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Research Article

Exploring the spectral properties of multivariable (m, n)-isosymmetric operators

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Abstract

Drawing from recent advancements in the study of m-isometric and n-symmetric completely positive operators on Hilbert spaces, this paper introduces the concept of $(m; n)$-isosymmetric multivariable operators. This new class of operators serves as a generalization of both m-isometric and n-isosymmetric multioperators. We explore the fundamental properties of these operators, demonstrating that if $\mathbf{R} \in \mathcal{B}^{(d)}(\mathcal{H})$ is an $(m.n)$-isosymmetric multioperator and $\mathcal{Q} \in \mathcal{B}^{(d)}(\mathcal{H})$ is a q-nilpotent multioperators, then the sum $\mathbf{R} + \mathcal{Q}$ is an $(m+2q-2; n+2q-2)$-isosymmetric multioperator under appropriate conditions. Additionally, we present results concerning the joint approximate spectrum of (m, n)-isosymmetric multioperators.

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1. Introduction

We set below the notations used throughout this paper. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. We use the notations $\mathbb{N}$ the set of natural numbers, $\mathbb{N}_0$ the set of nonnegative integers, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. An operator $R \in \mathcal{B}(\mathcal{H})$ is said to be m-isometric operator if

$$\sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} R^{*k} R^k = 0, \tag{1.1}$$

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for some positive integer m, or

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|R^kx\|^2=0,\qquad\forall\ x\in\mathcal{H}. \tag{1.2}$$

Such m-isometric operators were introduced by J. Agler and were studied in great detail by J. Alger and M. Stankus in the papers [1–3]. For more rich theory on m-isometric operators and related classes, we invite the most interesting readers to consult the references [4,6,7] and [19,20,24].

The concept of n-symmetric operators has been introduced and study in [16,21]. Let R be an operator on a Hilbert space, R is said to be an n-symmetric operator if R satisfies

$$\sum_{0\leq j\leq n}(-1)^j\binom{n}{j}R^{*j}R^{n-j}=0, \tag{1.3}$$

for some positive integer n. It has proved that a power of n-symmetric operator is a again n-symmetric and the product of two n-symmetric operator is also n-symmetric under suitable conditions (see [21]).

Based on (1.1) and (1.3) the authors in ([22,23]) introduced the class of (m,n)-isosymmetric operators. An operator $R\in\mathcal{B}(\mathcal{H})$ is said to be (m,n)-isosymmetric operator if

$$\begin{aligned}&\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}R^{*(m-j)}\Big(\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}R^{*(n-k)}R^k\Big)R^{m-j}\\ =\ &\sum_{0\leq k\leq n}(-1)^k\binom{n}{k}R^{*(n-k)}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}R^{*(m-j)}R^{m-j}\Big)R^k\\ =\ &0.\end{aligned}$$

The study of multi-operators has received great interest from many authors in recent years. The investigation of multioperators belonging to some specific classes has been quite fashionable since the beginning of the century, and sometimes it is indeed relevant. Some developments on these subjects have been made in [5,6,10,15,17,21] and the references therein.

For $d\in\mathbb{N}$, let $\mathbf{R}=(R_1,\cdots,R_d)\in\mathcal{B}^{(d)}(\mathcal{H}):\underbrace{\mathcal{B}(\mathcal{H})\times\ldots\times\mathcal{B}(\mathcal{H})}_{\text{d-times}}$ be a tuple of commuting bounded linear operators. Let $\gamma=(\gamma_1,\cdots,\gamma_d)\in\mathbb{N}_0^d$ and set $|\gamma|:=\sum_{1\leq j\leq d}\gamma_j$ and $\gamma!:=\prod_{1\leq k\leq d}\gamma_k!$. Furthermore, define $\mathbf{R}^\gamma:=R_1^{\gamma_1}R_2^{\gamma_2}\cdots R_d^{\gamma_d}$ where $R^{\gamma_j}=\underbrace{R_j\cdots R_j}_{\gamma_j-\text{times}}$ $(1\leq j\leq d)$ and $\mathbf{R}^*=(R_1^*,\cdots,R_d^*)$.

Let $\mathbf{R}=(R_1,\cdots,R_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ be a commuting multioperators and set for $l\in\mathbb{N}_0$:

$$\mathbf{S}_l(\mathbf{R})=\sum_{0\leq k\leq l}(-1)^{l-j}\binom{l}{k}\big(R_1^*+\cdots+R_d^*\big)^k\big(R_1+\cdots+R_d\big)^{l-k}, \tag{1.4}$$

and

$$\mathbf{M}_l(\mathbf{R})=\sum_{0\leq k\leq l}(-1)^{l-k}\binom{l}{k}\Big(\sum_{|\gamma|=k}\frac{k!}{\gamma!}\mathbf{R}^{*\gamma}\mathbf{R}^{\gamma}\Big), \tag{1.5}$$

we have $M_0(\mathbf{R})=I$ and $M_1(\mathbf{R})=\sum_{1\leq j\leq d}R_j^*R_j-I$.

Gleason and Richter [12] considered the multivariable setting of m-isometries and studied their properties. A commuting d-tuple of operators $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ is said to be an m-isometric multi operators if it satisfies the operator equation

$$\mathbf{M}_m(\mathbf{R}) = \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\gamma|=k} \frac{k!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{R}^{\gamma} \Big) = 0. \tag{1.6}$$

It holds

$$\mathbf{M}_{l+1}(\mathbf{R}) = \sum_{1 \le j \le d} R_j^* \mathbf{M}_l(\mathbf{R}) R_j - \mathbf{M}_l(\mathbf{R}). \tag{1.7}$$

In particular if $\mathbf{R}$ is an m -isometric multioperators, then $\mathbf{R}$ is an $(m+k)$-isometric multioperators for all $k \ge 0$. The authors in [8] considered the multivariable setting of m-msymetries and studied their properties. A commuting d-tuple of operators $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ is said to be an m-symmetric multioperators if it satisfies the operator equation

$$\mathbf{S}_m(\mathbf{R}) = \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \big(R_1^* + \cdots + R_d^*\big)^k (R_1 + \cdots + R_d)^{m-k} = 0. \tag{1.8}$$

Similarly,

$$\mathbf{S}_{l+1}(\mathbf{R}) = \Big(\sum_{1 \le k \le d} R_k^* \Big) \mathbf{S}_l(\mathbf{R}) - \mathbf{S}_l(\mathbf{R}) \Big(\sum_{1 \le k \le d} R_k \Big), \tag{1.9}$$

and moreover if $\mathbf{R}$ is an m-symmetric multioperator, then $\mathbf{R}$ is an $(m+k)$-symmmetric multioperators for all $k \ge 0$.

2. $(m,\ n)$-isosymmetric commuting multioperators

The aim of this section is to present the study of certain properties of $(m;n)$-isosymmetric multioperators, a class of operators that includes both n-symmetric multioperators and m-isometric multioperators. We will explore several key properties of these operator classes and their relationships.

The following definition is quoted from [11] where the concept was initially introduced in Banach spaces. However, our approach differs from the one presented in [11].

Definition 2.1. Let $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ be a commuting multioperators. $\mathbf{R}$ is said to be an (m,n)-isosymmetric if $\Lambda_{m,\,n}(\mathbf{R}) = 0$, where

$$\begin{aligned}\Lambda_{m,\,n}(\mathbf{R}) &= \sum_{0 \le k \le n} (-1)^{n-k} \binom{n}{k} \big(R_1^* + \cdots + R_d^*\big)^k \mathbf{M}_m(\mathbf{R}) \big(R_1 + \cdots + R_d\big)^{n-k} \\ &= \sum_{0 \le k \le m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\gamma|=k} \frac{k!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{S}_n(\mathbf{R}) \mathbf{R}^{\gamma} \Big).\end{aligned}$$

Remark 2.2. (1) Every m-isometric multioperators is an (m,n)-isosymmetric multioperators and every n-isosymetric multioperators is an (m,n)-isosymmetric multioperators.

Remark 2.3. We make the following observations

$$\Lambda_{1,\,0}(\mathbf{R}) = \sum_{1 \le k \le d} R_k^* R_k - I, \tag{2.1}$$

$$\Lambda_{0,\,1}(\mathbf{R}) = \sum_{1 \le k \le d} (R_k^* - R_k), \tag{2.2}$$

$$\Lambda_{1,\,1}(\mathbf{R}) = \Big(\sum_{1 \le k \le d} R_k^* \Big) \Big(\sum_{1 \le j \le d} R_j^* R_j - I \Big) - \Big(\sum_{1 \le j \le d} R_j^* R_j - I \Big) \Big(\sum_{1 \le k \le d} R_k \Big) \tag{2.3}$$

or

$$\Lambda_{1,\,1}(\mathbf{R})=\sum_{1\leq k\leq d}\Big(R_k^*\sum_{1\leq j\leq d}(R_j^*-R_j)R_k\Big)-\sum_{1\leq j\leq d}(R_j^*-R_j). \tag{2.4}$$

Example 2.4. Let $R\in\mathcal{B}(\mathcal{H})$ be an (m,n)-isosymmetric single operator, $d\in\mathbb{N}$ and $\beta=(\beta_1,\cdots,\beta_d)\in(\mathbb{R}^d,\ \|.\|_2)$ with $\|\beta\|_2^2=\sum\limits_{1\leq j\leq d}\beta_j^2=1$. Then the multioperator $\mathbf{R}=(R_1,\cdots,R_d)$ where $R_j=\beta_jR$ for $j=1,\cdots,d$ is an (m,n)-isosymmetric multioperators.

In fact, it is obvious that $R_jR_k=R_kR_j$ for all $1\leq j,\ k\leq d$. From the multinomial expansion, we get for any natural number q

$$\begin{aligned}1=\Big(\beta_1^2+\beta_2^2+\cdots+\beta_d^2\Big)^q &= \sum_{\gamma_1+\gamma_2+\cdots+\gamma_d=q}\binom{q}{\gamma_1,\gamma_2,\cdots,\gamma_p}\prod_{1\leq l\leq d}\beta_l^{2\gamma_i}\\ &=\sum_{|\gamma|=q}\frac{q!}{\gamma!}|\beta^\gamma|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathbf{M}_m(\mathbf{R}) &= \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{R}^{*\gamma}\mathbf{R}^\gamma\Big)\\ &= \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}\Big(\sum_{|\gamma|=j}\frac{j!}{\gamma!}\prod_{1\leq j\leq d}\beta_j^{2\gamma_j}R^{*|\gamma|}R^{|\gamma|}\Big)\\ &= \sum_{0\leq j\leq m}(-1)^{m-j}\binom{m}{j}R^{*j}R^j.\end{aligned}$$

$$\Lambda_{m,\,n}(\mathbf{R})=\sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}\big(R_1^*+\cdots+R_d^*\big)^k\mathbf{M}_m(\mathbf{R})\big(R_1+\cdots+R_d\big)^{n-k}$$

$$\begin{aligned}&= \Big(\sum_{1\leq j\leq d}\beta_j\Big)^n\Big(\sum_{0\leq k\leq n}(-1)^{n-k}\binom{n}{k}R^{*(k)}\Big(\sum_{0\leq j\leq m}(-1)^j\binom{m}{j}R^{*(m-j)}R^{m-j}\Big)R^{n-k}\Big)\\ &= 0.\end{aligned}$$

Therefore $\mathbf{R}$ is a (m,n)-isosymmetric multioperators as required.

In the following example we show that there is a multioperators which is (m,n)-isosymmetric, but neither m-isometric nor n-isosymmetric multioperators for some multiindex and a positive integer m and n. Thus, the proposed new class of multioperators contains the classes of m-isometric mutioperatros and n-symmetric multioperators as proper subsets.

Example 2.5. Consider $\mathbf{R}=(R_1,\ R_2)$ where $R_1=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$ and $R_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

A simple computation shows that

$$(R_1+R_2)^*\Big(R_1^*R_1+R_2^*R_2-I\Big)-\Big(R_1^*R_1+R_2^*R_2-I\Big)(R_1+R_2)=0.$$

Therefore, $\mathbf{R}$ is a $(1,1)$-isosymmetric pairs of operators. However $\mathbf{R}$ is not a 1-isometric and not a 1-symmetric pairs due to the following facts

$$R_1^*R_1+R_2^*R_2-I\neq 0\ \text{ and }\ (R_1+R_2)^*-(R_1+R_2)\neq 0.$$

Theorem 2.6. *Let $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ be a commuting multioperators. Then the following statements hold.*

$$\Lambda_{m+1,\,n}(\mathbf{R}) = \sum_{1\le j\le d} R_j^*\Lambda_{m,\,n}(\mathbf{R})R_j - \Lambda_{m,\,n}(\mathbf{R}). \tag{2.5}$$

$$\Lambda_{m,\,n+1}(\mathbf{R}) = \sum_{1\le j\le d} R_j^*\Lambda_{m,\,n}(\mathbf{R}) - \sum_{1\le j\le d} \Lambda_{m,\,n}(\mathbf{R})R_j. \tag{2.6}$$

Proof. By taking into account (1.7) we have

$$\begin{aligned}
\Lambda_{m+1,\,n}(\mathbf{R}) &= \sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\big(R_1^*+\cdots+R_d^*\big)^k\mathbf{M}_{m+1}(\mathbf{R})\big(R_1+\cdots+R_d\big)^{n-k}\\
&= \sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\big(R_1^*+\cdots+R_d^*\big)^k\Big(\sum_{1\le j\le d}R_j^*\mathbf{M}_m(\mathbf{R})R_j-\mathbf{M}_m(\mathbf{R})\Big)\big(R_1+\cdots+R_d\big)^k\\
&= \sum_{1\le j\le d}R_j^*\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\big(R_1^*+\cdots+R_d^*\big)^k\mathbf{M}_m(\mathbf{R})\big(R_1+\cdots+R_d\big)^{n-k}R_j\\
&\quad-\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\big(R_1^*+\cdots+R_d^*\big)^k\mathbf{M}_m(\mathbf{R})\big(R_1+\cdots+R_d\big)^{n-k}\\
&= \sum_{1\le j\le d}R_j^*\Lambda_{m,\,n}(\mathbf{R})R_j-\Lambda_{m,\,n}(\mathbf{R}).
\end{aligned}$$

Using a similar argument to that employed above we can prove the identity (2.6). □

Corollary 2.7. *Let $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ be a commuting multioperators. If $\mathbf{R}$ is an (m,n)-isosymmetric, then $\mathbf{R}$ is (m',n')-isosymmetric for all $n' \ge n$ and $m' \ge m$.*

The following example shows that there exists an operator tuple R that is (m,n)-isosymmetric for $m,n \ge 2$, but not $(m-1,n-1)$-isosymmetric.

Example 2.8. Let $R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \sqrt{2}\\ 0 & 1\end{pmatrix}$ $R_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\sqrt{2}\\ 0 & 1\end{pmatrix}$. Direct calculation shows that $\mathbf{R} = (R_1, R_2)$ is a $(2,2)$-isosymmetric tuple but not a $(1,1)$-isosymmetric tuple.

Lemma 2.9 ([6])**.** *Let $\alpha = (\alpha_1, \cdots, \alpha_d), \gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ be such that $|\alpha| + |\gamma| + k = n + 1$. For $1 \le i \le d$, let $\epsilon(i) \in \mathbb{N}_0^d$ where $\epsilon(i)$ is the d-tuple with 1 in the i-th entry and zeros elsewhere. Then,*

$$\binom{n+1}{\alpha,\gamma,k} = \sum_{1\le i\le d}\left(\binom{n}{\alpha-\epsilon(i),\gamma,k}+\binom{n}{\alpha,\gamma-\epsilon(i),k}\right)+\binom{n}{\alpha,\gamma,k-1}. \tag{2.7}$$

Proposition 2.10. *Let $\mathbf{R} = (R_1, \cdots, R_d)$ and $\mathcal{Q} = (\mathcal{Q}_1, \cdots, \mathcal{Q}_d)$ be two commuting multioperators for which $[R_j, \mathcal{Q}_i] = [R_j, \mathcal{Q}_i^*] = 0$ for all $j, i \in \{1, \cdots, d\}$. Then, for a positive integers m and n, the following identity holds:*

$$\Lambda_{m,\,n}(\mathbf{R}+\mathcal{Q}) = \sum_{0\le j\le n}\sum_{|\alpha|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,n-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}, \tag{2.8}$$

where $\binom{n}{j} = \dfrac{n!}{(n-j)!j!}$ *and* $\binom{m}{\alpha,\gamma,k} = \dfrac{m!}{\alpha!\gamma!k!}$.

Proof. We prove (2.8) by two-dimensional induction principle on $(m,n)\in\mathbb{N}^2$. We first check that (2.8) is true for $(m,n)=(1,1)$. In fact,

$$\begin{aligned}
&\sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=1}\binom{1}{j}\binom{1}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\alpha}\mathcal{Q}^{\gamma}\\
=&\sum_{0\le j\le 1}\Big\{\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathcal{Q}_i+\sum_{1\le i\le d}\mathcal{Q}_i^*\Lambda_{0,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})R_i\\
&+\Lambda_{1,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathbf{Q})\Big\}\\
=&\Big\{\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1}(\mathbf{R})\mathbf{S}_0(\mathcal{Q})\mathcal{Q}_i+\sum_{1\le i\le d}\mathcal{Q}_i^*\Lambda_{0,\,1}(\mathbf{R})\mathbf{S}_0(\mathcal{Q})R_i+\Lambda_{1,\,1}(\mathbf{R})\mathbf{S}_0(\mathcal{Q})\Big\}\\
&+\Big\{\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,0}(\mathbf{R})\mathbf{S}_1(\mathcal{Q})\mathcal{Q}_i+\sum_{1\le i\le d}\mathcal{Q}_i^*\Lambda_{0,\,0}(\mathbf{R})\mathbf{S}_1(\mathcal{Q})R_i+\Lambda_{1,\,0}(\mathbf{R})\mathbf{S}_1(\mathcal{Q})\Big\}\\
=&\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})\mathcal{Q}_i+\sum_{1\le i\le d}+\mathcal{Q}_i^*\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})R_i\\
&+\Lambda_{1,\,1}(\mathbf{R})+\Lambda_{1,\,0}(\mathbf{R})\mathbf{S}_1(\mathcal{Q}).
\end{aligned}$$

Based on (2.1),(2.2) and (2.5) we have

$$\begin{aligned}
\Lambda_{1,\,1}(\mathbf{R})+\Lambda_{1,\,0}(\mathbf{R})\mathbf{S}_1(\mathcal{Q})&=\sum_{1\le i\le d}R_i^*\Lambda_{0,\,1}(\mathbf{R})R_i-\Lambda_{0,\,1}(\mathbf{R})+\Big(\sum_{1\le i\le d}R_i^*R_i-I\Big)\mathbf{S}_1(\mathcal{Q})\\
&=\sum_{1\le i\le d}R_i^*\Lambda_{0,\,1}(\mathbf{R})R_i-\Lambda_{0,\,1}(\mathbf{R})+\sum_{1\le i\le d}R_i^*R_i\mathbf{S}_1(\mathcal{Q})-\mathbf{S}_1(\mathcal{Q})\\
&=\sum_{1\le i\le d}R_i^*\Lambda_{0,\,1}(\mathbf{R})R_i+\sum_{1\le i\le d}R_i^*\mathbf{S}_1(\mathcal{Q})R_i-\Lambda_{0,\,1}(\mathbf{R})-\mathbf{S}_1(\mathcal{Q})\\
&=\sum_{1\le i\le d}R_i^*(\Lambda_{0,\,1}(\mathbf{R})+\mathbf{S}_1(\mathcal{Q}))R_i-\Big(\Lambda_{0,\,1}(\mathbf{R})+\mathbf{S}_1(\mathcal{Q})\Big)\\
&=\sum_{1\le i\le d}R_i^*\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})R_i-\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q}).
\end{aligned}$$

$$\begin{aligned}
&\sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=1}\binom{1}{j}\binom{1}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\alpha}\mathcal{Q}^{\gamma}\\
=&\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})\mathcal{Q}_i+\sum_{1\le i\le d}+Q_i^*\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})R_i\\
&+\sum_{1\le i\le d}R_i^*\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})R_i-\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})\\
=&\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})\mathcal{Q}_i+\sum_{1\le i\le d}+(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})R_i-\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})\\
=&\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})(R_i+\mathcal{Q}_i)-\Lambda_{0,\,1}(\mathbf{R}+\mathcal{Q})\\
=&\ \Lambda_{1,\,1}(\mathbf{R}+\mathcal{Q})\qquad\text{(by (2.5))}.
\end{aligned}$$

So the identity (2.8) holds for $(m, n) = (1, 1)$. Assume that the identity (2.8) holds for $(m, 1)$ and prove it for $(m+1, 1)$. According to (2.5) and the induction hypothesis we have

$$\Lambda_{m+1,\,1}\big(\mathbf{R}+\mathcal{Q}\big) = \sum_{1\le i\le d}\big(R_i^*+\mathcal{Q}_i^*\big)\Lambda_{m,\,1}\big(\mathbf{R}+\mathbf{Q}\big)\big(R_i+\mathcal{Q}_i\big) - \Lambda_{m,\,1}\big(\mathbf{R}+\mathcal{Q}\big).$$

$$\begin{aligned}
&= \sum_{1\le i\le d}\big(R_i^*+\mathcal{Q}_i^*\big)\Big\{\sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\Big\}\big(R_i+\mathcal{Q}_i\big)\\
&\quad - \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\\
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Big[\sum_{1\le i\le d}R_i^*\Lambda_{k,\,1-j}(\mathbf{R})R_i - \Lambda_{k,\,1-j}(\mathbf{R})\\
&\quad + \sum_{1\le i\le d}\Big\{R_i^*\Lambda_{k,\,1-j}(\mathbf{R})\mathcal{Q}_i + \mathcal{Q}_i^*\Lambda_{k,\,1-j}(\mathbf{R})R_i + \mathcal{Q}_i^*\Lambda_{k,\,1-j}(\mathbf{R})\mathcal{Q}_i\Big\}\Big]\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\\
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k+1,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+Q)^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\Big\{\\
&\quad \big(R_i^*+\mathcal{Q}_i^*\big)\Lambda_{k,\,1-j}(\mathbf{R})\mathcal{Q}_i + \mathcal{Q}_i^*\Lambda_{k,\,1-j}(\mathbf{R})R_i\Big\}\Big]\mathbf{S}_j(\mathcal{Q})\Big\}\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\\
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\cdot\Lambda_{k+1,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}\sum_{1\le i\le d}(\mathbf{R}+\mathbf{Q})^{*\alpha}\big(R_i^*+Q_i^*\big)\cdot\mathbf{Q}^{*\gamma}\Lambda_{k,\,1-j}(\mathbf{R})Q_i\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}\sum_{1\le i\le d}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}Q_i^*\Lambda_{k,\,1-j}(\mathbf{R})R_i\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m+1}\binom{1}{j}\binom{m}{\alpha,\gamma,k-1}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}\sum_{1\le i\le d}(\mathbf{R}+\mathbf{Q})^{*\alpha+\epsilon(i)}\cdot\mathbf{Q}^{*\gamma}\Lambda_{k,\,1-j}(\mathbf{R})Q_i\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m}\binom{1}{j}\binom{m}{\alpha,\gamma,k}\sum_{1\le i\le d}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma+\epsilon(i)}\Lambda_{k,\,1-j}(\mathbf{R})R_i\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m+1}\binom{1}{j}\binom{m}{\alpha,\gamma,k-1}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m+1}\Bigg(\sum_{1\le i\le d}\binom{1}{j}\binom{m}{\alpha-\epsilon(i),\gamma,k}\Bigg)(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\Lambda_{k,\,1-j}(\mathbf{R})Q_i\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&\quad + \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m+1}\binom{1}{j}\Bigg(\sum_{1\le i\le d}\binom{m}{\alpha,\gamma-\epsilon(i),k}\Bigg)\sum_{1\le i\le d}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\Lambda_{k,\,1-j}(\mathbf{R})R_i\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m+1}\binom{1}{j}\Bigg(\sum_{1\le i\le d}\Bigg(\binom{m}{\alpha-\epsilon(i),\gamma,k}+\binom{m}{\alpha,\gamma-\epsilon(i),k}\Bigg)+\binom{m}{\alpha,\gamma,k-1}\Bigg)(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\\
&\quad \cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}\\
&= \sum_{0\le j\le 1}\sum_{|\alpha|+|\gamma|+k=m+1}\binom{1}{j}\binom{m+1}{\alpha,\gamma,k}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathbf{Q}^{*\gamma}\cdot\Lambda_{k,\,1-j}(\mathbf{R})\mathbf{S}_j(\mathbf{Q})\mathbf{R}^{\gamma}\mathbf{Q}^{\alpha}.
\end{aligned}$$

So (2.8) holds for $(m,1)$ implies (2.8) holds for $(m+1,1)$.

Now assume that (2.8) holds for (m,n) and we prove that (2.8) holds for $(m,n+1)$. According to (2.6) and the induction hypothesis we have

$$\begin{aligned}
&\Lambda_{m,\,n+1}(\mathbf{R}+\mathcal{Q}) \\
=\;& \sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{m,\,n}(\mathbf{R}+\mathcal{Q})-\sum_{1\le i\le d}\Lambda_{m,\,n}(\mathbf{R}+\mathcal{Q})(R_i+\mathcal{Q}_i) \\
=\;& \sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Big[\sum_{0\le j\le n}\sum_{|\alpha|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathbf{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,n-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\Big] \\
&-\sum_{1\le i\le d}\Big[\sum_{0\le j\le n}\sum_{|\alpha|+|\gamma|+k=m}\binom{n}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,n-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}\Big](R_i+\mathcal{Q}_i) \\
=\;& \sum_{|\alpha|+|\gamma|+k=m}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\Big[\mathbf{A}(n,\mathbf{R},\,\mathcal{Q})\Big]\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha},
\end{aligned}$$

where

$$\mathbf{A}(n,\mathbf{R},\,\mathcal{Q})=\sum_{0\le j\le n}\binom{n}{j}\sum_{1\le i\le d}(R_i^*+\mathcal{Q}_i^*)\Lambda_{k,\,n-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})-\sum_{0\le j\le n}\binom{n}{j}\Lambda_{k,\,n-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\sum_{1\le i\le d}(R_i+\mathcal{Q}_i).$$

Based on (1.9) and (2.6) we have

$$\begin{aligned}
\mathbf{A}(n,\mathbf{R},\,\mathcal{Q}) &= \sum_{0\le j\le n}\binom{n}{j}\Big\{\sum_{1\le i\le d}R_i^*\Lambda_{k,\,n-j}(\mathbf{R})-\Lambda_{k,\,n-j}(\mathbf{R})\big(\sum_{1\le i\le d}R_i\big)\Big\}\mathbf{S}_j(\mathcal{Q}) \\
&\quad+\sum_{0\le j\le n}\binom{n}{j}\Lambda_{k,\,n-j}(\mathbf{R})\Big\{\big(\sum_{1\le i\le d}\mathcal{Q}_i^*\big)\mathbf{S}_j(\mathcal{Q})-\mathbf{S}_j(\mathcal{Q})\big(\sum_{1\le i\le d}\mathcal{Q}_i\big)\Big\} \\
&= \sum_{0\le j\le n}\binom{n}{j}\Lambda_{k,\,n+1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})+\sum_{0\le j\le n}\binom{n}{j}\Lambda_{k,\,n-j}(\mathbf{R})\mathbf{S}_{j+1}(\mathcal{Q}) \\
&= \Lambda_{k,\,n+1}(\mathbf{R})+\sum_{1\le j\le n}\left(\binom{n}{j}+\binom{n}{j-1}\right)\Lambda_{k,\,n+1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})+\Lambda_{k,\,0}(\mathbf{R})\mathbf{S}_{n+1}(\mathcal{Q}) \\
&= \sum_{0\le j\le n+1}\binom{n+1}{j}\Lambda_{k,\,n+1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q}).
\end{aligned}$$

By using the above relation we obtain

$$\begin{aligned}
&\Lambda_{m,\,n+1}(\mathbf{R}+\mathcal{Q}) \\
=\;& \sum_{0\le j\le n+1}\sum_{|\alpha|+|\gamma|+k=m}\binom{n+1}{j}\binom{m}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,n+1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}.
\end{aligned}$$

Consequently, the identity (2.8) holds for all m and n. This ends the proof. □

Let $\mathcal{Q}=(\mathcal{Q}_1,\cdots,\mathcal{Q}_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ be a commuting multioperators. Recall that $\mathbf{Q}$ is said to be p-nilpotent, $p\in\mathbb{N}$, if $\mathcal{Q}^{\alpha}=\mathcal{Q}_1^{\alpha_1}\cdots\mathcal{Q}_d^{\alpha_d}=0$ for all $\alpha=(\alpha_1,\cdots,\alpha_d)\in\mathbb{N}_0^d$ with $|\alpha|=p$. (See [14]).

Remark 2.11. Let $\mathcal{Q}=(\mathcal{Q}_1,\cdots,\mathcal{Q}_d)$ be a commuting multioperators. If $\mathcal{Q}$ is nilpotent of order k,then $\mathbf{S}_r(\mathcal{Q})=0$ for all $r\ge 2k$.

Theorem 2.12. *Let $\mathbf{R}=(R_1,\cdots,R_d)$ and $\mathcal{Q}=(\mathcal{Q}_1,\cdots,\mathcal{Q}_d)$ be two commuting multioperators for which $[R_j,\mathcal{Q}_i]=[R_j,\mathcal{Q}_i^*]=0$ for all $j,i\in\{1,\cdots,d\ \}$. If $\mathbf{R}$ is an (m,n)-isosymmetric multioperators and $\mathcal{Q}$ is a nilpotent of order q, then $\mathbf{R}+\mathcal{Q}$ is an $(m+2q-2,n+2q-2)$-isosymmetric multioperators.*

Proof. We need to show that $\Lambda_{m+2q-2,\,n+2q-2}(\mathbf{R}+\mathcal{Q})=0$. According to (2.8) we have

$$\Lambda_{m+2q-2,\,n+2q-2}(\mathbf{R}+\mathcal{Q})=\sum_{0\le j\le n+2q-2}\ \sum_{|\alpha|+|\gamma|+k=m+2k-2}\binom{n+2q-2}{j}\binom{m+2q-2}{\alpha,\gamma,k}(\mathbf{R}+\mathcal{Q})^{*\alpha}\cdot\mathcal{Q}^{*\gamma}\cdot\Lambda_{k,\,n+2k-1-j}(\mathbf{R})\mathbf{S}_j(\mathcal{Q})\mathbf{R}^{\gamma}\mathcal{Q}^{\alpha}.$$

(i) If $j\ge 2q$ or $\max\{|\alpha|,\ |\gamma|\ \}\ge q$ we have $\mathbf{S}_j(\mathcal{Q})=0$ or $\mathcal{Q}^{*\gamma}=0$ or $\mathcal{Q}^{\alpha}=0$.

(ii) If $j\le 2q-1$ and $\max\{|\alpha|,\ |\gamma|\ \}\le q-1$ we have

$k=m+2q-2-|\alpha|-|\gamma|\ge m$ and $n+2q-1-j\ge n$ and therefore $\Lambda_{k,\,n+2k-1-j}(\mathbf{R})=0$ by Corollary 2.7.

By combining (i) and (ii) we can conclude that $\Lambda_{m+2q-2,\,n+2q-2}(\mathbf{R})=0$. $\square$

A particularly interesting consequences of Theorems 2.12 are the following.

Corollary 2.13. *Let $\mathbf{R}=(R_1,\ldots,R_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ be an (m,n)-isosymmetric commuting multioperators and let $\mathcal{Q}=(\mathcal{Q}_1,\cdots,\mathcal{Q}_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ be a q-nilpotent commuting multoperators. Then*
$\mathbf{R}\otimes\mathbf{I}+\mathbf{I}\otimes\mathcal{Q}:=(R_1\otimes I+I\otimes\mathcal{Q}_1,\cdots,R_d\otimes I+I\otimes\mathcal{Q}_d)\in\mathcal{B}^{(d)}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is an $(m+2q-2,n+2q-2)$-isosymmetric multioperators.

Proof. It is obviously that $\big[(R_k\otimes I),(I\otimes N_j)\big]=\big[(R_k\otimes I),(I\otimes N_j)^*\big]=0$ for all $j,k=1,\cdots,d$. Moreover, it is easy to check that $\mathbf{R}\otimes\mathbf{I}=(R_1\otimes I,\ \cdots,R_d\otimes I)\in\mathcal{B}(\mathcal{H}^{(d)}\overline{\otimes}\mathcal{H})$ is an (m,n)-isosymmetric commuting multioperators and $\mathbf{I}\otimes\mathcal{Q}\in\mathcal{B}^{(d)}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is nilpotent commuting multioperators of order q. Therefore $\mathbf{R}\otimes\mathbf{I}$ and $\mathbf{I}\otimes\mathbf{Q}$ satisfy the conditions of Theorem 2.12. Consequently, $\mathbf{R}\otimes\mathbf{I}+\mathbf{I}\otimes\mathcal{Q}$ is an $(m+2q-2,n+2q-2)$-isosymmetric multioperators. $\square$

Corollary 2.14. *Let $\mathbf{A}=(A_1,\cdots,A_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ be an (m,n)-isomsymetric multioperators. If $\mathbf{B}=(B_1,\cdots,B_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ is defined by*

$$B_k=\begin{pmatrix}A_k&\mu_kI&0&\cdots\\0&\ddots&\ddots&\ddots\\\ddots&\ddots&\ddots&\mu_kI\\0&\ddots&0&A_k\end{pmatrix}\ \textit{on}\ \ \mathcal{H}^{(q)}:=\mathcal{H}\oplus\cdots\oplus\mathcal{H}$$

where $\mu_k\in\mathbb{C}$ for $k=1,\cdots,d$ and $\mathcal{H}^{(q)}$ is the sum of q-copies of $\mathcal{H}$, then $\mathbf{B}$ is an $(m+2q-2,n+2q-1)$-isosymmetric multioperators.

Proof. Obviously we have

$$B_k=\begin{pmatrix}A_k&0&0&\cdots\\0&\ddots&\ddots&\ddots\\\ddots&\ddots&\ddots&0\\0&\ddots&0&A_k\end{pmatrix}+\begin{pmatrix}0&\mu_kI&0&\cdots\\0&\ddots&\ddots&\ddots\\\ddots&\ddots&\ddots&\mu_kI\\0&\ddots&0&0\end{pmatrix}=R_k+Q_k\ \text{ for }\ k=1,\cdots,d,$$

and so we my write $\mathbf{B}=\mathbf{R}+\mathcal{Q}=(R_1+\mathcal{Q}_1,\cdots,R_d+\mathcal{Q}_d)$. By Direct computations, we show that $\mathbf{R}$ is an (m,n)-isosymmetic multioperators, $\mathbf{Q}$ is q-nilpotent and

$$\big[R_k,\mathcal{Q}_j\big]=\big[R_k,\mathcal{Q}_j^*\big]=0,\ ;\ \text{for}\ \ k,j\in\{1,\cdots,d\}.$$

According to Theorem 2.12, $\mathbf{B}$ is a $(m+2q-2,n+2q-2)$-isosymmetric multioperators. $\square$

In [4, Theorem 3.1] it has been proved that if $R\in\mathcal{B}(\mathcal{H})$ is a strict m-isometry, then the list of operators $\{\ \sum_{0\le l\le k}(-1)^l\binom{k}{l}R^{*k-l}R^{k-l},k=0,1,\ \cdots m-1\ \}$ is linearly independent.

However in [21, Theorem 4.1] it has been proved that if R is a strict n-symmetric operator, then the list of operators

$$\{ \sum_{0\leq l\leq k}(-1)^l\binom{k}{l}R^{*l}R^{k-l}, k=0,1,\ \cdots n-1\ \},$$

is linearly independent.

In the following proposition we extend these results to m-isometric and n-symmetric multioperatros.

Proposition 2.15. *Let* $\mathbf{R}=(R_1,\cdots,R_d)\in\mathcal{B}^{(d)}(\mathcal{H})$ *be a commuting multioperators such that* $\Lambda_{m-1,;n-1}(\mathbf{R})\neq 0$ *for some positive integers* $m\geq 2$ *and* $n\geq 2$. *The following properties hold.*

1) *If* $\mathbf{R}$ *is* m*-isometric multioperators, then the list of operators*

$$\{\ \Lambda_{k,\,n-1}(\mathbf{R}), k=0,1,\ \cdots,m-1\ \}$$

is linearly independent.

2) *If* $\mathbf{R}$ *is* n*-symmetricc multioperators, then the list of operators*

$$\{\ \Lambda_{m-1,\,l}(\mathbf{R}), l=0,1,\ \cdots,n-1\ \}$$

is linearly independent.

Proof. (1) The proof of the statement (1) is essentially based on the multiple use of the identity (2.5)

$$\Lambda_{m+1,\,n}(\mathbf{R})=\sum_{1\leq j\leq d}R_j^*\Lambda_{m,\,n}(\mathbf{R})R_j-\Lambda_{m,\,n}(\mathbf{R}).$$

Assume that

$$\sum_{0\leq k\leq m-1}a_k\Lambda_{k,\,n-1}(\mathbf{R})=0,$$

for some complex numbers a_k. Multiplying the above equation on the left by R_i^* and right by R_i for $i=1,\cdots,d$ we obtain the following relation

$$\sum_{0\leq k\leq m-1}a_k\Big(\sum_{1\leq i\leq d}R_i^*\Lambda_{k,\,n-1}(\mathbf{R})R_i\Big)=0$$

and subtracting two equations, we have

$$\sum_{0\leq k\leq m-1}a_k\Big(\sum_{1\leq i\leq d}R_i^*\Lambda_{k,\,n-1}(\mathbf{R})R_i-\Lambda_{k,\,n-1}(\mathbf{R})\Big)=\sum_{0\leq k\leq m-1}a_k\Lambda_{k+1,\,n-1}(\mathbf{R})=0$$

From an argument similar to the above applied to the equation

$$\sum_{0\leq k\leq m-1}a_k\Lambda_{k+1,\,n-1}(\mathbf{R})=0,$$

we get

$$\sum_{0\leq k\leq m-1}a_k\Lambda_{k+2,\,n-1}(\mathbf{R})=0.$$

Following the same steps, we can obtain

$$\sum_{0\leq k\leq m-1}a_k\Lambda_{k+l,\,n-1}(\mathbf{R})=0\quad\text{for all}\ \ l\in\mathbb{N}.$$

In the case that $\mathbf{R}$ is m-isometric multioperators, it follows that $\Lambda_{j,\,n-1}(\mathbf{R})=0$ for all $j\geq m$. By considering the following cases, we get

For $l = m-1$, $\sum_{0\leq k\leq m-1} a_k\Lambda_{k+l,\, n-1}(\mathbf{R}) = 0 \Rightarrow a_0\Lambda_{m-1,\, n-1}(\mathbf{R}) = 0$, So we have that $a_0 = 0$.

For $l = m-2$, $\sum_{0\leq k\leq m-1} a_k\Lambda_{k+l,\, n-1}(\mathbf{R}) = 0 \Rightarrow a_1\Lambda_{m-1,\, n-1}(\mathbf{R}) = 0$, So we have that $a_1 = 0$.

Continuing this process we see that all $a_k = 0$ for $k = 2, \cdots, m-1$. Hence the result is proved.

(2) The proof is similar to the above one by using the identity ()2.6

$$\Lambda_{m,\, n+1}(\mathbf{R})) = \sum_{1\leq j\leq d} R_j^*\Lambda_{m,\, n}(\mathbf{R}) - \sum_{1\leq j\leq d} \Lambda_{m,\, n}(\mathbf{R})R_j.$$

□

3. Spectral properties of (m,n)-isosymmetric multioperators

Let $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ be a commuting multioperators. Following [18], the authors noted that

(1) A point $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d$ is in the joint eigenvalue of $\mathbf{R}$ if there exists a nonzero vector $u \in \mathcal{H}$ such that

$$(R_l - \mu_l)u = 0 \text{ for all } \quad l = 1, 2, \cdots, d.$$

The joint point spectrum, denoted by $\sigma_{jp}(\mathbf{R})$ of $\mathbf{R}$ is the set of all joint eigenvalues of $\mathbf{R}$, that is,

$$\sigma_{jp}(\mathbf{R}) = \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d : \bigcap_{1\leq l\leq d} \mathcal{N}(R_l - \mu_l) \neq \{0\}\}.$$

(2) A point $\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d$ is in the joint approximate point spectrum $\sigma_{ja}(\mathbf{R})$ if and only if there exists a sequence $\{u_k\}_k \subset \mathcal{H}$ such that $\|u_k\| = 1$ and

$$(R_l - \mu_l)u_k \longrightarrow 0 \text{ as } k \longrightarrow \infty \text{ for every } \quad l = 1, \cdots, d.$$

For additional information on these concepts, see [9, 10].

In the following results we examine some spectral properties of an (m,n) isosymmetric commuting multioperators. That extend the cases of m-isometries and n-symmetric multioperators studied in [12] and [5].

We put

$$\mathbb{B}(\mathbb{C}^d) := \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \;\; \|\mu\|_2 = \Big(\sum_{1\leq l\leq d} |\mu_l|^2\Big)^{\frac{1}{2}} < 1\,\}$$

and

$$\partial\mathbb{B}(\mathbb{C}^d) := \{\mu = (\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \;/\; \|\mu\|_2 = \Big(\sum_{1\leq l\leq d} |\mu_l|^2\Big)^{\frac{1}{2}} = 1\,\}.$$

In ([12], Lemma 3.2), the authors have proved that if $\mathbf{R}$ is an m-isometric multioperators, then the joint approximate point spectrum of $\mathbf{R}$ is in the boundary of the unit ball $\mathbb{B}(\mathbb{C}^d)$. However in [5, Theorem 4.1], the authors have proved that if $\mathbf{R}$ is an n-symmetric multioperators, then the joint approximate point spectrum of $\mathbf{R}$ is in the set $\Big\{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \;/\; \sum_{1\leq l\leq d} \mu_l \in \mathbb{R}\Big\}$.

Proposition 3.1. *Let* $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ *be a commuting multioperators and* $\mu = (\mu_1, \cdots, \mu_d) \in \sigma_{ja}(\mathbf{R})$*. If* $\mathbf{R}$ *is an* (m,n)*-isosymmetric, then we have*

$$\sum_{1\leq k\leq d} \mu_k \notin \sigma_a\Big(\sum_{1\leq k\leq d} R_k^*\Big) \Longrightarrow (\mu_1, \cdots, \mu_d) \notin [0],$$

where

$$[0] := \{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d : \prod_{1\leq l\leq d} \mu_l = 0 \}.$$

Proof. Assume that $\mu = (\mu_1, \cdots, \mu_d) \in [0] \cap \sigma_{ja}(\mathbf{R})$, then there exists a sequence $(u_k)_{k\geq 1} \subset \mathcal{H}$, with $||u_k|| = 1$ such that $(R_l - \mu_l I)u_k \longrightarrow 0$ for all $l = 1, 2, \cdots, d$. Since for $\gamma_l > 1$,

$$R_j^{\gamma_l} - \mu_l^{\gamma_l} = (R_l - \mu_l) \sum_{1\leq k\leq \mu_l} \mu_l^{k-1} R_l^{\gamma_l - k}.$$

By induction, for $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{N}_0^d$, we have

$$(\mathbf{R}^\gamma - \mu^\gamma I) = \sum_{1\leq k\leq d} \Big(\prod_{i<k} \mu_i^{\gamma_i}\Big)\Big(R_k^{\gamma_k} - \mu_k^{\gamma_k}\Big) \prod_{i>k} R_i^{\gamma_i}.$$

We deduce that $(\mathbf{R}^\gamma - \mu^\gamma I)u_k \to 0$ as $k \longrightarrow \infty$ for all $\gamma = (\gamma_1, \cdots, \gamma_D) \in \mathbb{N}_0^d$. Since $\Lambda_{m,n}(\mathbf{R}) = 0$, it follows that

$$\begin{aligned} 0 &= \sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \Big(\sum_{|\gamma|=j} \frac{j!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{S}_n(\mathbf{R}) \mathbf{R}^\gamma \Big) u_k \\ &= \mathbf{S}_n(\mathbf{R})u_k + \sum_{1\leq j\leq m} (-1)^{m-j} \binom{m}{j} \Big(\sum_{|\gamma|=j} \frac{j!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{S}_n(\mathbf{R}) \mathbf{R}^\gamma \Big) u_k \\ &= \mathbf{S}_n(\mathbf{R})u_k + \sum_{1\leq j\leq m} (-1)^{m-j} \binom{m}{j} \Big(\sum_{|\gamma|=j} \frac{j!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{S}_n(\mathbf{R}) (\mathbf{R}^\gamma - \mu^\gamma) \Big) u_k. \end{aligned}$$

By taking $k \to \infty$ we get $\lim\limits_{k\to\infty} \mathbf{S}_n(\mathbf{R})u_k = 0$. Hence,

$$\begin{aligned} \lim_{k\to\infty} \mathbf{S}_n(\mathbf{R})u_k = 0 \quad &\Rightarrow \quad \lim_{k\to\infty} \Big(\sum_{0\leq j\leq n} (-1)^j \binom{n}{j} (R_1^* + \cdots + R_d^*)^j (R_1 + \cdots + R_d)^{n-j} u_k \Big) = 0 \\ &\Rightarrow \quad \lim_{k\to\infty} (\mu_1 + \cdots + \mu_d - R_1^* - \cdots - R_d^*)^n u_k = 0. \end{aligned}$$

If $(\mu_1 + \cdots + \mu_d - R_1^* - \cdots - R_d^*)$ is bounded from below, then so is $(\mu_1 + \cdots + \mu_d - R_1^* - \cdots - R_d^*)^n$ is bounded from below and therefore

$$\|(\mu_1 + \cdots + \mu_d - R_1^* - \cdots - R_d^*)^n u\| \geq C\|u\|,$$

for some constant $C > 0$ and all $u \in \mathcal{H}$. In particular,

$$\|(\mu_1 + \cdots + \mu_d - R_1^* - \cdots - R_d^*)^n u_k\| \geq C\|u_k\| = C.$$

If $k \to \infty$ we get $C = 0$ which is a contradiction.

Theorem 3.2. *Let* $\mathbf{R} = (R_1, \cdots, R_d) \in \mathcal{B}^{(d)}(\mathcal{H})$ *be an* (m,n)*-isosymmetric multioperators, then the following properties hold.*

(1) $\sigma_{ja}(\mathbf{R}) \subset \partial\mathbb{B}(\mathbb{C}^d) \bigcup \Big\{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \ / \ \sum\limits_{1\leq k\leq d} \mu_k \in \mathbb{R} \Big\}.$

(2) $\sigma_{jp}(\mathbf{R}) \subset \partial\mathbb{B}(\mathbb{C}^d) \bigcup \Big\{(\mu_1, \cdots, \mu_d) \in \mathbb{C}^d \ / \ \sum\limits_{1\leq k\leq d} \mu_k \in \mathbb{R} \Big\}.$

(3) *Let* $\mu = (\mu_1, \cdots, \mu_d)$ *and* $\mu' = (\mu'_1, \cdots, \mu'_d) \in \sigma_{ja}(\mathbf{R})$ *such that* $\sum_{1\leq j\leq d} \mu_j \overline{\mu'_j} \neq 1$ *and* $\sum_{1\leq j\leq d} (\mu_j - \overline{\mu'_j}) \neq 0$. *If* $\{u_k\}_k$ *and* $\{v_k\}_k$ *are two sequences of unit vectors in* $\mathcal{H}$ *such that* $\|(R_j - \mu_j)u_k\| \longrightarrow 0$ *and* $\|(R_j - \mu'_j)v_k\| \longrightarrow 0$ *(as* $k \longrightarrow \infty$*) for* $j = 1, \cdots, d$, *then*

$$\langle u_k |\, v_k\rangle \longrightarrow 0 \ \ (as\ \ k \longrightarrow \infty).$$

(4) *Let* $\mu = (\mu_1, \cdots, \mu_d)$ *and* $\mu' = (\mu'_1, \cdots, \mu'_d) \in \sigma_{jp}(\mathbf{R})$ *such that*

$$\sum_{1\leq j\leq d} \mu_j \mu'_j \neq 1 \ and \ \sum_{1\leq j\leq d} (\mu_j - \overline{\mu'_j}) \neq 0.$$

If $(R_j - \lambda_j)u = 0$ *and* $(R_j - \mu'_j)v = 0$ *for* $j = 1, \cdots, d$, *then*

$$\langle x \mid y\rangle = 0.$$

□

Proof. (1) Let $\mu = (\mu_1, \cdots, \mu_d) \in \sigma_{ja}(\mathbf{R})$, and $\{u_k\}_k \subset \mathcal{H}$, such that $\|u_k\| = 1$ and $(R_l - \mu_l)u_k \longrightarrow 0$ for all $l = 1, \cdots, d$. It is easy to see that for all $\gamma_j \geq 0$ we have $(R_j^{\gamma_j} - \mu_j^{\gamma_j})u_k \longrightarrow 0$ and $(\mathbf{R}^\gamma - \mu^\gamma)u_k \longrightarrow 0$ as $k \longrightarrow \infty$.

In the case that $\mathbf{R}$ is an (m,n)-isosymmetric multioperators, then we have

$$0 = \sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \Big(\sum_{|\gamma|=j} \frac{j!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{S}_n(\mathbf{R}) \mathbf{R}^\gamma \Big) u_k$$

and so we may write

$$\begin{aligned}
0 &= \Big\langle \sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \Big(\sum_{|\gamma|=j} \frac{j!}{\gamma!} \mathbf{R}^{*\gamma} \mathbf{S}_n(\mathbf{R}) \mathbf{R}^\gamma \Big) u_k \mid u_k \Big\rangle \\
&= \sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \sum_{|\gamma|=j} \frac{j!}{\gamma!} \Big\langle \mathbf{S}_n(\mathbf{R}) \mathbf{R}^\gamma u_k \mid \mathbf{R}^\gamma u_k \Big\rangle \\
&= \sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \sum_{|\gamma|=j} \frac{j!}{\gamma!} \Big\langle \mathbf{S}_n(\mathbf{R}) \Big((\mathbf{R}^\gamma - \mu^\gamma)u_k + \mu^\gamma u_k \Big) \mid (\mathbf{R}^\gamma - \mu^\gamma u_k) + \mu^\gamma u_k \Big\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \sum_{0\leq j\leq m} (-1)^{m-j} \binom{m}{j} \sum_{|\gamma|=j} \frac{j!}{\gamma!} |\mu|^{2\gamma} \lim_{k\to\infty} \Big\langle \mathbf{S}_n(\mathbf{R}) u_k \mid u_k \Big\rangle = 0 \\
\Rightarrow\ & \Big(1 - \sum_{1\leq j\leq d} |\mu_j|^2\Big)^m \sum_{0\leq j\leq n} (-1)^j \binom{n}{j} \lim_{k\to\infty} \Big\langle \big(R_1 + \cdots + R_d\big)^{n-j} u_k \big|\ \big(R_1 + \cdots + R_d\big)^j u_k \Big\rangle = 0 \\
\Rightarrow\ & \Big(1 - \sum_{1\leq j\leq d} |\mu_j|^2\Big)^m \Big(\mu_1 + \cdots + \mu_d - \overline{\mu_1} - \cdots - \overline{\mu_d}\Big)^n = 0 \\
\Rightarrow\ & \Big(1 - \sum_{1\leq j\leq d} |\mu_j|^2\Big)^m \Big(\mu_1 + \cdots + \mu_d - \overline{\mu_1 + \cdots + \mu_d}\Big)^n = 0 \\
\Rightarrow\ & \Big(1 - \sum_{1\leq j\leq d} |\mu_j|^2\Big)^m \Big(2iIm\big(\mu_1 + \cdots + \mu_d\big)\Big)^n = 0 \\
\Rightarrow\ & \Big(1 - \sum_{1\leq j\leq d} |\mu_j|^2\Big)^m = 0 \ \text{ or } \Big(2iIm\big(\mu_1 + \cdots + \mu_d\big)\Big)^n = 0.
\end{aligned}$$

This shows that $\sigma_{ja}(\mathbf{R}) \subset \partial\mathbb{B}(\mathbb{C}^d)\bigcup\Big\{(\mu_1,\cdots,\mu_d)\in\mathbb{C}^d\ \ /\ \mu_1+\cdots+\mu_d\in\mathbb{R}\ \Big\}$.

(2) Assume that $\mu=(\mu_1,\cdots,\mu_d)\in\sigma_{jp}(\mathbf{R})$, then there exists a nonzero vector u for which $(R_l-\mu_l)u=0$ for $l=1,\cdots,d$. Since $\mathbf{R}$ is an (m,n)-isosymmetric multioperators, then it follows by a similar calculation as in the proof of statement (1) that

$$\Big(1-\sum_{1\le j\le d}|\mu_j|^2\Big)^m\Big(2iIm(\mu_1+\cdots+\mu_d)\Big)^n=0,$$

and so that

$$\Big(1-\sum_{1\le j\le d}|\mu_j|^2\Big)^m=0\ \text{ or }\ \Big(2iIm(\mu_1+\cdots+\mu_d)\Big)^n=0.$$

This justifies the statement (2).

(3) Assume that $\{u_k\}_k$ and $\{v_k\}_k$ be two sequences in $\mathcal{H}$ such that $\|u_k\|=\|v_k\|=1$,

$$\|(R_l-\mu_l)u_k\|\longrightarrow 0 \text{ and } \|(R_l-\mu_j')v_k\|\longrightarrow 0\ \ (\text{as }\ k\longrightarrow\infty)\ \text{ for }\ \ l=1,\cdots,d.$$

We have $\lim\limits_{k\to\infty}(R_l^{\gamma_l}-\mu_l^{\mu_l})u_k=0$ and $\lim\limits_{k\to\infty}(R_l^{\gamma_l}-\mu_l'^{\gamma_l})v_k=0$, which implies

$$\lim_{k\to\infty}(\mathbf{R}^\gamma-\mu^\gamma)u_k=0\quad\text{and}\quad\lim_{k\to\infty}(\mathbf{R}^\gamma-\mu'^\gamma)v_k=0.$$

Since $\mathbf{R}$ is an (m,n)-isosymmetric multioperators, it thus follows that

$$\begin{aligned}
0 &= \lim_{k\to\infty}\Big\langle\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\sum_{|\gamma|=j}\frac{j!}{\gamma!}\mathbf{R}^{*\gamma}\mathbf{S}_n(\mathbf{R})\mathbf{R}^\gamma\Big)u_k\mid v_k\Big\rangle\\
&= \sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\sum_{|\gamma|=j}\frac{k!}{\gamma!}\lim_{k\to\infty}\Big\langle\mathbf{S}_n(\mathbf{R})\big(\mathbf{R}^\gamma-\mu^\gamma+\mu^\gamma\big)u_k\mid\big((\mathbf{R}^\gamma-\mu'^\gamma+\mu'^\gamma\big)v_k\Big\rangle\\
&= \sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\sum_{|\gamma|=j}\frac{k!}{\gamma!}\mu^\gamma(\overline{\mu'})^\gamma\lim_{k\to\infty}\Big\langle\mathbf{S}_n(\mathbf{R})u_k\mid v_k\Big\rangle\\
&= \Big(1-\mu_1.\overline{\mu_1'}-\cdots-\mu_d.\overline{\mu_d'}\Big)^m\lim_{k\to\infty}\sum_{0\le j\le n}(-1)^j\binom{n}{j}\Big\langle\big(R_1+\cdots+R_d\big)^{n-j}u_k\mid\big(R_1+\cdots+R_d\big)^jv_k\Big\rangle\\
&= \Big(1-\mu_1.\overline{\mu_1'}-\cdots-\mu_d.\overline{\mu_d'}\Big)^m\Big(\mu_1+\cdots+\mu_d-\overline{\mu_1'}-\cdots-\overline{\mu_d'}\Big)^n\lim_{k\to\infty}\langle u_k\mid v_k\rangle\\
&= \Big(1-\sum_{1\le j\le d}\mu_j\overline{\mu_j'}\Big)^m\Big(\sum_{1\le j\le d}(\mu_j-\overline{\mu_j'})\Big)^n\lim_{k\to\infty}\langle u_k\mid v_k\rangle,
\end{aligned}$$

and this, since $1-\sum\limits_{1\le j\le d}\mu_j\overline{\mu_j'}\neq 0$ and $\sum\limits_{1\le j\le d}(\mu_j-\overline{\mu_j'})\neq 0$, implies $\lim\limits_{k\to\infty}\langle u_k\mid v_k\rangle=0$ as claimed. The proof of the statement (4) follows from an argument similar to that used in (3). This ends the proof. □

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