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Research Article

Some Novel Proximal Point Results and Applications

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Abstract

The current study provides significant findings on coincidence the best proximity points for proximal-contractions in the context of extended b-metric spaces. To substantiate our assertions, we present illustrative examples across several circumstances. The conclusions presented in this study provide an expanded and more nuanced viewpoint, extending and generalizing multiple previous findings in optimal proximity theory. These findings present a new method for comprehending proximal coincidence locations in multi-valued mappings, representing a substantial advancement in the existing research landscape. This work gives practical benchmarks for implementing optimal proximity results, provided that existence and uniqueness constraints are met.

1. Introduction

The theory of metric spaces has been a fundamental aspect of mathematical analysis since its inception. One of the earliest and most significant contributions to this area was made by Fréchet (1906) [1], who introduced the foundational concepts of metric spaces in his seminal work on functional calculus. This laid the groundwork for the subsequent exploration of generalized metric spaces, which have been extensively studied and applied in various branches of mathematics and science. Over time, the classical metric space structure has been extended to encompass a range of generalizations, including b-metric spaces and their variants, as explored by Kamran et al. (2017) [2], who established fixed point results in extended b-metric spaces.

The concept of the best proximity points emerged as an important generalization of fixed point theory, addressing situations where mappings lack fixed points but exhibit minimal distances. This line of inquiry gained prominence with the pioneering work of Basha and Veeramani (1997) [3], who introduced the concept of best approximations, providing a framework for analyzing such mappings. Subsequently, their work was extended in 2000 [4], where they proposed theorems for best proximity pairs in multifunctions. These studies established a foundation for connecting approximation theory with optimization problems, marking an important milestone in the development of best proximity point theory; see, for example, [5].

The integration of contraction principles into the study of best proximity points has been another significant advancement. For instance, Suzuki et al. (2009) [6] investigated the existence of best proximity points in metric spaces satisfying the UC property, introducing new methods to address proximity problems. These foundational results were further enhanced by Asadi and Afshar (2022) [7], who introduced fixed point theorems for rational C-class functions in b-metric spaces and demonstrated their application to integral equations. Similarly, Alqahtani et al. (2018) [8] explored extended b-metric spaces to establish common fixed point results, offering a deeper understanding of the relationships between different types of metric spaces.

Further advancements in fixed point theory have been made through the study of generalized contraction mappings. For example, Chandok and Karapinar (2012) [9] examined generalized rational-type contractions in partially ordered metric spaces. This approach provided new perspectives on the structural properties of metric spaces and their implications for fixed point existence. The study of fuzzy metric spaces also gained traction with contributions like those by Jabeen et al. (2020) [10], who applied weakly compatible and quasi-contraction results to fuzzy cone metric spaces.

The application of these theories has extended beyond pure mathematics to practical and computational domains. For instance, McConnell et al. (1991) [11] applied metric concepts to dynamic time warping techniques in geosciences, demonstrating the utility of these mathematical structures in real-world problems. More recently, Younis et al. (2021) [12] utilized graphical structures in b-metric spaces to study the

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transverse oscillations of homogeneous bars, while Younis et al. (2023) [13] explored applications in elastic beam deformations. For further synthesis in this direction, we refer to [14, 15].
The evolution of the best proximity point theory has also been marked by innovative applications in optimization and computational modelling. The work of Savanovic et al. (2022) [16] on multi-valued mappings in b-metric spaces illustrates the growing adaptability of proximity point results to diverse mathematical contexts. Moreover, the recent study by Younis et al. (2024) [17] connected best proximity points to the equations of motion, highlighting the increasing relevance of these concepts in applied mathematics and physics. The chronological development of metric space theories and best proximity point research demonstrates a profound progression from foundational concepts to modern applications. These contributions, spanning more than a century, underscore the importance of metric spaces and their generalizations in addressing both theoretical and practical challenges in mathematics and related fields.
This research is inspired by the current literature regarding rational-type contractive conditions in diverse metric spaces. This work presents a series of coincidence best proximity point theorems that are specifically designed for extended b-metric spaces. The results for coincidence points in this paper are new because they include contraction requirements for both multivalued and single-value mappings in extended b-metric spaces. The main goal is to come up with coincidence best proximity point theorems for generalized and modified proximal-contractions in extended b-metric spaces. This will help to find the best solutions for the equation $\nabla(x) = x$. Secondly, it utilizes these findings to establish adequate conditions for the existence of solutions to nonlinear differential and integral equations, demonstrating the practical significance of the proposed theory. The uniqueness of these conclusions is emphasized by instances that validate their generalization as noteworthy outcomes in the current state of the art.

2. Preliminaries

Before delving into the main content of the article, we review some fundamental concepts and notations that will be utilized throughout this study.

Definition 2.1. *[2] Let $\mathbb{J}$ be a nonempty set, and let $\pi : \mathbb{J}\times\mathbb{J} \to [1,\infty)$ be a mapping. A function $\partial : \mathbb{J}\times\mathbb{J} \to [0,\infty)$ is said to be an* extended b-metric *if it satisfies the following conditions:*

1. $\partial(\tau_1,\tau_2) = 0$ if and only if $\tau_1 = \tau_2$,
2. $\partial(\tau_1,\tau_2) = \partial(\tau_2,\tau_1)$ for all $\tau_1,\tau_2 \in \mathbb{J}$,
3. $\partial(\tau_1,\tau_2) \le \pi(\tau_1,\tau_2)[\partial(\tau_1,\tau_3)+\partial(\tau_3,\tau_2)]$ for all $\tau_1,\tau_2,\tau_3 \in \mathbb{J}$.

The pair $(\mathbb{J},\partial)$ is then called an *extended b-metric space*.

Remark 2.2. *If $\pi(\tau_1,\tau_2) = s$ for $s \ge 1$, the extended b-metric space $(\mathbb{J},\partial)$ reduces to a b-metric space.*

Let $(\mathbb{J},\partial)$ be an extended b-metric space, and let $\complement$ and $\backepsilon$ be two nonempty subsets of $\mathbb{J}$.

$$\complement_0 = \{\tau \in \complement : \partial(\tau,\tau^*) = \partial(\complement,\backepsilon) \text{ for some } \tau^* \in \backepsilon\},$$
$$\backepsilon_0 = \{\tau^* \in \backepsilon : \partial(\tau,\tau^*) = \partial(\complement,\backepsilon) \text{ for some } \tau \in \complement\},$$

where

$$\partial(\complement,\backepsilon) = \inf\{\partial(\tau,\tau^*) : \tau \in \complement, \tau^* \in \backepsilon\}.$$

Definition 2.3. *[2] Let $(\mathbb{J},\partial)$ be an extended b-metric space. A sequence $\{\tau_n\}$ in $\mathbb{J}$ is said to* converge *to $\tau \in \mathbb{J}$ if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $\partial(\tau_n,\tau) < \varepsilon$ for all $n \ge N_\varepsilon$. This is denoted as*

$$\lim_{n\to\infty} \tau_n = \tau.$$

Definition 2.4. *[2] A sequence $\{\tau_n\}$ in an extended b-metric space $(\mathbb{J},\partial)$ is called a* Cauchy sequence *if for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that $\partial(\tau_n,\tau_m) < \varepsilon$ for all $m,n \ge N_\varepsilon$.*

Definition 2.5. *[2] An extended b-metric space $(\mathbb{J},\partial)$ is said to be* complete *if every Cauchy sequence in $\mathbb{J}$ converges to a point in $\mathbb{J}$.*

Lemma 2.6. *[18] Let $(\mathbb{J},\partial)$ be a complete extended b-metric space. If ∂ is continuous, then every convergent sequence in $\mathbb{J}$ has a unique limit.*

Definition 2.7. *[18] Let $(\complement,\backepsilon)$ be a pair of nonempty subsets of a metric space such that $\complement_0$ is nonempty. The pair $(\complement,\backepsilon)$ is said to have the* P-property *if*

$$\partial(\tau_1,\tau_1^*) = \partial(\complement,\backepsilon) \text{ and } \partial(\tau_2,\tau_2^*) = \partial(\complement,\backepsilon) \implies \partial(\tau_1,\tau_2) = \partial(\tau_1^*,\tau_2^*),$$

where $\tau_1,\tau_2 \in \complement$ and $\tau_1^*,\tau_2^* \in \backepsilon$.

Definition 2.8. *[19] Let $\mathscr{CB}(\mathbb{J})$ denote the set of all closed and bounded subsets of $\mathbb{J}$. The Pompeiu–Hausdorff metric $\mathscr{H}$ induced by ∂ is defined as*

$$\mathscr{H}(\complement,\backepsilon) = \max\{\sup_{a\in\complement} \wp(a,\backepsilon), \sup_{b\in\backepsilon} \wp(b,\complement)\},$$

for $\complement,\backepsilon \in \mathscr{CB}(\mathbb{J})$, where

$$\wp(a,\backepsilon) = \inf\{\partial(\sigma_1,\sigma_2) : b \in \backepsilon\}.$$

Additionally, we define

$$\wp^*(\sigma_1,\sigma_2) = \wp(\sigma_1,\sigma_2) - \partial(\complement,\backepsilon), \quad \forall a \in \complement, b \in \backepsilon.$$

From this point onward, let $\complement$ and $\eth$ be nonempty subsets of a complete extended b-metric space $(\mathbb{J},\partial)$ define $\pi_* : \mathbb{J}^2 \to [1,\infty)$ as

$$\pi_*(\tau,\eth) = \inf\{\pi(\tau,\tau^*) : \tau^* \in \eth\}$$

and

$$\pi_*(\complement,\eth) = \inf\{\pi(\tau,\tau^*) : \tau \in \complement, \tau^* \in \eth\},$$

where $\pi : \mathbb{J}\times\mathbb{J} \to [1,\infty)$.

3. Coincidence Proximal Point Results for Multi-Valued Mappings

In this section, we will discuss some of the best coincidence proximity point theorems using multi-valued concepts on extended b-metric space.

Definition 3.1. *Given $\nabla : \complement \to \mathscr{CB}(\eth)$ and $\mho : \complement \to \complement$ be mappings. The pair $(\mho,\nabla)$ is said to be a $\varphi_{max}-$proximal-contraction if there exists a real number $\varphi \in [0,1)$ such that*

$$\left.\begin{array}{r}\wp(\mho u,\nabla\tau) = \partial(\complement,\eth)\\ \wp(\mho v,\nabla\tau^*) = \partial(\complement,\eth)\end{array}\right\} \text{ implies } \mathscr{H}(\nabla\tau,\nabla\tau^*) \le \varphi\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*) = \max\left\{ \partial(\mho u,\mho v), \frac{\wp(\mho\tau,\nabla v)-\pi_*(\mho\tau,\nabla v)\wp(\mho u,\nabla v)}{\pi_*(\mho\tau,\nabla v)}, \wp^*(\mho v,\nabla u) \right\}$$

for all u,v,τ,τ^* in $\complement$.

Example 3.2. *Let $\mathbb{J} = \{1,2,3,4\}$. Consider the function ∂ given as $\partial(\tau,\tau^*) = \partial(\tau^*,\tau)$ and $\partial(\tau,\tau) = 0$, where*

∂	1	2	3	4
1	0	3	5	7
2	3	0	4	6
3	5	4	0	2
4	7	6	2	0

Take $\pi_* : \mathbb{J}\times\mathbb{J} \to [1,\infty)$ to be symmetric, defined as $\pi(\tau,\tau^*) = 10\tau + 5\tau^*$. It is easy to see that $(\mathbb{J},\partial)$ is an extended $b-$metric space. Let $\complement = \{1,2\}$ and $\eth = \{3,4\}$ be two non-empty subsets of the extended $b-$metric space $(\mathbb{J},\partial)$. After routine calculations, we get $\partial(\complement,\eth) = 4$, $\complement_0 = \complement$ and $\eth_0 = \eth$. Since $\nabla : \complement \to \mathscr{CB}(\eth)$, we define

$$\nabla\tau = \left\{ \{3,4\}, \quad \text{if } \tau \in \{1,2\} \right\},$$

and for $\mho : \complement \to \complement$, we have

$$\mho\tau = \begin{cases} 1, & \text{if } \tau = 2,\\ 2, & \text{if } \tau = 1.\end{cases}$$

We show that the pair $(\mho,\nabla)$ satisfies φ_M-proximal-contraction

$$\mathscr{H}(\nabla\tau,\nabla\tau^*) \le \varphi\,\mathbb{M}(u,v,\tau,\tau^*),$$

for all $u,v,\tau,\tau^* \in \complement$, where $\mathbb{M}(u,v,\tau,\tau^*)$ is defined in Definition 3.1.
Now,

$$\left.\begin{array}{r}\wp(\mho 2,\nabla 1) = \partial(\complement,\eth),\\ \wp(\mho 1,\nabla 2) = \partial(\complement,\eth),\end{array}\right\} \text{ implies } \mathscr{H}(\nabla 1,\nabla 2) \le \varphi\mathbb{M}(u,v,\tau,\tau^*).$$

Hence, $\mathscr{H}(\nabla 1,\nabla 2) = 0$, for every $\varphi \in [0,1)$, and Definition 3.1 is satisfied.

Definition 3.3. *Given $\nabla : \complement \to \mathscr{CB}(\eth)$ and $\mho : \complement \to \complement$ be mappings. The pair $(\mho,\nabla)$ is said to be a $\varphi_\nabla-$proximal-contraction if there exists a real number $\varphi \in [0,1)$ such that*

$$\left.\begin{array}{r}\wp(u,\nabla\tau) = \partial(\complement,\eth)\\ \wp(v,\nabla\tau^*) = \partial(\complement,\eth)\end{array}\right\} \text{ implies } \mathscr{H}(\nabla\tau,\nabla\tau^*) \le \varphi\mathbb{M}(u,v,\tau,\tau^*),$$

where

$$\mathbb{M}(u,v,\tau,\tau^*) = \max\left\{ \partial(u,v), \frac{\wp(\tau,\nabla v)-\pi_*(\tau,\nabla v)\wp(u,\nabla v)}{\pi_*(\tau,\nabla v)}, \wp^*(v,\nabla u) \right\}$$

for all u,v,τ,τ^* in $\complement$.
Note that, if we take $\mho = I_\complement$ ($\mho$ as an identity mapping on $\complement$), then every φ_M-proximal-contraction will reduce to a $\varphi_\nabla-$proximal-contraction.

Definition 3.4. *A mapping $\nabla : \mathbb{J} \to \mathscr{CB}(\mathbb{J})$ is continuous in an extended b-metric space $(\mathbb{J}, \partial)$ at $\tau \in \mathbb{J}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\nabla(K(\tau, \delta)) \subseteq K(\nabla\tau, \varepsilon),$$

where $K(\tau, \varepsilon)$ is given as

$$K(\tau, \varepsilon) = \{\tau^* \in \mathbb{J}, \partial(\tau, \tau^*) < \varepsilon\}.$$

Clearly, if ∇ is continuous at τ, then $\tau_n \to \tau$ implies that $\nabla\tau_n \to \nabla\tau$ as $n \to \infty$.

The following theorem is based on Definition 3.1, which is more general than the results discussed in the introduction within the context of extended $b-$metric spaces. Furthermore, Example 3.7 illustrates our fact.

Theorem 3.5. *Let $\nabla : \mathsf{C} \to \mathscr{CB}(\text{Ↄ})$, $\mho : \mathsf{C} \to \mathsf{C}$ and $\pi_* : \mathbb{J}^2 \to [1, \infty)$ be mappings, where C is closed subset of an extended $b-$metric space $(\mathbb{J}, \partial)$. Let the pair $(\mathsf{C}, \text{Ↄ})$ satisfies the P-property with $\nabla(\mathsf{C}_0) \subseteq \text{Ↄ}_0$ and $\mathsf{C}_0 \subseteq \mho(\mathsf{C}_0)$. Assume that the pair of continuous mappings $(\mho, \nabla)$, where $\mho$ is one-to-one, satisfies φ_M-proximal-contraction the following statement holds*

$$\lim_{n,m\to\infty} \pi(\mho\tau_n, \mho\tau_m) < \frac{1}{k}, \quad \textit{where} \quad k \in (0,1).$$

Then, there exists a unique coincidence best proximity point of the pair $(\mho, \nabla)$.

Proof. Let τ_0 be an arbitrary element in C_0. Since $\nabla(\mathsf{C}_0)$ is contained in Ↄ_0 and C_0 is contained in $\mho(\mathsf{C}_0)$, there exists an element τ_1 in C_0 such that

$$\wp(\mho\tau_1, \nabla\tau_0) = \partial(\mathsf{C}, \text{Ↄ}).$$

Again, since $\nabla\tau_1$ is an element of $\nabla(\mathsf{C}_0)$ which is contained in Ↄ_0, and C_0 is contained in $\mho(\mathsf{C}_0)$, it follows that there is an element τ_2 in C_0 such that

$$\wp(\mho\tau_2, \nabla\tau_1) = \partial(\mathsf{C}, \text{Ↄ}).$$

Making use of the property P, we acquire

$$\partial(\mho\tau_1, \mho\tau_2) = \mathscr{H}(\nabla\tau_0, \nabla\tau_1).$$

Since the pair of mappings $(\mho, \nabla)$ satisfies φ_M-proximal-contraction, we obtain

$$\partial(\mho\tau_1, \mho\tau_2) \leq \varphi\mathbb{M}(\tau_1, \tau_2, \tau_0, \tau_1),$$

where

$$\begin{aligned} M_\nabla(\tau_1, \tau_2, \tau_0, \tau_1) &\leq \max\left\{ \partial(\tau_1, \tau_2), \frac{\wp(\tau_0, \nabla\tau_2) - \pi_*(\tau_0, \nabla\tau_2)\wp(\tau_1, \nabla\tau_2)}{\pi_*(\tau_0, \nabla\tau_2)}, \wp^*(\tau_2, \nabla\tau_1) \right\} \\ &\leq \max\left\{ \begin{array}{c} \partial(\tau_1, \tau_2), \frac{\pi_*(\tau_0, \nabla\tau_2)[\partial(\tau_0, \tau_1) + \wp(\tau_1, \nabla\tau_2)] - \pi_*(\tau_0, \nabla\tau_2)\wp(\tau_1, \nabla\tau_2)}{\pi_*(\tau_0, \nabla\tau_2)}, \\ \wp(\tau_2, \nabla\tau_1) - \partial(\mathsf{C}, \text{Ↄ}) \end{array} \right\} \\ &\leq \max\left\{ \partial(\tau_1, \tau_2), \frac{\pi_*(\tau_0, \nabla\tau_2)\partial(\tau_0, \tau_1)}{\pi_*(\tau_0, \nabla\tau_2)}, 0 \right\} \\ &\leq \max\{\partial(\tau_1, \tau_2), \partial(\tau_0, \tau_1), 0\}. \end{aligned}$$

Hence, we have

$$M_\nabla(\tau_1, \tau_2, \tau_0, \tau_1) \leq \max\{\partial(\mho\tau_1, \mho\tau_2), \partial(\mho\tau_0, \mho\tau_1)\}.$$

If $\max\{\partial(\mho\tau_1, \mho\tau_2), \partial(\mho\tau_0, \mho\tau_1)\} = \partial(\mho\tau_1, \mho\tau_2)$, then the above inequality implies

$$\partial(\mho\tau_1, \mho\tau_2) < \varphi\partial(\mho\tau_1, \mho\tau_2),$$

which is a contradiction. Therefore, we conclude that

$$\max\{\partial(\mho\tau_1, \mho\tau_2), \partial(\mho\tau_0, \mho\tau_1)\} = \partial(\mho\tau_0, \mho\tau_1).$$

Further, by the fact that $\nabla\tau_2$ is a member of $\nabla(\mathsf{C}_0)$, which is contained in Ↄ_0, and C_0 is contained in $\mho(\mathsf{C}_0)$, there exists $\tau_3 \in \mathsf{C}_0$ such that

$$\wp(\mho\tau_2, \nabla\tau_1) = \partial(\mathsf{C}, \text{Ↄ}),$$

$$\wp(\mho\tau_3, \nabla\tau_2) = \partial(\mathsf{C}, \text{Ↄ}).$$

Again utilizing the P-property, we get

$$\partial(\mho\tau_2, \mho\tau_3) = \mathscr{H}(\nabla\tau_1, \nabla\tau_2).$$

Also, the pair of mappings $(\mho, \nabla)$ satisfies φ_M-proximal-contraction, we obtain

$$\partial(\mho\tau_2, \mho\tau_3) \leq \varphi\mathbb{M}(\tau_2, \tau_3, \tau_1, \tau_2),$$

where

$$\begin{aligned}\mathbb{M}(\tau_2,\tau_3,\tau_1,\tau_2) &\leq \max\left\{\ \partial(\mho\tau_2,\mho\tau_3),\frac{\wp(\mho\tau_1,\nabla\tau_3)-\pi_*(\mho\tau_1,\nabla\tau_3)\wp(\mho\tau_2,\nabla\tau_3)}{\pi_*(\mho\tau_1,\nabla\tau_3)},\wp^*(\mho\tau_3,\nabla\tau_2)\ \ )\right\}\\ &\leq \max\left\{\begin{array}{c}\partial(\mho\tau_2,\mho\tau_3),\frac{\pi_*(\mho\tau_1,\nabla\tau_3)[\partial(\mho\tau_1,\mho\tau_2)+\wp(\mho\tau_2,\nabla\tau_3)]-\pi_*(\mho\tau_1,\nabla\tau_3)\wp(\mho\tau_2,\nabla\tau_3)}{\pi_*(\mho\tau_1,\nabla\tau_3)},\\ \wp(\mho\tau_3,\nabla\tau_2)-\partial(\mathsf{C},\backepsilon)\end{array}\right\}\\ &\leq \max\left\{\ \partial(\mho\tau_2,\mho\tau_3),\frac{\pi_*(\mho\tau_1,\nabla\tau_3)\partial(\mho\tau_1,\mho\tau_2)}{\pi_*(\mho\tau_1,\nabla\tau_3)},0\ \right\}\\ &\leq \max\left\{\partial(\mho\tau_2,\mho\tau_3),\partial(\mho\tau_1,\mho\tau_2),0\right\}.\end{aligned}$$

That is

$$\mathbb{M}(\tau_2,\tau_3,\tau_1,\tau_2)\leq \max\left\{\partial(\mho\tau_2,\mho\tau_3),\partial(\mho\tau_1,\mho\tau_2)\right\}.$$

If $\max\left\{\partial(\mho\tau_2,\mho\tau_3),\partial(\mho\tau_1,\mho\tau_2)\right\}=\partial(\mho\tau_2,\mho\tau_3)$, then the above inequality implies

$$\partial(\mho\tau_2,\mho\tau_3)<\varphi\partial(\mho\tau_2,\mho\tau_3),$$

which is a contradiction. Thus we conclude that

$$\max\left\{\partial(\mho\tau_2,\mho\tau_3),\partial(\mho\tau_1,\mho\tau_2)\right\}=\partial(\mho\tau_1,\mho\tau_2).$$

This process could be continued further. Having chosen $\tau_n\in\mathsf{C}_0$, it is clear that there exists an element $\tau_{n+1}\in\mathsf{C}_0$. Since, τ_{n+1} is a member of $\nabla(\mathsf{C}_0)$ which is contained in $\backepsilon_0$ and C_0 is contained in $\mho(\mathsf{C}_0)$ such that

$$\wp(\mho\tau_n,\nabla\tau_{n-1})=\partial(\mathsf{C},\backepsilon),$$

$$\wp(\mho\tau_{n+1},\nabla\tau_n)=\partial(\mathsf{C},\backepsilon).$$

Making use of the property P, we obtain the following

$$\partial(\mho\tau_n,\mho\tau_{n+1})=\mathscr{H}(\nabla\tau_n,\nabla\tau_{n-1})$$

and

$$\partial(\mho\tau_n,\mho\tau_{n+1})\leq\varphi\mathbb{M}(\tau_n,\tau_{n+1},\tau_{n-1},\tau_n),$$

where

$$\begin{aligned}\mathbb{M}(\tau_n,\tau_{n+1},\tau_{n-1},\tau_n) &\leq \max\left\{\ \partial(\mho\tau_n,\mho\tau_{n+1}),\frac{\wp(\mho\tau_{n-1},\nabla\tau_{n+1})-\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})\wp(\mho\tau_n,\nabla\tau_{n+1})}{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})},\wp^*(\mho\tau_{n+1},\nabla\tau_n)\ \right\}\\ &\leq \max\left\{\begin{array}{c}\partial(\mho\tau_n,\mho\tau_{n+1}),\wp(\mho\tau_{n+1},\nabla\tau_n)-\partial(\mathsf{C},\backepsilon),\\ \frac{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})[\partial(\mho\tau_{n-1},\mho\tau_n)+\wp(\mho\tau_n,\nabla\tau_{n+1})]-\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})\wp(\mho\tau_n,\nabla\tau_{n+1})}{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})}\end{array}\right\}\\ &\leq \max\left\{\ \partial(\mho\tau_n,\mho\tau_{n+1}),0,\frac{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})\partial(\mho\tau_{n-1},\mho\tau_n)}{\pi_*(\mho\tau_{n-1},\nabla\tau_{n+1})}\ \right\}\\ &\leq \max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}),0,\partial(\mho\tau_{n-1},\mho\tau_n),\partial(\mho\tau_n,\mho\tau_{n+1})\right\}.\end{aligned}$$

Hence,

$$\mathbb{M}(\tau_n,\tau_{n+1},\tau_{n-1},\tau_n)\leq\max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}),\partial(\mho\tau_{n-1},\mho\tau_n)\right\}.$$

If $\max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}),\partial(\mho\tau_{n-1},\mho\tau_n)\right\}=\partial(\mho\tau_n,\mho\tau_{n+1})$, then the above inequality implies

$$\partial(\mho\tau_n,\mho\tau_{n+1})<\varphi\partial(\mho\tau_n,\mho\tau_{n+1}),$$

which is a contradiction. Hence, we have

$$\max\left\{\partial(\mho\tau_n,\mho\tau_{n+1}),\partial(\mho\tau_{n-1},\mho\tau_n)\right\}=\partial(\mho\tau_{n-1},\mho\tau_n).$$

Keeping with the same pattern, we assert that

$$\partial(\mho\tau_n,\mho\tau_{n+1})<\varphi^n\partial(\mho\tau_0,\mho\tau_1).$$

Now, we show that $\{\mho\tau_n\}$ is a Cauchy sequence. Since $(\mathbb{J},\partial)$ is a complete extended $b-$metric space, for all $n,m\in\mathbb{N}$ with $n<m$, we have

$$\begin{aligned}\partial(\mho\tau_n,\mho\tau_m) &\leq \pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_n,\mho\tau_{n+1})+\pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_{n+1},\mho\tau_m)\\
&\leq \pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_n,\mho\tau_{n+1})+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\partial(\mho\tau_{n+1},\mho\tau_{n+2})\\
&+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\partial(\mho\tau_{n+2},\mho\tau_m)\\
&\leq \pi(\mho\tau_n,\mho\tau_m)\partial(\mho\tau_n,\mho\tau_{n+1})+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\partial(\mho\tau_{n+1},\mho\tau_{n+2})+\ldots\\
&+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)...\pi(\mho\tau_{m-2},\mho\tau_m)\pi(\mho\tau_{m-1},\mho\tau_m)\partial(\mho\tau_{m-1},\mho\tau_m)\\
&\leq \pi(\mho\tau_n,\mho\tau_m)\varphi^n\partial(\mho\tau_0,\mho\tau_1)+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)\varphi^{n+1}\partial(\mho\tau_0,\mho\tau_1)+...\\
&+\pi(\mho\tau_n,\mho\tau_m)\pi(\mho\tau_{n+1},\mho\tau_m)...\pi(\mho\tau_{m-2},\mho\tau_m)\pi(\mho\tau_{m-1},\mho\tau_m)\varphi^{m-1}\partial(\mho\tau_0,\mho\tau_1)\\
&=\partial(\mho\tau_0,\mho\tau_1)\sum_{i=1}^{m-1}\varphi^i\prod_{j=1}^{i}\pi(\mho\tau_i,\mho\tau_m).\end{aligned}$$

Assume that

$$S_n=\sum_{i=1}^{n}\varphi^i\prod_{j=1}^{i}\pi(\mho\tau_i,\mho\tau_m).$$

We can write

$$\partial(\mho\tau_n,\mho\tau_m)\leq[S_{m-1}-S_n]. \tag{3.1}$$

Using the ratio test, we get

$$a_i=\varphi^i\prod_{j=1}^{i}\pi(\mho\tau_i,\mho\tau_m),\ \text{where}\ \frac{a_{i+1}}{a_i}<\frac{1}{k}.$$

Further taking the limit as $n\to\infty$ in inequality (3.1), we infer

$$\lim_{n\to\infty}\partial(\mho\tau_n,\mho\tau_m)=0,$$

which implies that $\{\mho\tau_n\}$ is a Cauchy sequence in a complete extended $b-$metric space $(\mathbb{J},\partial)$, hence it is convergent. Suppose that $\{\mho\tau_n\}$ converges to some τ^* in $\mathtt{C}$ (as the set $\mathtt{C}$ is closed), assuring that the sequence $\{\tau_n\}\subseteq\mathtt{C}_0$, since $\{\tau_n\}\to\tau^*$. Therefore, $(\mho,\nabla)$ is a pair of continuous mappings, so one can write

$$\partial(\mho\tau^*,\nabla\tau^*)=\partial(\mathtt{C},\ni).$$

Therefore, τ^* is a coincidence best proximity point of the pair of mappings $(\mho,\nabla)$.
Uniqueness: Suppose that there are two distinct coincidence best proximity points of $(\mho,\nabla)$ with $\tau\neq\tau^*$. Then $q=\partial(\mho\tau,\mho\tau^*)>0$. Since $\wp(\mho\tau,\nabla\tau)=\wp(\mho\tau^*,\nabla\tau^*)=\partial(\mathtt{C},\ni)$, using the P-property, we conclude that $q=\mathscr{H}(\nabla\tau,\nabla\tau^*)$. Moreover the pair of mappings $(\mho,\nabla)$ satisfies φ_M-proximal-contraction, then we obtain $q\leq\varphi q$.
Hence, $\varphi\geq1$. Since, $\varphi\leq1$, we conclude that $\varphi=1$, which is again a contradiction. Hence the uniqueness is certified. □

Corollary 3.6. *Let $\nabla:\mathtt{C}\to\mathscr{CB}(\ni)$ and $\pi_*:\mathbb{J}\times\mathbb{J}\to[1,\infty)$ be mappings, where $\mathtt{C}$ is closed subset of an extended $b-$metric space $(\mathbb{J},\partial)$ and the pair $(\mathtt{C},\ni)$ satisfies the P-property with $\nabla(\mathtt{C}_0)\subseteq\ni_0$. Assume that the continuous mapping ∇ satisfies $\varphi_\nabla-$proximal contraction and the following expression holds*

$$\lim_{n,m\to\infty}\pi(\tau_n,\tau_m)<\frac{1}{k},\ \textit{where}\ k\in(0,1).$$

Then there exists a unique best proximity point of the mapping ∇.

Proof. Taking into account the identity mapping $\mho=I_{\mathtt{C}}$ ($\mho$ is the identity on $\mathtt{C}$), the proof can be completed on the similar lines as in Theorem 3.5.

□

Example 3.7. *Let $\mathbb{J}=\{0,1,2,3,4\}$ and consider the function $\partial:\mathbb{J}\times\mathbb{J}\to[0,\infty)$ defined as $\partial(\tau,\tau^*)=\partial(\tau^*,\tau)$ and $\partial(\tau,\tau)=0$, where*

∂	0	1	2	3	4
0	0	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{13}$
1	$\frac{1}{7}$	0	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$
2	$\frac{1}{9}$	$\frac{1}{5}$	0	$\frac{1}{4}$	$\frac{1}{3}$
3	$\frac{1}{11}$	$\frac{1}{6}$	$\frac{1}{4}$	0	$\frac{1}{2}$
4	$\frac{1}{13}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{2}$	0

.

Take $\pi_* : \mathbb{J} \times \mathbb{J} \to [1,\infty)$ to be symmetric and defined as $\pi_*(\tau,\tau^*) = 2\tau + 3\tau^* + 1$. It is straightforward to verify that $(\mathbb{J},\partial)$ is an extended b-metric space. Let $\complement = \{0,1\}$ and $\daleth = \{2,3,4\}$ be nonempty subsets of $\mathbb{J}$. After simple calculations, we attain $\partial(\complement,\daleth) = \frac{1}{7}$. It can also be shown that the pair $(\complement,\daleth)$ satisfies the P-property, with $\complement_0 = \complement$ and $\daleth_0 = \daleth$.
Now, consider the mappings $\nabla : \complement \to \mathscr{CB}(\daleth)$ and $\mho : \complement \to \complement$ defined as follows

$$\nabla(\tau) = \begin{cases} \{2,3\}, & \text{if } \tau = 0, \\ \{3,4\}, & \text{if } \tau = 1, \end{cases}$$

and

$$\mho(\tau) = \begin{cases} 1, & \text{if } \tau = 0, \\ 0, & \text{if } \tau = 1. \end{cases}$$

Clearly, $\nabla(\complement_0) \subseteq \daleth_0$ and $\complement_0 \subseteq \mho(\complement_0)$. We now show that the pair $(\mho,\nabla)$ satisfies φ_M-proximal-contraction

$$\mathscr{H}(\nabla(\tau),\nabla(\tau^*)) \leq \varphi \mathbb{M}(u,v,\tau,\tau^*),$$

for all $u,v,\tau,\tau^* \in \complement$, where $\mathbb{M}(u,v,\tau,\tau^*)$ is as defined in Definition 3.1. Since

$$\wp(\mho 0,\nabla 1) = \partial(\complement,\daleth)$$

$$\wp(\mho 1,\nabla 0) = \partial(\complement,\daleth),$$

and

$$\wp(\mho 0,\nabla 0) = \partial(\complement,\daleth)$$

$$\wp(\mho 1,\nabla 1) = \partial(\complement,\daleth),$$

then we analyze the following cases:

Case (i): If $\wp(\mho 0,\nabla 1) = \wp(\mho 1,\nabla 0) = \partial(\complement,\daleth)$, let $u = 0, v = 1, \tau = 0, \tau^* = 1$, then

$$\mathscr{H}(\nabla(0),\nabla(1)) = \mathscr{H}(\{2,3\},\{3,4\}) = \frac{1}{4}.$$

Also,

$$\mathbb{M}(0,1,0,1) = \max\left\{\frac{1}{7}, \frac{\wp(1,\{3,4\}) - \pi_*(1,\{3,4\})\wp(0,\{3,4\})}{\pi_*(1,\{3,4\})}, \frac{1}{4}\right\} = \frac{1}{4}.$$

Thus,

$$\mathscr{H}(\nabla(0),\nabla(1)) \leq \varphi \mathbb{M}(0,1,0,1),$$

for $\varphi \in [0,1)$.

Case (ii): If $\wp(\mho 0,\nabla 0) = \wp(\mho 1,\nabla 1) = \partial(\complement,\daleth)$, let $u = 0, v = 1, \tau = 1, \tau^* = 0$, then

$$\mathscr{H}(\nabla(1),\nabla(0)) = \mathscr{H}(\{3,4\},\{2,3\}) = \frac{1}{4}.$$

Similarly,

$$\mathbb{M}(0,1,1,0) = \frac{1}{4}.$$

Thus,

$$\mathscr{H}(\nabla(1),\nabla(0)) \leq \varphi \mathbb{M}(0,1,1,0),$$

for $\varphi \in [0,1)$. Since $\mathscr{H}(\nabla(\tau),\nabla(\tau^*)) \leq \varphi \mathbb{M}(u,v,\tau,\tau^*)$ holds in all cases, it is concluded that $(\mho,\nabla)$ satisfies φ_M-proximal-contraction. Finally, $\wp(\mho 0,\nabla 0) = \partial(\complement,\daleth)$, so 0 is the unique coincidence best proximity point of $(\mho,\nabla)$. Therefore, Theorem 3.5 is validated.

Remark 3.8. *It may be noted that the pair of mapping $(\mho,\nabla)$ does not satisfy the proximal conditions of the main results in [20, 21] within the realm of extended b-metric spaces. Hence the results of [20, 21] can not be applied on the pair $(\mho,\nabla)$, showing that under the context of extended b−metric spaces, our generalizations are applicable and generalizations in the true sense.*

4. Coincidence Proximal Point Results for Single-Valued Mappings

This section is devoted to discussing some of best coincidence proximity point theorems for single-valued mappings in the framework of extended $b-$metric spaces.

Definition 4.1. *Given $\nabla : \complement \to \eth$ and $\mho : \complement \to \complement$ be mappings. The pair $(\mho, \nabla)$ is said to be a φ_S-proximal-contraction if there exists a real number $\varphi \in [0,1)$ such that*

$$\left.\begin{array}{c} \partial(\mho u, \nabla\tau) = \partial(\complement, \eth) \\ \partial(\mho v, \nabla\tau^*) = \partial(\complement, \eth) \end{array}\right\} \text{ implies } \partial(\nabla\tau, \nabla\tau^*) \leq \varphi\, \mathbb{M}(u, v, \tau, \tau^*),$$

where

$$\mathbb{M}(u, v, \tau, \tau^*) = \max\left\{ \partial(\mho u, \mho v), \frac{\partial(\mho\tau, \nabla v) - \pi(\mho\tau, \nabla v)\partial(\mho u, \nabla v)}{\pi(\mho\tau, \nabla v)}, \partial^*(\mho v, \nabla u) \right\},$$

for all u, v, τ, τ^* in $\complement$.

Definition 4.2. *Given $\nabla : \complement \to \eth$ and $\mho : \complement \to \complement$ be mappings. The pair $(\mho, \nabla)$ is said to be a $\varphi_{S^*}-$proximal-contraction if there exists a real number $\varphi \in [0,1)$ such that*

$$\left.\begin{array}{c} \partial(u, \nabla\tau) = \partial(\complement, \eth) \\ \partial(v, \nabla\tau^*) = \partial(\complement, \eth) \end{array}\right\} \text{ implies } \partial(\nabla\tau, \nabla\tau^*) \leq \varphi\mathbb{M}(u, v, \tau, \tau^*),$$

where

$$\mathbb{M}(u, v, \tau, \tau^*) = \max\left\{\partial(u, v), \frac{\partial(\tau, \nabla v) - \pi(\tau, \nabla v)\partial(u, \nabla v)}{\pi(\tau, \nabla v)}, \partial^*(v, \nabla u)\right\},$$

for all u, v, τ, τ^* in $\complement$.
Note that, if we take $\mho = I_{\complement}$ ($\mho$ as an identity mapping on $\complement$), then every φ_S-proximal-contraction will reduce to a $\varphi_{S^*}-$proximal-contraction.

Theorem 4.3. *Let $\nabla : \complement \to \eth$, $\mho : \complement \to \complement$ and $\pi : \mathbb{J} \times \mathbb{J} \to [1, \infty)$ be mappings, where $\complement$ is closed subset of an extended $b-$metric space $(\mathbb{J}, \partial)$ and the pair $(\complement, \eth)$ satisfies the P-property with $\nabla(\complement_0) \subseteq \eth_0$ and $\complement_0 \subseteq \mho(\complement_0)$. Assume that the pair of continuous mappings $(\mho, \nabla)$, where $\mho$ is one-to-one satisfies φ_S-proximal-contraction and the following statement holds*

$$\lim_{n,m\to\infty} \pi(\mho\tau_n, \mho\tau_m) < \frac{1}{k}, \text{ where } k \in (0,1).$$

Then, there exists a coincidence best proximity point of the pair $(\mho, \nabla)$.

Proof. Since every single valued mapping is multi-valued mapping, the remaining proof is the same as Theorem 3.5. □

Corollary 4.4. *Let $\nabla : \complement \to \eth$ and $\pi : \mathbb{J} \times \mathbb{J} \to [1, \infty)$ be mappings, where $\complement$ is closed subset and the pair $(\complement, \eth)$ satisfies the P-property with $\nabla(\complement_0) \subseteq \eth_0$. Assume that the continuous mapping ∇ satisfies $\varphi_{S^*}-$proximal-contraction and the following statement holds*

$$\lim_{n,m\to\infty} \pi(\tau_n, \tau_m) < \frac{1}{k}, \text{ where } k \in (0,1).$$

Then, there exists a best proximity point of the mapping ∇.

Proof. If we take identity mapping $\mho = I_{\complement}$ ($\mho$ is identity on $\complement$), the remaining proof is the same as Theorem 4.3.

□

Corollary 4.5. *Let $(\mathbb{J}, \partial)$ be an extended $b-$metric space and $\nabla : \mathbb{J} \to \mathbb{J}$, $\pi : \mathbb{J} \times \mathbb{J} \to [1, \infty)$ be mappings. If $\lim_{n,m\to\infty} \pi(\tau_n, \tau_m) < \frac{1}{k}$, where $k \in (0,1)$, the mapping ∇ is continuous and there exists a real number $\varphi \in [0,1)$ such that the following $\varphi_{S^*}-$proximal-type contraction is satisfied*

$$\partial(\nabla\tau, \nabla\tau^*) \leq \varphi\, \mathbb{M}(u, v, \tau, \tau^*),$$

where

$$\mathbb{M}(u, v, \tau, \tau^*) = \max\left\{\partial(u, v), \frac{\partial(\tau, \nabla v) - \pi(\tau, \nabla v)\partial(u, \nabla v)}{\pi(\tau, \nabla v)}, \partial^*(v, \nabla u), \frac{\partial(u, \nabla\tau^*) - \pi(u, \nabla\tau^*)\partial(v, \nabla\tau^*)}{\pi(u, \nabla\tau^*)}\right\},$$

for all u, v, τ, τ^* in $\mathbb{J}$, then there exists a unique fixed point of the mapping ∇ in $(\mathbb{J}, \partial)$.

Example 4.6. *Let $\mathbb{J}=\{5,6,7,8,9,10\}$ be a complete extended b-metric space $(\mathbb{J},\partial)$, where the distance function is defined as*

$$\partial(\tau,\tau^*)=|\tau-\tau^*|^3, \text{ for all } \tau,\tau^*\in\mathbb{J}.$$

Additionally, let $\mathsf{C}=\{5,7,9\}$ and $\backepsilon=\{6,8,10\}$ be two non-empty subsets of $\mathbb{J}$. It can be verified through straightforward calculations that $\partial(\mathsf{C},\backepsilon)=1$, and the pair $(\mathsf{C},\backepsilon)$ satisfies the P-property. Here $\mathsf{C}_0=\mathsf{C}$, $\backepsilon_0=\backepsilon$, and we define

$$\pi(\tau,\tau^*)=12\tau^2+10\tau^{*2}+5.$$

Next, we define the mappings $\nabla:\mathsf{C}\to\backepsilon$ and $\mho:\mathsf{C}\to\mathsf{C}$ as follows

$$\nabla(\tau)=\begin{cases}6, & \text{if } \tau\in\{5,7\},\\ 8, & \text{if } \tau=9,\end{cases}$$

and

$$\mho(\tau)=\begin{cases}5, & \text{if } \tau=7,\\ 7, & \text{if } \tau=9,\\ 9, & \text{if } \tau=5.\end{cases}$$

Clearly, $\nabla(\mathsf{C}_0)\subseteq\backepsilon_0$ and $\mho(\mathsf{C}_0)\subseteq\mathsf{C}_0$. Now, we verify that the pair $(\mho,\nabla)$ satisfies the φ_S-proximal-contraction condition

$$\partial(\nabla\tau,\nabla\tau^*)\le\varphi\mathbb{M}(u,v,\tau,\tau^*), \text{ for all } u,v,\tau,\tau^*\in\mathsf{C}.$$

Let us consider the case where

$$\partial(\mho5,\nabla7)=\partial(\mathsf{C},\backepsilon),\quad \partial(\mho7,\nabla9)=\partial(\mathsf{C},\backepsilon),$$

and take $u=5$, $v=7$, $\tau=7$, $\tau^*=9$. After routine calculations, we derive that

$$\partial(\nabla7,\nabla9)=\partial(6,8)=8,$$

and

$$\mathbb{M}(5,7,7,9)=\max\left\{\ \partial(\mho5,\mho7),\tfrac{\partial(\mho7,\nabla7)-\pi(\mho7,\nabla7)\partial(\mho5,\nabla7)}{\pi(\mho7,\nabla7)},\partial^*(\mho7,\nabla5)\ \right\}.$$

Breaking this down

$$\partial(\mho5,\mho7)=\partial(5,9)=64,$$

$$\frac{\partial(\mho7,\nabla7)-\pi(\mho7,\nabla7)\partial(\mho5,\nabla7)}{\pi(\mho7,\nabla7)}=\frac{\partial(7,6)-\pi(7,6)\partial(5,6)}{\pi(7,6)}=\frac{1-(615)\partial(5,6)}{615}=\frac{-3097}{615},$$

and

$$\partial^*(\mho7,\nabla5)=\partial(9,6)=27.$$

Thus, we have

$$\mathbb{M}(5,7,7,9)=\max\{64,\frac{-3097}{615},27\}=64.$$

Finally, we attain

$$\partial(\nabla7,\nabla9)=8\le\varphi\mathbb{M}(5,7,7,9),$$

which holds for $\varphi=\frac{1}{8}$, thereby satisfying the φ_S-proximal-contraction condition. Therefore, $\partial(\mho5,\nabla5)=\partial(\mathsf{C},\backepsilon)$, it is concluded that 5 is a coincidence best proximity point of the mappings $(\mho,\nabla)$. This confirms that all the conditions of Theorem 4.3 are fulfilled.

5. Applications

Fundamental instruments in science, including basic and differential conditions, significantly influence various logical domains. These conditions illustrate relationships regarding rates of advancement or accumulation, rendering them essential for representing dynamic systems. Differential conditions highlight the aspects of characteristics that are subject to change, whereas essential conditions enhance these analyses by incorporating cumulative effects or boundary conditions. Fundamental and differential conditions are essential in mathematics for the advancement of hypotheses in applied science, control systems, and optimization. They constitute the foundation of mathematical inquiry and computational modelling. In practical contexts, their applications encompass a remarkable variety: from assessing meteorological anomalies and examining liquid substances to simulating biological structures, such as disease propagation, and engineering designs like bridges and aircraft. Furthermore, the partial plans of these scenarios simplify the demonstration of processes involving memory effects, such as the transformation or uneven distribution of materials in chemistry and physical science. Recently, necessary and differential conditions have gained prominence for their capacity to elucidate phenomena across various domains, including epidemiology, finance, and energy. In 2021 Khan et al. [22] employed partial differential equations to illustrate the transmission of Coronavirus, integrating the effects of isolation on individuals with diabetes. Abdou [23] studied nonlinear fragmentary differential conditions in symmetrical measurement spaces in 2023. He used a fixed-direct hypothesis to show that complex frameworks were possible. These models demonstrate the sufficiency of fundamental and differential conditions as quantitative tools for evaluating and resolving intricate problems in science and engineering. Fundamental and differential equations are essential numerical tools for characterizing and analyzing systems that evolve over time or space. Differential conditions show how quickly things are changing in different frameworks, while fundamental conditions show how effects add up across a domain, which is common in boundary value or inverse problems. Collectively, they present a theoretical rationale for valuing diverse physical, natural, and financial characteristics. The fixed-point hypothesis has emerged as a crucial tool for analyzing these situations, providing robust methods to ascertain the existence and uniqueness of solutions. Experts transform necessary and sufficient conditions into corresponding fixed-point problems, utilizing fixed-point theorems such as Banach's and Schauder's principles to verify the solveability of these problems. This procedure has demonstrated considerable progress in both theoretical and practical mathematics. Hamdan and Kechil [24] illustrate through numerical simulations that fractional-order models can accurately characterize the dissemination of COVID-19, providing insights into the effects of different control methods. Their research indicates that fractional-order differential equations are effective instruments for comprehending and forecasting the dynamics of infectious diseases, especially those characterized by intricate transmission patterns such as COVID-19. Abdou [23] employed fixed-direct hypotheses in symmetrical measurement spaces to address nonlinear fragmentary differential equations and manage intricate limit conditions. Cabada and Hamdi [25] examined essential limit value problems, highlighting the importance of fixed-point theory in nonlinear fractional differential equations. These commitments highlight the importance of fundamental and differential conditions in the dynamics of numerical speculations and the pursuit of viable issues. Their versatility and significance guarantee their essential role in both fundamental and applied research, propelling advancements across science, engineering, and various disciplines.

Let $\mathbb{J}=\mathscr{C}[\sigma_1,\sigma_2]$ be a set of all real valued continuous functions on $[\sigma_1,\sigma_2]$. Define the mappings $\partial:\mathbb{J}\times\mathbb{J}\to[0,+\infty)$ by

$$\partial(\tau,\tau^*)=\sup_{c\in[\sigma_1,\sigma_2]}|\tau(c)-\tau^*(c)|^p,$$

for all $\tau,\tau^*\in\mathbb{J}$ and

$$e(\tau,\tau^*)=r+\tau+\tau^*,\;\; p\geq 2,\;\; r>2.$$

Then, $(\mathbb{J},\partial)$ is a complete extended b-metric space. Consider the Fredholm integral equation given by

$$\tau(\varpi)=\varphi(\varpi)+\sqsupset\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau(\kappa))d\kappa, \tag{5.1}$$

where $t\in[\sigma_1,\sigma_2]$, $|\sqsupset|>0$ and $\mathscr{I}:[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]\times\mathbb{J}\to\mathfrak{R}$ and $\varphi:[\sigma_1,\sigma_2]\to\mathfrak{R}$ are continuous functions. Let $\nabla:\mathbb{J}\to\mathbb{J}$ be an integral operator defined by

$$\nabla\tau(\varpi)=f(\varpi)+\sqsupset\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau(\kappa))d\kappa. \tag{5.2}$$

Then $\tau(\varpi)$ is a fixed point of ∇ if and only it is a solution of the Fredholm integral equation (5.1).
We now offer the following subsequent theorem to establish the existence of a solution to the Fredholm integral equation (5.1).

Theorem 5.1. *Let $\nabla:\mathbb{J}\to\mathbb{J}$ be an integral operator defined in (5.2). Suppose that the following assumptions hold*

1. *for any $\tau_0\in\mathbb{J}$, $\lim_{n,m\to\infty}\pi(\nabla^n\tau_0,\nabla^m\tau_0)<\frac{1}{\varphi}$, where $\varphi=\frac{1}{2^p}$*
2. *for any $\tau,\tau^*\in\mathbb{J},\tau\neq\tau^*$, $\mathscr{I}:[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]\times\mathbb{J}\to\mathfrak{R}$ satisfies*

$$|\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|\leq\xi(\varpi,\kappa)\mid\tau(\kappa)-\tau^*(\kappa))\mid,$$

where $(\kappa,t)\in[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]$ and $\xi:[\sigma_1,\sigma_2]\times[\sigma_1,\sigma_2]\to\mathfrak{R}$ is a continuous function satisfying

$$\sup_{t\in[\sigma_1,\sigma_2]}\int_{\sigma_1}^{\sigma_2}\xi^p(\varpi,\kappa)|\tau(\kappa)-\tau^*(\kappa)|^p d\kappa\leq\frac{1}{2^p|\sqsupset|^p(\sigma_2-\sigma_1)^{p-1}}\mathbb{M}(\varpi,\kappa,\tau,\tau^*),$$

where $\mathbb{M}(\varpi,\kappa,\tau,\tau^*)$ is defined as in Definition 4.2.
Then, the integral operator ∇ has a unique solution in $\mathbb{J}$.

Proof. Let $\tau_0\in\mathbb{J}$ and define a sequence $\{\tau_n\}$ in $\mathbb{J}$ by $\tau_n=\nabla^n\tau_0,\;\; n\geq 1$. From (5.2), we obtain

$$\tau_{n+1}=\nabla\tau_n(\varpi)=L(\varpi)+\sqsupset\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau_n(\kappa))d\kappa.$$

Let $q>1$ be a constant with $\frac{1}{p}+\frac{1}{q}=1$. By the Holder's inequality, we speculate that

$$\begin{aligned}|\nabla\tau(\varpi)-\nabla\tau^*(\varpi)|^p &=\left|\sqsupset\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau(\kappa))d\kappa-\sqsupset\int_{\sigma_1}^{\sigma_2}\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))d\kappa\right|^p\\ &\leq\left(\int_{\sigma_1}^{\sigma_2}|\sqsupset||\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|d\kappa\right)^p\\ &\leq\left(\int_{\sigma_1}^{\sigma_2}|\sqsupset|^q d\kappa\right)^{\frac{p}{q}}\left(\left(\int_{\sigma_1}^{\sigma_2}|\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|^p d\kappa\right)^{\frac{1}{p}}\right)^p\\ &=|\sqsupset|^p(\sigma_2-\sigma_1)^{p-1}\left(\int_{\sigma_1}^{\sigma_2}|\mathscr{I}(\varpi,\kappa,\tau(\kappa))-\mathscr{I}(\varpi,\kappa,\tau^*(\kappa))|^p d\kappa\right)\\ &\leq|\sqsupset|^p(\sigma_2-\sigma_1)^{p-1}\int_{\sigma_1}^{\sigma_2}\xi^p(\varpi,\kappa)|\tau(\kappa)-\tau^*(\kappa)|^p d\kappa.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\partial(\nabla\tau,\nabla\tau^*) &= \sup_{t\in[\sigma_1,\sigma_2]}|\nabla\tau(\varpi)-\nabla\tau^*(\varpi)|^p \\ &\leq |\beth|^p(\sigma_2-\sigma_1)^{p-1}\sup_{t\in[\sigma_1,\sigma_2]}\left[\int_{\sigma_1}^{\sigma_2}\xi^p(\varpi,\kappa)|\tau(\kappa)-\tau^*(\kappa)|^p d\kappa\right] \\ &\leq \frac{1}{2^p}\mathscr{M}(\varpi,\kappa,\tau,\tau^*).\end{aligned}$$

Setting $\varphi=\frac{1}{2^p}$, we obtain

$$\partial(\nabla\tau,\nabla\tau^*)\leq\varphi\mathscr{M}(\varpi,\kappa,\tau,\tau^*).$$

Thus, all the conditions of Corollary 4.5 are satisfied and hence ∇ possesses a unique fixed point in $\mathbb{J}$, which means the integral operator ∇ has a unique solution in $\mathbb{J}$.

□

5.1. An application to the solution of a second-order differential equation

The existence of a solution for the preceding second-order boundary value problem is manifested in this section

$$\begin{cases} u''(\varpi)=W\big(\varpi,u(\varpi),\kappa(\varpi)\big), & \varpi\in[0,1]; \\ u(0)=u_0, u(1)=u_1, \end{cases} \tag{5.3}$$

where $W:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
Initially, consider the space $\mathbb{J}=C(\mathbb{Q})(\mathbb{Q}=[0,1],\mathbb{R})$ of continuous functions defined on $\mathbb{Q}$. Obviously this space with metric given by

$$\partial(u,v)=\sup_{c\in[\sigma_1,\sigma_2]}|u(c)-v(c)|^p,$$

for all $u,v\in\mathbb{J}$ and $e(u,v)=r+u+v, p\geq 2, r>2$ is a complete extended b-metric space.

Theorem 5.2. *Consider the boundary value problem given in (5.3). Suppose that for any $\tau_0\in\mathbb{J}$, $\lim_{n,m\to\infty}\pi(\nabla^n\tau_0,\nabla^m\tau_0)<\frac{1}{\varphi}$, where $\varphi=\frac{1}{2^p}$, and for suitable value of $\beth$ if $|\beth|^p\leq\frac{1}{2^p}$, then the second-order boundary value problem given in (5.3) has a unique solution.*

Proof. The boundary value problem given in (5.3) is equivalent to the second kind Fredholm integral equation

$$u(\varpi)=L(\varpi)+\beth\int_0^1\mathfrak{D}(\varpi,\kappa)u(\kappa)d\kappa,\quad \varpi\in[0,1], \tag{5.4}$$

in which $L(\varpi)=u_0+\varpi(u_1-u_0)$ and $\mathfrak{D}(\varpi,\kappa)$ is the Green's function, given by

$$\mathfrak{D}(\varpi,\kappa)=\begin{cases}\kappa(1-\kappa) & 0\leq\kappa\leq\varpi; \\ \varpi(1-\kappa) & \varpi\leq\kappa\leq 1.\end{cases} \tag{5.5}$$

Note that if $u\in C(\mathbb{Q})$ is a fixed point of ∇, then u is a solution of (5.4), consequently a solution of (5.3).
Let $u,v\in C(\mathbb{Q})$ and $q>1$ be a constant with $\frac{1}{p}+\frac{1}{q}=1$. By the Hölder's inequality, we acquire that

$$\begin{aligned}|\nabla u(\varpi)-\nabla v(\varpi)|^p &= \left|\beth\int_0^1\mathfrak{D}(\varpi,\kappa)u(\kappa)d\kappa-\beth\int_0^1\mathfrak{D}(\varpi,\kappa)v(\kappa)d\kappa\right|^p \\ &\leq\left(\int_0^1|\beth||\mathfrak{D}(\varpi,\kappa)u(\kappa)-\mathfrak{D}(\varpi,\kappa)v(\kappa)|d\kappa\right)^p \\ &\leq\left(\int_0^1|\beth|^q d\kappa\right)^{\frac{p}{q}}\left(\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)u(\kappa)-\mathfrak{D}(\varpi,\kappa)v(\kappa)|^p d\kappa\right)^{\frac{1}{p}}\right)^p \\ &=|\beth|^p\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)u(\kappa)-\mathfrak{D}(\varpi,\kappa)v(\kappa)|^p d\kappa\right) \\ &\leq|\beth|^p|u(\varpi)-v(\varpi)|^p\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)|^p d\kappa\right).\end{aligned}$$

Then we conclude that

$$\begin{aligned}\partial(\nabla\tau,\nabla\tau^*) &= \sup_{t\in[\sigma_1,\sigma_2]}|\nabla\tau(\varpi)-\nabla\tau^*(\varpi)|^p \\ &\leq|\beth|^p\sup_{t\in[\sigma_1,\sigma_2]}|u(\varpi)-v(\varpi)|^p\left(\int_0^1|\mathfrak{D}(\varpi,\kappa)|^p d\kappa\right) \\ &\leq|\beth|^p\partial(u,v),\ \text{for any value of } p \text{ and utilizing (5.5)} \\ &\leq\frac{1}{2^p}\mathscr{M}(\varpi,\kappa,\tau,\tau^*).\end{aligned}$$

Fixing $\varphi = \frac{1}{2^p}$, we get

$$\partial(\nabla\tau, \nabla\tau^*) \leq \varphi \mathscr{M}(\varpi, \kappa, \tau, \tau^*).$$

Hence, we conclude that the proximal condition of Corollary 4.5 is satisfied, so ∇ has a unique fixed point, which is the solution of the integral equation (5.4). Consequently, the differential equation (5.3) has a solution.

□

6. Conclusion

This study elucidated the ideas of the proximal-contractions by developing some generalized proximal-contractions for both multivalued and single-valued mappings and demonstrated their applicability in extended b-metric spaces. The findings extended traditional proximity point theorems, guaranteeing the existence of unique coincidence-best proximity points under less stringent conditions. This framework streamlines current hypotheses and sets a basis for investigation in generalized metric structures, flexible mappings, and practical applications, presenting the substantial potential for advancing the theory of metric fixed points and the best proximity point results.

Some open problems for future research:

- Is it possible to apply the findings to real-world optimization problems, such as those in architecture or finance, where constraints inherently prompt a framework that incorporates proximal mappings?
- Is it possible to relax the continuity condition for $\mho$ and ∇ or substitute it with milder forms of continuity, such as lower semi-continuity or upper semi-continuity, while still obtaining comparable outcomes?
- Can the findings shown in this article be used to generate a solution to the following first-order periodic boundary value problem:

$$\begin{cases} \xi'(\tau) = f(\tau, \xi(\tau)), \\ \xi(0) = \xi(\nabla), \end{cases}$$

  where $\tau \in I = [0, \nabla]$, $\xi(\tau)$ is a real-valued function on I, and $f : I \times \mathbb{C} \to \mathbb{C}$ is a continuous function.
- Is it possible to explore effective numerical techniques to approximate optimal proximity points for large-scale data sets represented in an extended b-metric space?

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