Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Sayı: 16 Eylül 2008 ISSN - 1302 - 3055

SOLUTION OF THE DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH INDEX 3 USING DIFFERENTIAL TRANSFORM METHOD

Murat OSMANOĞLU* & Muhammet KURULAY* & Mustafa BAYRAM*

*Yildiz Technical University, Faculty of Arts and Sciences,Department of Mathematics,
34210-Davutpasa-Esenler-Istanbul, Turkey E-mail: msbayram@yildiz.edu.tr

Geliş tarihi: 05.06.2008 Kabul tarihi: 29.07.2008

ÖZET

Bu makalede, 3 indexli diferensiyel cebirsel denklemleri çözmek için diferensiyel transform yöntemini kullandık. İki farklı diferensiyel cebirsel denklem nümerik olarak çözüldü. Nümerik ve analitik çözümler karşılaştırıldı. Bu örnekler, diferensiyel transform yönteminin, diferensiyel cebirsel denklemlerin çözümü için uygun bir yöntem olduğunu gösterir. Çözümler için MAPLE bilgisayar cebiri programı kullanıldı.

Anahtar Kelimeler: *Diferensiyel Transform Yöntemi, Diferensiyel-Cebirsel denklem, Diferensiyel-cebirsel denklemlerin indexi, Kuvvet serisi, MAPLE.*

ABSTRACT

In this paper, we have applied the differential transform method to solve differential-algebraic equations with index 3. Two kind of differential-algebraic equations have been considered and solved numerically, then we compared numerical and analytical solution of the given equations. Examples were presented to show the ability of the method for differential-algebraic equations. We have used MAPLE computer algebra system to solve given problems[4].

Key Words: *Differential transform method, Differential-algebraic equation, Index of differential-algebraic equation, Power series, MAPLE.*

1. INTRODUCTION

The Differential transform method has been successfully used by Zhou [1] to solve a linear and nonlinear initial value problems in electric circuit analysis. Using one-dimensional differential transform, Chen and Ho [5] proposed a method to solve eigenvalue problems. The method has been applied to the partial differential equation[6,7,10], and the system of partial differential equation[11]. Hassan applied the differential transform method to solve eigenvalues and normalized eigenfunctions for a Sturm-Liouville eigenvalue problem[8,9]. The differential transform method has been extended to solve differential-difference equations by Arıkoglu[14]. Chen used the Differential transform method to predict the advective-dispersive transport problems[12]. The numerical solution of the differential-algebraic equation systems has been found using Differential transform method[13,15]. We have used the Differential transform method to solve differential-algebraic equation with index 3.

2. THE DIFFERENTIAL TRANSFORM METHOD

The differential transform of the k th derivate of function $y(x)$ in one dimensional is defined as follows:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

where $y(x)$ is original function and $Y(k)$ is transformed function and the differential inverse transform of $Y(k)$ is defined as

$$y(x)=\sum_{k=0}^{\infty}(x-x_0)^k Y(k) \tag{2.2}$$

From (2.1) and (2.2) is defined

$$y(x)=\sum_{k=0}^{\infty}(x-x_0)^k \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{2.3}$$

Equation (2.3) is obtained from Taylor series expansion at $x=x_0$. From the definitions of equations (2.1) and (2.2), it is easily proven that transformed functions comply with the basic mathematics operations shown in Table 1.

Original function	**Transformed function**
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=cw(x)$	$Y(k)=cW(k)$
$y(x)=dw/dx$	$Y(k)=(k+1)W(k+1)$
$y(x)=d^j w/dx^j$	$Y(k)=(k+1)(k+2)...(k+j)W(k+j)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{r=0}^{k}U(r)V(k-r)$
$y(x)=u_1(x)u_2(x)...u_n(x)$	$Y(k)=\sum_{r=0}^{r_1}\sum_{r=r_1}^{r_2}...\sum_{r=r_{n-1}}^{k}U_1(r)U_2(r_1-r)...U_n(k-r_{n-1})$
$y(x)=x^j$	$Y(k)=\delta(k-j)=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}, \quad \text{if} \quad x_0=0$

Table 1. The fundamental operation of one–dimensional differential transform method.

3. APPLICATIONS

We first considered the following index-3 differential-algebraic equations

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix}+\begin{pmatrix}1&1&t\\e^t&t+1&0\\0&t^2&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2t\\t^2+t+2\\t^3\end{pmatrix}, \quad t\in[0,\infty] \tag{3.1}$$

with initial conditions

$$x(0)=\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

The exact solutions are

$$x_1(t)=e^{-t},$$
$$x_2(t)=t,$$
$$x_3(t)=1.$$

By using the basic properties of differential transform method and taking the transform of differential algebraic equations given (3.1), we obtained

$$(k+1)X_1(k+1)+X_1(k)+X_2(k)+\sum_{r=0}^{k}\delta(r-1)X_3(k-r)=2.\delta(k-1) \tag{3.2}$$

$$(k+1)X_2(k+1)+\sum_{r=0}^{k}\frac{1}{r!}X_1(k-r)+\sum_{r=0}^{k}\delta(r-1)X_2(k-r)+X_2(k)=\delta(k-2)+\delta(k-1)+2.\delta(k) \tag{3.3}$$

$$\sum_{r=0}^{k}\delta(r-2)X_2(k-r)=\delta(k-3) \tag{3.4}$$

Equations (3.2)-(3.4) can be simplified as

$$X_1(k+1)=\frac{1}{k+1}\left[2.\delta(k-1)-X_1(k)-X_2(k)-\sum_{r=0}^{k}\delta(r-1)X_3(k-r)\right] \tag{3.5}$$

$$X_2(k+1)=\frac{1}{k+1}\left[\delta(k-2)+\delta(k-1)+2.\delta(k)-\sum_{r=0}^{k}\frac{1}{r!}X_1(k-r)-\sum_{r=0}^{k}\delta(r-1)X_2(k-r)-X_2(k)\right] \tag{3.6}$$

$$\sum_{r=0}^{k}\delta(r-2)X_2(k-r)=\delta(k-3) \tag{3.7}$$

For $k=0,1,2,\ldots$ $X_1(k)$, $X_2(k)$, $X_3(k)$ coefficients can be calculated from equations (3.5)-(3.7)

$$X_1(1)=-1,\ X_1(2)=\frac{1}{2},\ X_1(3)=-\frac{1}{6},\ X_1(4)=\frac{1}{24},\ X_1(5)=-\frac{1}{120},\ldots$$

$$X_2(1)=1,\ X_2(2)=0,\ X_2(3)=0,\ X_2(4)=0,\ X_2(5)=0,\ldots$$

$$X_3(1)=0,\ X_3(2)=0,\ X_3(3)=0,\ X_3(4)=0,\ X_3(5)=0,\ldots$$

By substituting the values of $X_1(k)$, $X_2(k)$, $X_3(k)$ into Equation (2.2), the solutions can be written as

$$x_1(t)=1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+\frac{1}{720}t^6-\frac{1}{5040}t^7+\frac{1}{40320}t^8-\frac{1}{362880}t^9+O(t^{10}),$$

$x_2(t)=t$,

$x_3(t)=1$.

t	x_1	$\tilde{x}_1$	$\lvert x_1-\tilde{x}_1\rvert$
0.1	0.9048374180	0.9048374181	$0,1.10^{-9}$
0.2	0.8187307531	0.8187307532	$0,1.10^{-9}$
0.3	0.7408182207	0.7408182206	$0,1.10^{-9}$
0.4	0.6703200460	0.6703200461	1.10^{-10}
0.5	0.6065306597	0.6065306595	2.10^{-10}
0.6	0.5488116361	0.5488116345	$1,6.10^{-9}$
0.7	0.4965853038	0.4965852966	$7,2.10^{-9}$
0.8.	0.4493289641	0.4493289365	$2,76.10^{-8}$
0.9	0.4065696597	0.4065695710	$8,87.10^{-8}$
1.0	0.3678794412	0.3678791888	$2,524.10^{-7}$

Table 2. Compared of the numerical and exact solution of the first test problem, where x_1 is exact solution, $\tilde{x}_1$ is numerical solution.

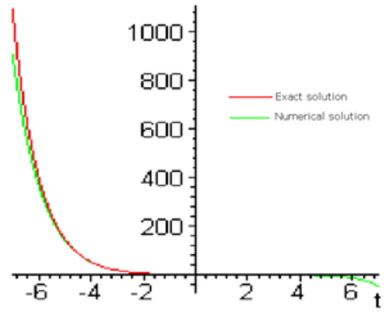

Figure 3.1. Graph of the functions x_1 and $\tilde{x}_1$ in the first test problem.

As a second application we consider the following index-3 DAEs

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & t & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}+\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & t & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 1 \\ 2t \\ e^t \end{pmatrix}, \quad t\in[0,\infty] \tag{3.8}$$

The initial values are

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}=\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and the exact solutions are

$$x_1(t)=e^t-1,$$
$$x_2(t)=2t-e^t,$$
$$x_3(t)=(1+t)e^t-2t^2.$$

By using the basic properties of differential transform method from Table 1 and taking the transform of differential algebraic equations given (3.8) it is obtained that

$$\begin{gathered}(k+1)X_2(k+1)+X_1(k)=\delta(k)\\ (k+1)X_2(k+1)+(k+1)X_3(k+1)+2X_2(k)=2\delta(k-1)\\ \sum_{r=0}^{k}\delta(r-1)X_2(k-r)+X_3(k)=\frac{1}{k!}.\end{gathered} \tag{3.9}$$

Equation (3.9) can be simplified

$$\begin{aligned} X_2(k+1)&=\frac{1}{k+1}\left[\delta(k)-X_1(k)\right],\\ X_3(k+1)&=\frac{1}{k+1}\left[2\delta(k-1)-2X_2(k)-(k+1)X_2(k+1)\right],\\ \sum_{r=0}^{k}\delta(r-1)X_2(k-r)&+X_3(k)=\frac{1}{k!}.\end{aligned} \tag{3.10}$$

For $k=0,1,2,\ldots$ $X_1(k), X_2(k), X_3(k)$ coefficients can be calculated from (3.10)

$$X_1(1)=1,\ X_1(2)=\frac{1}{2},\ X_1(3)=\frac{1}{6},\ X_1(4)=\frac{1}{24},\ X_1(5)=\frac{1}{120},\dots$$

$$X_2(0)=-1,\ X_2(1)=1,\ X_2(2)=-\frac{1}{2},\ X_2(3)=-\frac{1}{6},\ X_2(4)=-\frac{1}{24},\ X_2(5)=-\frac{1}{120},\dots$$

$$X_3(0)=1,\ X_3(1)=2,\ X_3(2)=-\frac{1}{2},\ X_3(3)=\frac{2}{3},\ X_3(4)=\frac{5}{24},\ X_3(5)=\frac{1}{20},\dots$$

By substituting the values of $X_1(k)$, $X_2(k)$, $X_3(k)$ into Equation (2.2), the solutions can be written as

$$x_1(t)=t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\frac{1}{120}t^5+\frac{1}{720}t^6+\frac{1}{5040}t^7+\frac{1}{40320}t^8+\frac{1}{362880}t^9+O(t^{10})$$

$$x_2(t)=-1+t-\frac{1}{2}t^2-\frac{1}{6}t^3-\frac{1}{24}t^4-\frac{1}{120}t^5-\frac{1}{720}t^6-\frac{1}{5040}t^7-\frac{1}{40320}t^8-\frac{1}{362880}t^9+O(t^{10})$$

$$x_3(t)=1+2t-\frac{1}{2}t^2+\frac{2}{3}t^3+\frac{5}{24}t^4+\frac{1}{20}t^5+\frac{7}{720}t^6+\frac{1}{630}t^7+\frac{1}{4480}t^8+\frac{1}{36288}t^9+O(t^{10})$$

t	x_3	$\tilde{x}_3$	$\lvert x_3-\tilde{x}_3\rvert$
0.1	1.195688010	1.195688010	0
0.2	1.385683310	1.385683309	$0,1.10^{-8}$
0.3	1.574816450	1.574816451	$0,1.10^{-8}$
0.4	1.768554577	1.768554576	$0,1.10^{-8}$
0.5	1.973081906	1.973081903	$0,3.10^{-8}$
0.6	2.195390080	2.195390061	$0,19.10^{-7}$
0.7	2.443379602	2.443379511	$0,91.10^{-7}$
0.8.	2.725973670	2.725973317	$0,353.10^{-6}$
0.9	3.053245911	3.053244751	$0,1160.10^{-5}$
1.0	3.436563656	3.436560295	$0,3361.\ 10^{-5}$

Table 3. Compared of the numerical and exact solution of the second test problem , where x_3 is exact solution, $\tilde{x}_3$ is numerical solution.

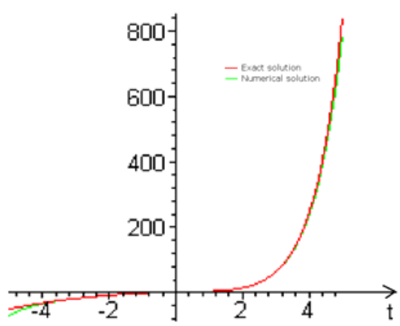

Figure 3.2. Graph of the functions x_3 and $\tilde{x}_3$ in the second test problem.

4. CONCLUSION

The method has been applied to the solution of differential-algrebraic equations. We have obtained approximant analytical solution of the given problem. If the numerical solution of the given problems are compared with their analytical solutions, the differential transform method is very effective and convergence are quite close. However, this example tell us the method can be alternative way for the solution of the differential-algebraic equations with index 2.

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