **Mean curvature flow solitons.** We recall that the mean curvature flow $\Psi : [0, T) \times \Sigma^n \to \overline{M}^{n+1}$ of an immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ in a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$, satisfying $\Psi(0, \cdot) = \psi(\cdot)$, looks for solutions of the equation

$$\frac{\partial \Psi}{\partial t} = \vec{H},$$

where $\vec{H}(t, \cdot)$ is the (non-normalized) mean curvature vector of $\Sigma_t^n = \Psi(t, \Sigma^n)$. In our context, according to [4, Definition (1.1)], a two-sided hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a warped product $\overline{M}^{n+1} = I \times_f M^n$ is said a *mean curvature flow soliton* with respect to $K = f(t)\partial_t$ with *soliton constant* $c \in \mathbb{R}$ if its (non-normalized) mean curvature function satisfies

$$H = cf(h)\Theta. \tag{2.7}$$

Adopting the terminology introduced in [4], we will also consider the *soliton function*

$$\zeta_c(t) = nf'(t) + cf(t)^2.$$

As it was observed in [4], a slice $M_{t_*} = \{t_*\} \times M^n$ is a mean curvature flow soliton with respect to $K = f(t)\partial_t$ and with soliton constant c given by

$$c = -n\frac{f'(t_*)}{f(t_*)^2}. \tag{2.8}$$

Moreover, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$.

2.3. **Standard examples.** In this subsection we quote important examples which will be addressed along the next two sections. In the first one, we consider a suitable warped product model for the Euclidean space minus a point.

Example 2.1. *Let $o = (0, \dots, 0)$ be the origin of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We have that $\mathbb{R}^{n+1} \setminus \{o\}$ is isometric to $\mathbb{R}_+ \times_t \mathbb{S}^n$ (see* [28, Section 4, Example 1]*), whose slices $\{t\} \times \mathbb{S}^n$ are isometric to n-dimensional Euclidean spheres $\mathbb{S}^n(t)$ of radius $t \in \mathbb{R}_+$. In this setting, the mean curvature flow solitons with respect to $K = t\partial_t$ with soliton constant $c = -1$ are just the self-shrinkers. So, from* (2.8) *we conclude that $\mathbb{S}^n(\sqrt{n}) \equiv \{\sqrt{n}\} \times \mathbb{S}^n$ is the only slice which is a self-shrinker.*

In our next example, we consider a suitable warped product model for the real projective space.

Example 2.2. *We recall that the $(n+1)$-dimensional real projective space is given by the quotient $\mathbb{RP}^{n+1} = \mathbb{S}^{n+1}/\{\pm 1\}$, where $\{\pm 1\}$ is the group of diffeomorphisms of $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$ consisting of the identity map $q \mapsto q$ and the antipodal map $q \mapsto -q$. We consider the Riemannian metric in $\mathbb{RP}^{n+1}$ in such a way that the natural projection $\pi : \mathbb{S}^{n+1} \to \mathbb{RP}^{n+1}$ becomes a local isometry. If P stands for the north pole of $\mathbb{S}^{n+1}$, then we denote by Cut_P the cut locus of $\pi(P) \in \mathbb{RP}^{n+1}$. We have that Cut_P is the image of the equator of $\mathbb{S}^{n+1}$ orthogonal to P via the natural projection, namely, $Cut_P = \pi(\mathbb{S}^n) = \mathbb{RP}^n$. Moreover, as it was proved in* [6, Section 9.111]*, $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$ is isometric to the warped product $\left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$. From* (2.8) *we conclude that the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$ is the only one that is a mean curvature flow soliton with respect to $K = \sin t\partial_t$ with soliton constant $c < 0$.*

Proceeding, we consider the so-called pseudo-hyperbolic spaces.

Example 2.3. *According to* [32]*, warped products of the type $I \times_{e^t} M^n$ are called* pseudo-hyperbolic spaces*. This terminology is due to the fact that the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is isometric to the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^n$, where the slices constitute a family of horospheres sharing a same fixed point in the asymptotic boundary $\partial_\infty \mathbb{H}^{n+1}$ and giving a complete foliation of $\mathbb{H}^{n+1}$ (for more details about pseudo-hyperbolic spaces see, for instance,* [2, 28, 32]*). From* (2.8) *we conclude that the slice $\{\log(-\frac{n}{c})\} \times M^n$ is the only one that is a mean curvature flow soliton with respect to $K = e^t\partial_t$ with soliton constant $c < 0$.*

In our last examples, we deal with the Schwarzschild and Reissner-Nordström spaces.

Example 2.4. *Given a mass parameter $\mathfrak{m} > 0$, the* Schwarzschild space *is defined to be the product $\overline{M}^{n+1} = (r_0(\mathfrak{m}), +\infty) \times \mathbb{S}^n$ furnished with the metric $\bar{g} = V_{\mathfrak{m}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric of $\mathbb{S}^n$, $V_{\mathfrak{m}}(r) = 1 - 2\mathfrak{m}r^{1-n}$ stands for its potential function and $r_0(\mathfrak{m}) = (2\mathfrak{m})^{1/(n-1)}$ is the unique positive root of $V_{\mathfrak{m}}(r) = 0$. Its importance lies in the fact that the manifold $\mathbb{R} \times \overline{M}^{n+1}$ equipped with the Lorentzian static metric $-V_{\mathfrak{m}}(r)dt^2 + \bar{g}$ is a solution of the Einstein field equation in vacuum with zero cosmological constant (see, for instance,* [30, Chapter 13] *for more details concerning Schwarzschild geometry).*

As it was observed in [16, Example 1.3], $\overline{M}^{n+1}$ *can be reduced in the form* $I \times_f \mathbb{S}^n$ *with metric* (2.1) *via the following change of variables:*

$$t = \int_{r_0(\mathfrak{m})}^{r} \frac{d\sigma}{\sqrt{V_{\mathfrak{m}}(\sigma)}}, \qquad f(t) = r(t), \qquad I = \mathbb{R}_+. \tag{2.9}$$

As it was noted in [16, Example 4.1], *since* $V_{\mathfrak{m}}(r)$ *is strictly increasing on* $(r_0(\mathfrak{m}), +\infty)$*, it follows from* (2.9) *that the warping function* f *satisfies:*

$$f'(t) = \frac{dr}{dt} = \sqrt{V_{\mathfrak{m}}(r(t))} > 0 \qquad \text{and} \qquad f''(t) = \frac{1}{2}\frac{dV_{\mathfrak{m}}}{dr}(r(t)) > 0. \tag{2.10}$$

Thus, from (2.8) *and* (2.10) *we can verify that a slice* $\{t_*\} \times \mathbb{S}^n$ *is a mean curvature flow soliton with respect to* $f(t)\partial_t = r\sqrt{V_{\mathfrak{m}}(r)}\partial_r$ *with soliton constant* $c < 0$ *when* $t_* = t(r_*)$ *with* $r_* > r_0(\mathfrak{m})$ *solving the following equation*

$$V_{\mathfrak{m}}(r) = \frac{c^2}{n^2} r^4. \tag{2.11}$$

We note that such a solution exists if and only if the function $\varphi_{\mathfrak{m}}(t) = \frac{c^2}{n^2}t^4 + \frac{2\mathfrak{m}}{t^{n-1}} - 1$ *has a zero on* $(r_0(\mathfrak{m}), +\infty)$*. Notice that* $\varphi_{\mathfrak{m}}$ *is a convex function which goes to infinity if* t *goes to* 0 *or* $+\infty$ *and so* $\varphi_{\mathfrak{m}}$ *has a unique minimal point in* $(0, \infty)$*. Such value* $\hat{r}$ *is given implicitly by* $\varphi'_{\mathfrak{m}}(\hat{r}) = 0$*, that is,*

$$\frac{4c^2}{n^2}\hat{r}^3 - \frac{2\mathfrak{m}(n-1)}{\hat{r}^n} = 0.$$

Therefore, the equation (2.11) *has a solution if and only if* $\hat{r} > r_0(\mathfrak{m})$ *and* $\varphi_{\mathfrak{m}}(\hat{r}) \le 0$*. The last condition can be rewritten in the following way:*

$$\hat{r} = \left(\frac{\mathfrak{m}(n-1)n^2}{2c^2}\right)^{1/(n+3)} \ge \left(\frac{\mathfrak{m}(n+3)}{2}\right)^{1/(n-1)}. \tag{2.12}$$

In particular, there are two solutions $r_0(\mathfrak{m}) < r_{*,-} < \hat{r} < r_{*,+}$ *if the strict inequality holds in* (2.12)*, and a unique solution* $r_* = \hat{r}$ *if equality holds.*

Example 2.5. *Given a mass parameter* $\mathfrak{m} > 0$ *and an electric charge* $\mathfrak{q} \in \mathbb{R}$*, with* $|\mathfrak{q}| \le \mathfrak{m}$*, the* Reissner-Nordström space *is defined to be the product* $\overline{M}^{n+1} = (r_0(\mathfrak{m}, \mathfrak{q}), +\infty) \times \mathbb{S}^n$ *endowed with the metric* $\bar{g} = V_{\mathfrak{m},\mathfrak{q}}(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^n}$*, where* $g_{\mathbb{S}^n}$ *is the standard metric of* $\mathbb{S}^n$*,* $V_{\mathfrak{m},\mathfrak{q}}(r) = 1 - 2\mathfrak{m}r^{1-n} + \mathfrak{q}^2 r^{2-2n}$ *stands for its potential function and* $r_0(\mathfrak{m}, \mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$ *is the largest positive zero of* $V_{\mathfrak{m},\mathfrak{q}}(r)$*. The importance of this model lies in the fact that the manifold* $\mathbb{R} \times \overline{M}^{n+1}$ *equipped with the Lorentzian static metric* $-V_{\mathfrak{m},\mathfrak{q}}(r)dt^2 + \bar{g}$ *is a charged black-hole solution of the Einstein field equation in vacuum with zero cosmological constant.*

As in the previous example, $\overline{M}^{n+1}$ *can be reduced in the form* $I \times_f \mathbb{S}^n$ *with metric* (2.1) *via the same change of variables as in* (2.9)*. Furthermore, following the same previous steps, the warping function* f *has positive first and second derivatives. Moreover, we can verify that a slice* $\{t_*\} \times \mathbb{S}^n$ *is a mean curvature flow soliton with respect to* $f(t)\partial_t = r\sqrt{V_{\mathfrak{m},\mathfrak{q}}(r)}\partial_r$ *with soliton constant* $c < 0$ *when* $t_* = t(r_*)$ *with* $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ *solving the following equation*

$$V_{\mathfrak{m},\mathfrak{q}}(r) = \frac{c^2}{n^2} r^4. \tag{2.13}$$

We observe that such a case is more complicated to make all values explicit, but qualitatively we can say that such a solution of (2.13) *exists if and only if the function* $\varphi_{\mathfrak{m},\mathfrak{q}}(x) = \frac{c^2}{n^2}x^4 + \frac{2\mathfrak{m}}{x^{n-1}} - \frac{\mathfrak{q}^2}{x^{2n-2}} - 1$ *has a zero on* $(r_0(\mathfrak{m}), +\infty)$*. Note that* $\varphi_{\mathfrak{m},\mathfrak{q}}$ *goes to positive infinity if* x *goes to positive infinity and* $\varphi_{\mathfrak{m},\mathfrak{q}}$

goes to negative infinity if x goes to zero. So, $\varphi_{\mathfrak{m},\mathfrak{q}}$ has at least one root in $(0,+\infty)$ and if such roots are greater than $r_0(\mathfrak{m},\mathfrak{q})$ we get the desired solutions r_*.

3. Nonexistence of complete mean curvature flow solitons

3.1. **Auxiliary results.** In order to investigate the nonexistence of complete mean curvature flow solitons, initially we introduce the following definition:

Definition 3.1. *The Laplacian operator Δ on a Riemannian manifold (Σ, g) satisfies the Omori-Yau maximum principle if for any $u \in C^2$ bounded from above, there exists a sequence $(p_k)_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k) = \sup_\Sigma u = u^*, \qquad \lim_k |\nabla u(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta u(p_k) \leq 0.$$

Now we recall the maximum principle due to Omori [29] and Yau [34]. Such concept gives us conditions to the validity of a maximum principle for the hessian or the Laplacian on a Riemannian manifold. Specifically, we quote the following result for the Laplacian:

Lemma 3.1 (Yau, [34]). *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Then the Laplacian Δ satisfies the Omori-Yau maximum principle on Σ.*

Denoting by K_M the sectional curvature of the fiber M^n, we will consider warped product spaces $I \times_f M^n$ satisfying the convergence condition

$$K_M \geq \sup_I (f'^2 - ff''). \tag{3.14}$$

Warped products satisfying (3.14) have been studying, for instance, in [4, 5, 17, 21]. The case that this condition holds for the Ricci curvature instead of the sectional curvature is also well known (see, for instance, [1, 3, 28]). Furthermore, it is not difficult to verify that there exists a wide class of warped product satisfying (3.14), including, for instance, the Euclidean space minus a point $\mathbb{R}^{n+1} \setminus \{o\} = \mathbb{R}_+ \times_t \mathbb{S}^n$, the real projective space (minus a suitable point and its cut locus) $\left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$, the pseudo-hyperbolic spaces $I \times_{e^t} M^n$ with fiber having nonnegative sectional curvature and the Schwarzschild and Reissner-Nordström spaces $I \times_f \mathbb{S}^n$ (see Examples 2.1, 2.2, 2.3, 2.4 and 2.5).

Indeed, this verification for the Euclidean, the real projective, the pseudo-hyperbolic and the Schwarzschild spaces is quite simple. In the case of the Reissner-Nordström space, with a straightforward computation we get that

$$f'(t)^2 - f(t)f''(t) = 1 - \mathfrak{m} r(t)^{1-n} - n\left\{\mathfrak{m} - \mathfrak{q}^2 r(t)^{1-n}\right\} r(t)^{1-n}. \tag{3.15}$$

But, since $r(t) > r_0(\mathfrak{m},\mathfrak{q}) = \left(\frac{\mathfrak{q}^2}{\mathfrak{m} - \sqrt{\mathfrak{m}^2 - \mathfrak{q}^2}}\right)^{1/(n-1)}$, it is not difficult to verify that we must have

$$\mathfrak{q}^2 r(t)^{1-n} < \mathfrak{m}. \tag{3.16}$$

Consequently, from (3.15) and (3.16) we conclude that the convergence condition (3.14) is also satisfied in the Reissner-Nordström space.

We recall that a hypersurface Σ^n lies in a *slab* of a warped product $I \times_f M^n$ when Σ^n is contained in a region of the type

$$[t_1, t_2] \times M^n = \{(t, p) \in I \times_f M^n : t_1 \leq t \leq t_2 \text{ and } p \in M^n\}.$$

We also recall the first and second Newton transformations, which are given by $P_1 = HI - A$ and $P_2 = S_2 I - AP_1$, and here S_2 stands for the second mean curvature, that is, $S_2 = \sum_{i<j} k_i k_j$,

where k_i are the principal curvatures of Σ^n. Finally, we say that an operator T on Σ is *f-bounded* whether there are continuous functions $G, H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$G(f, f') \circ h(p)|u|^2 \leq \langle Tu, u\rangle \leq H(f, f') \circ h(p)|u|^2$$

for all $u \in T_p\Sigma$ and $p \in \Sigma$.

Next, considering an immersed hypersurface Σ^n in a slab of a warped product space $I \times_f M^n$ satisfying (3.14), we will verify that the Omori-Yau maximum principle is satisfied.

Proposition 3.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product which satisfying the convergence condition (3.14), for $n \geq 3$, and, let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete hypersurface with second Newton transformation f-bounded and lying in a slab. Then, the Laplacian on Σ^n satisfies the Omori-Yau maximum principle.*

Proof. First, we recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $I \times_f M^n$ by the so-called Gauss' equation given by

$$g(R(X, Y)Z, W) = \bar{g}(\overline{R}(X, Y)Z, W) + g(A(X, Z), A(Y, W)) - g(A(X, W), A(Y, Z)) \tag{3.17}$$

for every tangent vector fields $X, Y, Z, W \in \mathfrak{X}(\Sigma)$.

Let us consider $X \in \mathfrak{X}(\Sigma)$ and take a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss equation (3.17) that the Ricci curvature Ric of Σ^n with respect to the induced metric g is given by

$$\begin{aligned} \mathrm{Ric}(X, X) &= \sum_i \bar{g}(\overline{R}(X, E_i)X, E_i) + H\langle AX, X\rangle - |AX|^2 \\ &= \sum_i \bar{g}(\overline{R}(X, E_i)X, E_i) - \langle (AP_1)X, X\rangle \\ &= \sum_i \bar{g}(\overline{R}(X, E_i)X, E_i) + S_2|X|^2 - \langle P_2X, X\rangle. \end{aligned} \tag{3.18}$$

Moreover, with a straightforward computation, we get

$$\begin{aligned} \overline{R}(X, E_i)X &= \overline{R}(X^*, E_i^*)X^* + \bar{g}(X, \partial_t)\overline{R}(X^*, E_i^*)\partial_t + \bar{g}(X, \partial_t)\bar{g}(E_i, \partial_t)\overline{R}(X^*, \partial_t)\partial_t \\ &\quad + \bar{g}(E_i, \partial_t)\overline{R}(X^*, \partial_t)X^* + \bar{g}(X, \partial_t)\overline{R}(\partial_t, E_i^*)X^* + \bar{g}(X, \partial_t)^2\overline{R}(\partial_t, E_i^*)\partial_t, \end{aligned} \tag{3.19}$$

where $X^* = X - \bar{g}(X, \partial_t)\partial_t$ and $E_i^* = E_i - \bar{g}(E_i, \partial_t)\partial_t$ are the projections of the tangent vector fields X and E_i onto the fiber M^n, respectively.

Thus, by repeated use of the formulas of [30, Proposition 7.42] and using equation (2.3), from (3.19) we get

$$\begin{aligned} &\sum_i \bar{g}(\overline{R}(X, E_i)X, E_i) \\ &= \sum_i \bar{g}(R_M(X^*, E_i^*)X^*, E_i^*) - (n-1)\frac{f'(h)^2}{f(h)^2}|X|^2 \\ &+ \left(\frac{f'(h)^2 - f(h)f''(h)}{f(h)^2}\right)|\nabla h|^2|X|^2 + (n-2)\left(\frac{f'(h)^2 - f(h)f''(h)}{f(h)^2}\right)g(X, \nabla h)^2, \end{aligned} \tag{3.20}$$

As in [30], the curvature tensor R of the hypersurface Σ^n is given by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, where $[\,]$ denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

where R_M denotes the curvature tensor of the fiber M^n. But, it is not difficult to verify that

$$\sum_i \bar{g}(R_M(X^*, E_i^*)X^*, E_i^*) = \frac{1}{f^2}\sum_i K_M(X^*, E_i^*)(|X|^2 - g(\nabla h, E_i)^2|X|^2 \\ -g(X,\nabla h)^2 - g(X,E_i)^2 + 2g(X,\nabla h)g(X,E_i)g(\nabla h, E_i)).$$

Thus, by using the convergence condition (3.14) and a direct computation, from (3.20) we obtain

$$\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) \geq -(n-1)\frac{f''(h)}{f(h)}|X|^2. \tag{3.21}$$

Thus, inserting the estimate (3.21) into the equation (3.18), and using the f-boundedness of P_2, we deduce that

$$\mathrm{Ric}(X,X) \geq \left(-(n-1)\frac{f''(h)}{f(h)} + \frac{n}{n-2}G(f,f') - H(f,f')\right)|X|^2. \tag{3.22}$$

Therefore, taking into account that Σ^n lies in a slab of the ambient space, from (3.22) we conclude that the Ricci curvature is bounded from below and by Lemma 3.1 the Laplacian satisfies the desired property. □

Corollary 3.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product which satisfying the convergence condition (3.14), and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$. If the second mean curvature is bounded from below and Σ lies in a slab, then the Laplacian on Σ^n satisfies the Omori-Yau maximum principle.*

Proof. Since the second mean curvature is bounded from below and $\psi(\Sigma)$ lies in a slab, notice that $k_i^2 \leq H^2 - 2S_2 \leq c^2 f(h)^2 + d$ is bounded on Σ for all i. So, P_2 is bounded and the result follows from Proposition 3.1. For $n = 2$, this result is immediate. □

3.2. **Nonexistence results via Omori-Yau maximum principle.** Into the scope of a warped product $I \times_f M^n$ we are in position to state and prove our first nonexistence result concerning mean curvature flow solitons immersed in a slab of a warped product.

Theorem 3.1. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n satisfies hypothesis (3.14). There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$, with second mean curvature bounded from below, lying in a slab $[t_1, t_2] \times M^n$ and $\zeta_c(t)$ having a strict sign on $[t_1, t_2]$.*

Proof. Let us suppose by contradiction the existence of such a mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$. From (2.6) we have

$$\begin{aligned} \Delta h &= n\frac{f'(h)}{f(h)} - \frac{f'(h)}{f(h)}|\nabla h|^2 + cf(h)\Theta^2 \\ &= n\frac{f'(h)}{f(h)}\Theta^2 + (n-1)\frac{f'(h)}{f(h)}|\nabla h|^2 + cf\Theta^2 \\ &= (n-1)\frac{f'(h)}{f(h)}|\nabla h|^2 + \frac{nf'(h) + cf^2(h)}{f}\Theta^2 \\ &= (n-1)\frac{f'(h)}{f(h)}|\nabla h|^2 + \frac{\zeta_c(h)}{f(h)}\Theta^2, \end{aligned} \tag{3.23}$$

where we used (2.4) in the second equality. Since the second mean curvature is bounded and the hypersurface is contained in a slab, from Corollary 3.1 we are able to apply the Omori-Yau maximum principle. Indeed, there are sequences $\{x_k\}$ and $\{p_k\}$ such that

$$\lim_k h(p_k) = \sup_\Sigma h = h^*, \qquad \lim_k |\nabla h(p_k)| = 0 \qquad \text{and} \qquad \limsup_k \Delta h(p_k) \leq 0,$$

and

$$\lim_k h(x_k) = \inf_\Sigma h = h_*, \qquad \lim_k |\nabla h(x_k)| = 0 \qquad \text{and} \qquad \liminf_k \Delta h(x_k) \geq 0,$$

and thus, using that Θ goes to 1 along the sequences $\{p_k\}$ and $\{x_k\}$, we deduce from equation (3.23) that

$$\zeta_c(h^*) \leq 0 \leq \zeta_c(h_*),$$

which contradict our hypothesis on the function ζ_c.

□

Remark 3.1. *It is worth to point out that complete mean curvature flow solitons immersed in a slab of a warped product $I \times_f M^n$ and with second mean curvature bounded from below constitute natural generalizations of the compact ones, and they have already been studied by Alías, de Lira and Rigoli in* [4].

Taking into account Example 2.1, it is not difficult to verify that we get from the proof of Theorem 3.1 the following result concerning the nonexistence of complete self-shrinkers:

Corollary 3.2. *There exists no complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ with second mean curvature bounded from below and lying in the closure of an n-dimensional annulus with either inner radius $r_{ir} > \sqrt{n}$ or outer radius $r_{or} < \sqrt{n}$.*

Remark 3.2. *We point out that the sphere of radius $\sqrt{n}$ satisfies all the hypotheses if we allow the inner radius r_{ir} (or outer radius r_{or}) equal to $\sqrt{n}$. We also notice that the self-shrinkers $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, for $1 \leq k \leq n-1$, of $\mathbb{R}^{n+1}$ have bounded second mean curvature but they do not belong to any n-dimensional annuli.*

Considering the discussion made in Example 2.2, from Theorem 3.1 we have:

Corollary 3.3. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = \sin t \partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times M^n$, with either $\cos^{-1}\left(\frac{\sqrt{4c^2+n^2}-n}{2|c|}\right) < t_1 < \frac{\pi}{2}$ or $0 < t_2 < \cos^{-1}\left(\frac{\sqrt{4c^2+n^2}-n}{2|c|}\right)$.*

When the ambient space is a pseudo-hyperbolic space (see Example 2.3), from Theorem 3.1 we also obtain the following consequence:

Corollary 3.4. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n has nonnegative sectional curvature. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = e^t \partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times M^n$, with either $t_1 > \log\left(-\frac{n}{c}\right)$ or $t_2 < \log\left(-\frac{n}{c}\right)$.*

Considering the context of Example 2.4, from Theorem 3.1 we get:

Corollary 3.5. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space. There is no complete mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $K = f(t)\partial_t$ with soliton constant $c < 0$, having second mean curvature bounded from below and lying in a slab $[t_1, t_2] \times \mathbb{S}^n$, with $f(t_2) \geq \sqrt{-\frac{n}{c}}$.*

Proof. Using (2.10) and definition of ζ_c we have
$$n\sqrt{V_{\mathfrak{m}}(r(t_1))}+cr(t_1)^2 \le \zeta_c(t) \le n\sqrt{V_{\mathfrak{m}}(r(t_2))}+cr(t_2)^2.$$
Since $V_{\mathfrak{m}}(r(t))<1$ for all $t\in I$, $r(t_2)=f(t_2)\ge\sqrt{-\frac{n}{c}}$ implies
$$\zeta_c(t)=n\sqrt{V_{\mathfrak{m}}(r(t))}+cr(t)^2<0$$
for all $t\ge t_1$. Therefore, we can apply Theorem 3.1 to conclude our result. □

In the setting of Example 2.5, we can reason as in the proof of Corollary 3.5 to obtain the following nonexistence result:

Corollary 3.6. *Let $\overline{M}^{n+1}=I\times_f\mathbb{S}^n$ be the Reissner-Nordström space. There is no complete mean curvature flow soliton $\psi:\Sigma^n\to\overline{M}^{n+1}$ with respect to $K=f(t)\partial_t$ with soliton constant $c<0$, having second mean curvature bounded from below and lying in a slab $[t_1,t_2]\times\mathbb{S}^n$, with $V_{\mathfrak{m},\mathfrak{q}}(r(t))<\frac{c^2}{n^2}r(t)^4$ for all $t\in[t_1,t_2]$.*

Remark 3.3. *In Corollaries 3.3, 3.4, 3.5 and 3.6, if we assume $c>0$ the condition ζ_c positive is immediate and so the nonexistence results follows directly.*

4. Rigidity of mean curvature flow solitons

4.1. **Rigidity results via an extension of Hopf's maximum principle.** We initiate this section regarding an extension of Hopf's theorem on a complete Riemannian manifold (Σ^n,g) due to Yau in [35]. For this, let us consider $\mathcal{L}^1_g(\Sigma):=\{u:\Sigma^n\to\mathbb{R}:\int_\Sigma|u|d\Sigma<+\infty\}$, where $d\Sigma$ is the measure related to the metric g.

Lemma 4.2. *Let u be a smooth function defined on a complete Riemannian manifold (Σ^n,g), such that Δu does not change sign on Σ^n. If $|\nabla u|\in\mathcal{L}^1_g(\Sigma)$, then Δu vanishes identically on Σ^n.*

Using the previous lemma, we have the following result:

Theorem 4.2. *Let $\overline{M}^{n+1}=I\times_f M^n$ be a warped product. Let $\psi:\Sigma^n\to\overline{M}$ be a complete mean curvature flow soliton with respect to $K=f(t)\partial_t$ and soliton constant $c\neq0$, that lies in a slab $[t_1,t_2]\times M^n$, and whose $\zeta_c(t)$ does not change the sign. If $|\nabla h|\in\mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice M_{t_*}, for $t_*\in[t_1,t_2]$ given implicitly by $\zeta_c(t_*)=0$.*

Proof. Considering $F(t)=\int_{t_0}^t f(v)^{1-n}dv$ and compute the Laplacian of $F(h)$ as follows:
$$\begin{aligned}\Delta F(h)&=F'(h)\Delta h+F''(h)|\nabla h|^2\\&=\frac{1}{f(h)^{n-1}}\Delta h+(1-n)f(h)^{-n}f'(h)|\nabla h|^2\\&=\frac{\zeta_c(h)}{f(h)^n}\Theta^2+(n-1)\frac{f'(h)}{f(h)^n}|\nabla h|^2+(1-n)f(h)^{-n}f'(h)|\nabla h|^2\\&=f(h)^{-n}\zeta_c(h)\Theta^2,\end{aligned}$$
where we used equation (2.6) in the third equality. Thus $F(h)$ is either subharmonic or superharmonic. Since Σ is contained is a slab and $|\nabla h|\in\mathcal{L}^1(\Sigma)$, we have that $|\nabla F(h)|=f(h)^{1-n}|\nabla h|$ belongs to the 1-Lebesgue space too.

Applying Lemma 4.2, we deduce that $\Delta F(h)=0$ and thus $\zeta_c(h)\Theta^2=0$ along Σ. Next, note that
$$\Delta F(h)^2=2F(h)\Delta F(h)+2|\nabla F(h)|^2=2f(h)^{2-2n}|\nabla h|^2\ge0.$$

Applying Lemma 4.2 again, we deduce that $\nabla h = 0$ on Σ and from (2.4) we have $\Theta = 1$. Thus, $\zeta_c(h)$ vanishes on Σ, as we claimed.

□

From Theorem 4.2 we get the following rigidity result:

Corollary 4.7. *The only complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ that lies in the closure of an n-dimensional annulus with either inner radius $r_{ir} \geq \sqrt{n}$ or outer radius $r_{or} \leq \sqrt{n}$ and such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$ is $\mathbb{S}^n(\sqrt{n})$.*

Remark 4.4. *Related to Corollary 4.7, it is worth to mention that Pigola and Rimoldi [31] studied geometric properties of complete non-compact bounded self-shrinkers obtaining natural restrictions that force these hypersurfaces to be compact. In particular, they proved that the only complete bounded self-shrinker of $\mathbb{R}^3$ with $|A| \leq 1$ is $\mathbb{S}^2(\sqrt{2})$. Afterwards, Cavalcante and Espinar [8] showed that the only complete self-shrinker of $\mathbb{R}^{n+1}$ properly immersed in a closed cylinder $\overline{\mathbb{B}^{k+1}(r)} \times \mathbb{R}^{n-k}$, for some $k \in \{1, \cdots, n\}$ and radius $r \leq \sqrt{k}$, is the cylinder $\mathbb{S}^k(\sqrt{k}) \times \mathbb{R}^{n-k}$.*

Considering the setting of Example 2.2, from Theorem 3.1 we have:

Corollary 4.8. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \setminus \{\pi(P) \cup Cut_P\}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \sin t \partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times \mathbb{S}^n$, and either $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) \leq t_1 < \frac{\pi}{2}$ or $0 < t_2 \leq \cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$.*

When the ambient space is a pseudo-hyperbolic space, Theorem 4.2 reads as follows:

Corollary 4.9. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is complete. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = e^t \partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times M^n$, and $t_1 \geq \log\left(-\frac{n}{c}\right)$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is the slice $\{\log\left(-\frac{n}{c}\right)\} \times M^n$.*

Taking into account again the context of Example 2.4 and Example 2.5, from Theorem 4.2 we also obtain:

Corollary 4.10. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space and suppose that inequality (2.12) is satisfied. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times \mathbb{S}^n$, and $V_{\mathfrak{m}}(r(t)) \leq \frac{c^2}{n^2} r(t)^4$ for all $t \in [t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m})$ solves equation (2.11).*

and

Corollary 4.11. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Reissner-Nordström space and suppose that there is $r_* > r_0(\mathfrak{m}, \mathfrak{q})$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, that lies in a slab $[t_1, t_2] \times \mathbb{S}^n$, and $V_{\mathfrak{m},\mathfrak{q}}(r(t)) \leq \frac{c^2}{n^2} r(t)^4$ for all $t \in [t_1, t_2]$. If $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solves equation (2.13).*

4.2. **Rigidity results via a parabolicity criterion.** We recall that a Riemannian manifold is said to be *parabolic* if the only subharmonic functions on it that are bounded from above are the constants. On the other hand, given two Riemannian manifolds (Σ, g) and $(\Sigma^{'}, g^{'})$, a diffeomorphism ϕ from Σ onto $\Sigma^{'}$ is called a *quasi-isometry* if there exists a constant $\kappa \geq 1$ such that

$$\kappa^{-1}|v|_g \leq |d\phi(v)|_{g'} \leq \kappa|v|_g$$

for all $v \in T_p\Sigma$, $p \in \Sigma$. From [25, Theorem 1] (see also [22, Corollary 5.3]) we have the following:

Lemma 4.3. *Let (Σ, g) and $(\Sigma^{'}, g^{'})$ be two complete Riemannian manifolds. If Σ and $\Sigma^{'}$ are quasi-isometric, then Σ and $\Sigma^{'}$ are both parabolic or neither is parabolic.*

We can use the previous lemma to get the following parabolicity criterion:

Lemma 4.4. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete hypersurface immersed in a warped product $\overline{M}^{n+1} = I \times_f M^n$, whose fiber (M^n, g_M) is complete with parabolic universal covering. If Θ is bounded away from zero, then $(\Sigma^n, \hat{g})$, endowed with the conformal metric $\hat{g} = \frac{1}{f(h)^2} g$, is parabolic.*

Proof. Given $p \in \Sigma^n$ and $v \in T_p\Sigma^n$, from (2.1) and (2.4) we have

$$g(v, v) = g(v, \nabla h)^2 + f(h)^2 g_M(d\pi(v), d\pi(v)). \tag{4.24}$$

Thus, from (4.24) we get

$$\hat{g}(v, v) = \frac{1}{f(h)^2} g(v, v) \geq g_M(d\pi(v), d\pi(v)). \tag{4.25}$$

On the other hand, using (2.4) and the Cauchy-Schwarz inequality in (4.24), we also have

$$\Theta^2 g(v, v) \leq f(h)^2 g_M(d\pi(v), d\pi(v)). \tag{4.26}$$

Since Θ is bounded away from zero, there exists a positive constant β such that $\Theta^2 \geq \beta^2$. Consequently, from (4.26) we get

$$\beta^2 g(v, v) \leq \Theta^2 g(v, v) \leq f(h)^2 g_M(d\pi(v), d\pi(v)). \tag{4.27}$$

Thus, from (4.27) we have

$$\hat{g}(v, v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{4.28}$$

Hence, using inequalities (4.25) and (4.28), we get

$$g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v, v) \leq \frac{1}{\beta^2} g_M(d\pi(v), d\pi(v)). \tag{4.29}$$

So, taking the constant $\kappa = \frac{1}{\beta^2} \geq 1$, from (4.29) we obtain

$$\kappa^{-1} g_M(d\pi(v), d\pi(v)) \leq \hat{g}(v, v) \leq \kappa g_M(d\pi(v), d\pi(v)), \tag{4.30}$$

which means that π is a quasi-isometry between Σ and M.

Let $\Sigma^{'}$ be the universal Riemannian covering of Σ with projection $\pi_\Sigma : \Sigma^{'} \to \Sigma$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \Sigma^{'} \to M$ is a covering map. If $M^{'}$ is the universal Riemannian covering of M with projection $\pi^{'} : M^{'} \to M$, then there exists a diffeomorphism $\phi : \Sigma^{'} \to M^{'}$ such that $\pi^{'} \circ \phi = \pi_0$. Moreover, from (4.30) it is not difficult to verify that ϕ is also a quasi-isometry. Therefore, since the universal Riemannian covering of M is parabolic, it follows from Lemma 4.3 that the universal Riemannian covering of Σ is parabolic and, hence, Σ must be also parabolic with respect to the metric $\hat{g}$. □

For the next result, let us establish one notation. Define the *modified soliton function* as being the function

$$\bar{\zeta}_c(t) := f'(t)\zeta_c(t). \tag{4.31}$$

Using Lemma 4.4, we obtain the following result:

Theorem 4.3. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function f satisfies*

$$(\log f)'' \leq \gamma[(\log f)']^2 \tag{4.32}$$

for some constant $\gamma > -1$, holding the equality only at isolated points of I. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c \neq 0$, such that Θ is bounded away from zero and $\inf_\Sigma f(h) > 0$. If $\bar{\zeta}_c(h) \leq 0$, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.

Proof. Let us consider on Σ^n the metric $\hat{g} = \frac{1}{f(h)^2} g$, which is conformal to its induced metric g. If we denote by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (2.4) and (2.6) we get

$$\begin{aligned} \hat{\Delta} h &= f(h)^2 \Delta h - (n-2) f(h) f'(h) |\nabla h|^2 \\ &= n f(h) f'(h) \Theta^2 + f(h) f'(h) |\nabla h|^2 + H f(h)^2 \Theta. \end{aligned} \tag{4.33}$$

With a straightforward computation, from (4.33) we obtain

$$\begin{aligned} \hat{\Delta} f(h) &= f''(h) \hat{g}(\widehat{\nabla} h, \widehat{\nabla} h) + f'(h) \hat{\Delta} h \\ &= f''(h) f(h)^2 |\nabla h|^2 + f'(h) \left(n f(h) f'(h) \Theta^2 + f(h) f'(h) |\nabla h|^2 + H f(h) \Theta \right) \\ &= n f(h) f'(h)^2 + H f'(h) f(h)^2 \Theta + f(h)^3 \left((\log f)''(h) - (n-2) \frac{f'(h)^2}{f(h)^2} \right) |\nabla h|^2. \end{aligned} \tag{4.34}$$

Given a positive real number α, we have that

$$\hat{\Delta} f(h)^{-\alpha} = \alpha(\alpha+1) f(h)^{-\alpha-2} \hat{g}(\widehat{\nabla} f(h), \widehat{\nabla} f(h)) - \alpha f(h)^{-\alpha-1} \hat{\Delta} f(h). \tag{4.35}$$

Using (4.34) in (4.35), we get

$$\begin{aligned} \hat{\Delta} f(h)^{-\alpha} &= -\alpha n f(h)^{-\alpha} f'(h)^2 - \alpha H f'(h) f(h)^{-\alpha+1} \Theta + \alpha(\alpha+1) f(h)^{-\alpha} f'(h)^2 |\nabla h|^2 \\ &\quad - \alpha f(h)^{-\alpha+2} \left((\log f)''(h) - (n-2) \frac{f'(h)^2}{f(h)^2} \right) |\nabla h|^2. \end{aligned} \tag{4.36}$$

But, from (2.4) we have

$$-\alpha n f(h)^{-\alpha} f'(h)^2 = -\alpha n f(h)^{-\alpha} f'(h)^2 |\nabla h|^2 - \alpha n f(h)^{-\alpha} f'(h)^2 \Theta^2. \tag{4.37}$$

Thus, from (4.36), (4.37), (2.7) and (4.31) we obtain

$$\begin{aligned} \hat{\Delta} f(h)^{-\alpha} &= -\alpha f(h)^{-\alpha} \bar{\zeta}_c(h) \Theta^2 \\ &\quad - \alpha f(h)^{-\alpha+2} \left\{ (\log f)''(h) - (\alpha-1)[(\log f)'(h)]^2 \right\} |\nabla h|^2. \end{aligned} \tag{4.38}$$

First, we note that Lemma 4.4 guarantees that $(\Sigma^n, \hat{g})$ is parabolic. Moreover, it follows from (4.38) that $f(h)^{-\alpha}$ (where $\alpha = 1 + \gamma$) is subharmonic on Σ^n. Thus, since the hypothesis $\inf_\Sigma f(h) > 0$ implies that $f(h)^{-\alpha}$ is bounded from above, it follows from the parabolicity of $(\Sigma^n, \hat{g})$ that $f(h)$ is constant on Σ^n. Consequently, since we are assuming that the equality holds in (4.32) only at isolated points of I, returning to (4.38) we conclude that $|\nabla h| = 0$ on Σ^n, which means that Σ^n is a slice. □

In the following we present several half-space results. More precisely, in the context of self-shrinkers, Theorem 4.3 reads as follows:

Corollary 4.12. *The only complete n-dimensional self-shrinker of $\mathbb{R}^{n+1}$ that lies in the closure of the unbounded domain determined by $\mathbb{S}^n(\sqrt{n}) \subset \mathbb{R}^{n+1}$ and such that Θ is bounded away from zero, is $\mathbb{S}^n(\sqrt{n})$.*

Taking into account once more Example 2.2, from Theorem 4.3 we get:

Corollary 4.13. *Let $\overline{M}^{n+1} = \left(0, \frac{\pi}{2}\right) \times_{\sin t} \mathbb{S}^n$ be the warped product model of $\mathbb{RP}^{n+1} \backslash \{\pi(P) \cup Cut_P\}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = \sin t\partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|}) \leq h < \frac{\pi}{2}$, then Σ^n is the slice $\{\cos^{-1}(\frac{\sqrt{4c^2+n^2}-n}{2|c|})\} \times \mathbb{S}^n$.*

From Theorem 4.3 we obtain the following result:

Corollary 4.14. *Let $\overline{M}^{n+1} = I \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber M^n is complete with parabolic universal covering. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = e^t\partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $h \geq \log(-\frac{n}{c})$, then Σ^n is the slice $\{\log\left(-\frac{n}{c}\right)\} \times M^n$.*

In the setting of Example 2.4 and Example 2.5, we also have the following consequence of Theorem 4.3:

Corollary 4.15. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Schwarzschild space and suppose that inequality (2.12) is satisfied. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $V_{\mathfrak{m}}(r(h)) \leq \frac{c^2}{n^2}r(h)^4$ on Σ^n, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m})$ solves equation (2.11).*

and

Corollary 4.16. *Let $\overline{M}^{n+1} = I \times_f \mathbb{S}^n$ be the Reissner-Nordström space and suppose that there is $r_* > r_0(\mathfrak{m}, \mathfrak{q})$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete mean curvature flow soliton with respect to $K = f(t)\partial_t$ and soliton constant $c < 0$, such that Θ is bounded away from zero. If $V_{\mathfrak{m},\mathfrak{q}}(r(h)) \leq \frac{c^2}{n^2}r(h)^4$ on Σ^n, then Σ^n is a slice $\{t_*\} \times \mathbb{S}^n$, where $t_* = t(r_*)$ is such that $r_* > r_0(\mathfrak{m}, \mathfrak{q})$ solves equation (2.13).*

5. Entire mean curvature flow graphs

Ecker and Huisken [20] proved that if an entire graph with polynomial volume growth is a self-shrinker, then it is necessarily a hyperplane. Later on, Wang [33] removed the condition of polynomial volume growth in Ecker-Huisken's Theorem. More recently, Colombo, Mari and Rigoli [16] extended this study to the context of entire mean curvature flow graphs in warped products. Motivated by these works, the last section of this paper is devoted to establish new Moser-Bernstein type results concerning entire graphs constructed over the fiber M^n of a warped product $\overline{M}^{n+1} = I \times_f M^n$, which are mean curvature flow solitons with respect to $K = f(t)\partial_t$ with soliton constant $c \neq 0$.

5.1. A key nonlinear differential equation.

Let $\Omega \subseteq M^n$ be a domain. A function $u \in C^\infty(\Omega)$ such that $u(\Omega) \subseteq I$ defines a vertical graph in the warped product $\overline{M}^{n+1} = I \times_f M^n$. In such a case, $\Sigma(u)$ will denote the graph over Ω determined by u, that is,

$$\Sigma(u) = \{(u(p), p) : p \in \Omega\} \subset \overline{M}^{n+1}.$$

The graph is said to be entire if $\Omega = M^n$. Observe that $h(u(p),p) = u(p), p \in \Omega$. Hence, h and u can be identified in a natural way. The metric induced on Ω from the Riemannian metric of the ambient space via $\Sigma(u)$ is

$$g_u = du^2 + f(u)^2 g_M.$$

If M^n is complete and $\inf_M f(u) > 0$, then $\Sigma(u)$ furnished with the metric g_u is also complete. The unit vector field

$$N(p) = -\frac{f(u(p))}{\sqrt{f(u(p))^2 + |Du(p)|^2_M}}\left(\partial_t|_{(u(p),p)} - \frac{Du(p)}{f(u(p))^2}\right), \quad p \in \Omega, \tag{5.39}$$

where Du stands for the gradient of u in M and $|Du|_M = g_M(Du,Du)^{1/2}$, gives an orientation of $\Sigma(u)$ with respect to which we have $\Theta = \overline{g}(N,\partial_t) < 0$. The corresponding shape operator is given by

$$\begin{aligned} AX = &-\frac{1}{f(u)\sqrt{f(u)^2+|Du|^2_M}} D_X Du + \frac{f'(u)}{\sqrt{f(u)^2+|Du|^2_M}} X \\ &- \left(\frac{-g_M(D_X Du, Du)}{f(u)\left(f(u)^2+|Du|^2_M\right)^{3/2}} - \frac{f'(u) g_M(Du,X)}{\left(f(u)^2+|Du|^2_M\right)^{3/2}}\right) Du \end{aligned} \tag{5.40}$$

for any vector field X tangent to Ω, where D is the Levi-Civita connection in M^n.

Consequently, being $\Sigma(u)$ a vertical graph over a domain $\Omega \subseteq M^n$ and denoting by div_M the divergence operator computed in the metric g_M, it is not difficult to verify from (5.40) that the mean curvature function $H(u)$ of $\Sigma(u)$ is given by:

$$H(u) = -\operatorname{div}_M\left(\frac{Du}{f(u)\sqrt{f(u)^2+|Du|^2_M}}\right) + \frac{f'(u)}{\sqrt{f(u)^2+|Du|^2_M}}\left(n - \frac{|Du|^2_M}{f(u)^2}\right). \tag{5.41}$$

Hence, from (2.7) and (5.41) we have that $\Sigma(u)$ is a mean curvature flow soliton with respect to $K = f(t)\partial_t$ with soliton constant c if, and only if, u is a solution of the following nonlinear differential equation:

$$\operatorname{div}_M\left(\frac{Du}{f(u)\sqrt{f(u)^2+|Du|^2_M}}\right) = \frac{1}{\sqrt{f(u)^2+|Du|^2_M}}\left\{cf(u)^2 + f'(u)\left(n - \frac{|Du|^2_M}{f(u)^2}\right)\right\}. \tag{5.42}$$

5.2. **Moser-Bernstein type results.** We say that $u \in C^\infty(M)$ has finite C^2 norm when

$$||u||_{\mathcal{C}^2(M)} := \sup_{|\gamma|\le 2} |D^\gamma u|_{L^\infty(M)} < +\infty.$$

In this context, we establish our first Moser-Bernstein type result:

Theorem 5.4. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (3.14). Suppose in addition that $c \neq 0$ and $\zeta_c(t) \geq 0$. If $u \in C^\infty(M)$ is an entire solution of equation (5.42), with finite C^2 norm and such that $|Du|_M \leq \mathcal{C}\inf_M |\zeta_c(u)|$ for some positive constant $\mathcal{C}$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Let $u \in C^\infty(M)$ be such a solution of equation (5.42). It follows from (5.40) that the shape operator A of $\Sigma(u)$ is bounded, provided that u has finite C^2 norm. We note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\inf_M f(u) > 0$. Hence, since we are assuming that M^n is complete, we get that $(\Sigma(u), g_u)$ must be also complete.

Therefore, we can reason as in the proof of Theorem 3.1 obtaining that $\inf_M |\zeta_c(u)| = 0$ and, hence, the result follows from our constraint on $|Du|_M$. □

From the proof of Theorem 5.4 we also get the following nonexistence result:

Corollary 5.17. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with sectional curvature obeying the convergence condition (3.14). Suppose in addition that $c \neq 0$ and $\inf_I \zeta_c(t) > 0$. There exists no entire solution with finite C^2 norm of the equation (5.42).*

Proceeding, Theorem 4.2 allows us to obtain our next result.

Theorem 5.5. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete. Suppose in addition that $c \neq 0$ and $\zeta_c(t)$ does not change the sign. If $u \in C^\infty(M)$ is a bounded entire solution of equation (5.42) such that $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Let $u \in C^\infty(M)$ be such a bounded entire solution of equation (5.42). Denoting by dM and $d\Sigma$ the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, from [1, Equation (3.7)] we have that

$$|\nabla h| d\Sigma = f(u)^{n-1} |Du|_M dM. \tag{5.43}$$

Hence, since we are assuming that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, from relation (5.43) we conclude that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 4.2. □

From (5.39) we see that the assumption Θ bounded away from zero is equivalent to $|Du|_M \leq \mathcal{C} f(u)$ for some positive constant $\mathcal{C}$. So, Theorem 4.3 allows us to obtain our last Moser-Bernstein type result:

Theorem 5.6. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a warped product whose fiber M^n is complete with parabolic universal covering and such that its warping function f satisfies (4.32), holding the equality only at isolated points of I. Suppose in addition that $c \neq 0$ and $\zeta_c(t) \leq 0$. If $u \in C^\infty(M)$ is a bounded entire solution of equation (5.42) such that $|Du|_M \leq \mathcal{C} f(u)$ for some positive constant $\mathcal{C}$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Remark 5.5. Regarding all the nonexistence, rigidity and Moser-Bernstein type results which were established along our manuscript, it remains an interesting open problem to infer what is the geometric behavior of the mean curvature flow solitons in the *unbounded case*, that is, when it is not contained in a slab of the ambient space. Furthermore, it is worth noting that a natural future prospect related to our work is to extend it to the context of *multiply warped product spaces* (for details on these spaces, see [9, Section 3.6]).

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