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Research Article

Fixed Point Theorems for Generalized Integral Type Weak-Contraction Mappings in Convex Modular Metric Spaces

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Abstract: Fixed point theory in convex modular metric spaces has seen significant advancements due to its broad applicability in various fields. In 2020, Chaira et al. [5] extended fixed point theorems for weak contraction mappings within partially ordered modular metric spaces. Subsequently, in 2023, Mithun et al. [14] established fixed point theorems for integral-type weak contraction mappings in modular metric spaces. Building on these foundational results, this paper investigates fixed point results for four mappings under integral-type contraction conditions in convex modular metric spaces.

Keywords: Fixed point, Δ_2-condition, convex modular metric spaces.

1. Introduction

Let (X, d) be a metric space, and let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $Tx = x$. A mapping $T : X \to X$ is said to be a contraction if there exists a constant $\alpha \in [0, 1)$ such that $d(Tu, Tv) \leq \alpha d(u, v)$ for all $u, v \in X$. The Banach Contraction Principle, introduced by Banach in 1922, asserts that such a mapping has a unique fixed point in X if X is a complete metric space. This principle is foundational in fixed point theory and has inspired extensive research, leading to numerous extensions and generalizations under various contractive conditions. One significant generalization is the concept of modular metric spaces, which extends the traditional notion of metric spaces. Modular spaces on linear spaces were first introduced by Nakano in 1950 [11]. Later, in 2010, Chistyakov [6] developed the framework of modular metric spaces, also known as parameterized metric spaces, by incorporating a time parameter. More recently, Khamsi and Kozlowski [9] introduced a fixed point theorem in modular function spaces in 2015, further advancing this field. To continue exploring fixed point theorem in metric modular space, follow those articles [2, 3, 8, 10, 13].

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In this paper, we present new fixed point results for four mappings satisfying integral-type contraction conditions in convex modular metric spaces.

2. Preliminaries

Let X be a non-empty set and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ a function defined by:

$$\omega_\tau(x, y) = \omega(\tau, x, y)$$

for all $x, y \in X$ and $\tau > 0$.

Definition 2.1 [1, 7, 9] *A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be modular metric on X if it satisfies the following conditions:*

(1) $\omega_\tau(x, y) = 0$ if and only if $x = y$ for all x, y in X and for all $\tau > 0$;

(2) $\omega_\tau(x, y) = \omega_\tau(y, x)$ for all x, y in X and for all $\tau > 0$;

(3) $\omega_{\tau+\nu}(x, y) \le \omega_\tau(x, z) + \omega_\nu(z, y)$ for all $x, y, z \in X$, for all $\tau, \nu > 0$.

The pair (X, ω) is said to be modular metric space.

Proposition 2.2 *If the condition (1) is satisfy for some $\tau > 0$, then ω is called regular modular space.*

Proposition 2.3 [14] *If $\omega_{\tau+\nu}(x, y) \le \frac{\tau}{\tau+\nu}\omega_\tau(x, z) + \frac{\nu}{\tau+\nu}\omega_\nu(z, y)$ for all $x, y, z \in X$ and for all $\tau, \nu > 0$, then ω is said to be a convex modular metric.*

If $0 < \nu < \tau$, then for the modular metric ω on a set X, the function $\tau \to \omega_\tau(x, y)$ is non-increasing on $(0, \infty)$ since, for any $x, y \in X$,

$$\omega_\tau(x, y) \le \omega_{\tau-\nu}(x, x) + \omega_\nu(x, y) = \omega_\nu(x, y).$$

Definition 2.4 [12] *Let (X, ω) be a modular metric space and fix $z_0 \in X$. Set*

$$X_\omega = X_\omega(z_0) = \{z \in X : \omega_\tau(z, z_0) \to 0 \ as \ \tau \to \infty\},$$

$$X_\omega^* = X_\omega^*(z_0) = \{z \in X : \omega_\tau(z, z_0) < \infty \ for \ \tau > 0\};$$

then the two linear spaces X_ω and X_ω^* are called modular spaces centered at z_0.

Proposition 2.5 *In case of some metric modular ω on X, if $\omega_\tau(x, y) = \omega_\nu(x, y) < \infty$ for all $x, y \in X$ and for all $\tau, \nu > 0$, then there exists a function $\rho(x, y)$ defined by $\rho(x, y) = \omega_\tau(x, y)$ is a metric on X.*

Definition 2.6 [1] *Let ω be a modular metric on a set X. Then*

(1) *A sequence $\{x_n\} \subset X_\omega$ is called ω-convergent to some $x \in X_\omega$ if and only if $\lim_{n\to\infty} \omega_1(x_n, x) = 0$ and x is called the ω-limit of $\{x_n\}$.*

(2) *A sequence $\{x_n\} \subset X_\omega$ is ω-Cauchy if for $m, n \in \mathbb{N}$ such that $\lim_{m,n\to\infty} \omega_1(x_m, x_n) = 0$.*

(3) *A set $W \subset X_\omega$ is ω-closed if ω-limit of any ω-convergent sequence of W is in W.*

(4) *A subset $W \subset X_\omega$ is ω-complete if any ω-Cauchy sequence in W is ω-convergent in W.*

Definition 2.7 [7] *Let ω is a modular metric on X, then ω satisfies Δ_2-condition or simply ω is Δ_2 if for a given sequence $\{x_n\} \subset X_\omega$ and for $x \in X_\omega$, for some $\tau > 0$, $\lim_{n\to\infty} \omega_\tau(x_n, x) = 0$ implies $\lim_{n\to\infty} \omega_\tau(x_n, x) = 0$ for all $\tau > 0$.*

Definition 2.8 [5] *Let ω is a modular metric on X, then ω satisfies the Δ_2-type condition if there exists a positive number k_1 such that $\omega_{\frac{\tau}{2}}(x, y) \le k_1 \omega_\tau(x, y)$ for all $x, y \in X_\omega$ and for all $\tau > 0$.*

Lemma 2.9 [5] *If ω satisfies the Δ_2-type condition, then ω satisfies Δ_2-condition.*

Lemma 2.10 [5] *Let $\{x_n\}$ be a sequence in X_ω and $\tau > 0$. If ω satisfies Δ_2-type condition, then $\{x_n\}$ is ω-Cauchy if and only if $\lim_{m,n\to\infty} \omega_\tau(x_m, x_n) = 0$.*

Lemma 2.11 [5] *If ω holds Δ_2-type condition, then ω is regular.*

Lemma 2.12 [5] *Let ω be a modular metric on X. If a sequence $\{x_n\} \subset X$ is not ω-Cauchy, then there exists $\epsilon > 0$ and two sub-sequence of integers $\{m_k\}$ and $\{n_k\}$ such that*

$$for\ m_k > n_k \ge k, \quad \omega_1(x_{n_k}, x_{m_k}) \ge \epsilon\ and\ \omega_1(x_{n_k}, x_{m_k-1}) < \epsilon.$$

Lemma 2.13 [5] *Let (X, ω) be a modular metric space and $r, s \in \mathbb{N}^*$ such that ω holds Δ_2-type condition. If a sequence $\{x_n\}$ is not ω-Cauchy in X, then there exists $\epsilon > 0$ and two sub-sequence of integers $\{m_k\}$ and $\{n_k\}$ such that*

$$for\ m_k > n_k \ge k, \quad \omega_{2^r}(x_{n_k}, x_{m_k}) \ge \epsilon\ and\ \omega_{\frac{1}{2^s}}(x_{n_k}, x_{m_k-1}) < \epsilon.$$

Lemma 2.14 [5] *Let ω be a modular metric on X such that ω satisfies Δ_2-condition. If a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} \omega_1(x_n, x_{n+1}) = 0$ then, $\{x_n\}$ is said to be a ω-Cauchy .*

Theorem 2.15 [14] *Let (X,ω) be a convex modular space and F be a non-empty complete subset of X such that ω satisfies the Δ_2-type condition. Let f, g : $(F,\omega) \to (F,\omega)$ be two functions satisfying the following:*

$$\int_0^{\gamma_1\{\omega_1(fx,gy)\}} \lambda(t)dt \le \int_0^{\upsilon\{\Omega(x,y)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x,y)\}} \lambda(t)dt; \tag{1}$$

with

$$\int_0^{\gamma_1(t)} \lambda(z)dz - \int_0^{\upsilon(t)} \lambda(z)dz + \int_0^{\phi(t)} \lambda(z)dz > 0;$$

for all

$$r > 0,\ \lim_{t\to r} \int_0^{\gamma_1(t)} \lambda(z)dz - \lim_{t\to r} \int_0^{\upsilon(t)} \lambda(z)dz + \liminf_{t\to r} \int_0^{\phi(t)} \lambda(z)dz > 0;$$

where $\gamma_1 \in \Gamma$, $\upsilon \in \Upsilon$, $\phi \in \Phi$ and $\lambda \in \Lambda$, and

$$\Omega(x,y) = \max\Big\{\omega_1(x,fx), \omega_1(y,gy), \omega_1(x,y), \omega_2(fx,y), \omega_2(x,gy)\Big\}. \tag{2}$$

Then f and g have a unique fixed point in F.

3. Main Results

From reference [4], we consider $\Lambda = \{\lambda | \lambda : \mathbb{R}^+ \to \mathbb{R}^+\}$ which is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ such that

(a) $\int_0^{\epsilon} \lambda(t)dt > 0$ for each $\epsilon > 0$,

(b) $\int_0^{a+b} \lambda(t)dt \le \int_0^{a} \lambda(t)dt + \int_0^{b} \lambda(t)dt$.

Lemma 3.1 [4] *Let $\lambda \in \Lambda$ and $\{s_n\}$ be a non-negative sequence with $\lim_{n\to\infty} s_n = s$, then*

$$\lim_{n\to\infty} \int_0^{s_n} \lambda(t)dt = \int_0^{s} \lambda(t)dt.$$

Lemma 3.2 [4] *Let $\lambda \in \Lambda$ and $\{s_n\}$ be a non-negative sequence. Then*

$$\lim_{n\to\infty} \int_0^{s_n} \lambda(t)dt = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} s_n = 0.$$

Consider three classes of functions Γ, Υ and Φ are as follows:

(a) $\Gamma = \Big\{\gamma : [0,\infty) \to [0,\infty)$ such that (i) γ is strictly increasing; (ii) $\lim_{t\to r} \gamma(t) > 0$ for $r > 0$ and $\lim_{t\to 0^+} \gamma(t) = 0$; (iii) $\gamma(t) = 0$ if and only if $t = 0\Big\}$.

(b) $\Upsilon = \Big\{\upsilon : [0,\infty) \to [0,\infty)$ such that (i) υ is non-decreasing; (ii) $\lim_{t\to r}\upsilon(t) > 0$ for $r > 0$ and $\lim_{t\to 0^+}\upsilon(t) = 0$; (iii) $\upsilon(t) = 0$ if and only if $t = 0\Big\}$.

(c) $\Phi = \Big\{\phi : [0,\infty) \to [0,\infty)$ such that (i) $\lim_{t\to r} inf \phi(t) > 0$ for all $r > 0$; (ii) $\phi(t) \to 0 \Rightarrow t \to 0$; (iii) $\phi(t) = 0$ if and only if $t = 0\Big\}$.

Theorem 3.3 *Let (X,ω) be a convex modular space and F be a non-empty complete subset of X such that ω satisfies the Δ_2-type condition. Let P, Q, R, S $:(F,\omega) \to (F,\omega)$ be four functions satisfying the following:*

$$\int_0^{\gamma_1\{\omega_1(Rx,Sy)\}} \lambda(t)dt \le \int_0^{\upsilon\{\Omega(x,y)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x,y)\}} \lambda(t)dt; \tag{3}$$

with

$$\int_0^{\gamma_1(t)} \lambda(z)dz - \int_0^{\upsilon(t)} \lambda(z)dz + \int_0^{\phi(t)} \lambda(z)dz > 0;$$

for all

$$r > 0,\ \lim_{t\to r}\int_0^{\gamma_1(t)} \lambda(z)dz - \lim_{t\to r}\int_0^{\upsilon(t)} \lambda(z)dz + \liminf_{t\to r}\int_0^{\phi(t)} \lambda(z)dz > 0;$$

where $\gamma_1 \in \Gamma$, $\upsilon \in \Upsilon$, $\phi \in \Phi$ and $\lambda \in \Lambda$, and

$\Omega(x,y) = \max\Big\{\omega_1(Px,Rx), \omega_1(Qy,Sy), \omega_1(Qy,Rx), \omega_2(Px,Sy), \omega_2(Px,Qy)\Big\}$.

Also,

(a) $R \subseteq Q$ *and* $S \subseteq P$,

(b) $\{P,R\}$ *and* $\{Q,S\}$ *is weakly compatible; either P or R is continuous.*

Then P, Q, R and S have a unique common fixed point in F.

Proof. Let x_0 be any arbitrary element in X. From condition (a), there exist two elements x_1 and x_2 in X such that $Rx_0 = Qx_1 = y_0$ and $Sx_1 = Px_2 = y_1$. Proceeding inductively we can construct a sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Rx_{2n} = Qx_{2n+1} \ \ \textit{and} \ \ y_{2n+1} = Sx_{2n+1} = Px_{2n+2}$$

for all $n \in \mathbb{N}$.

Case-1: For some $n \in \mathbb{N}$, $y_n = y_{n+1} \Rightarrow y_{n+1} = y_{n+2}$.

If n is even, i.e., $n = 2k$, $k \in \mathbb{N}$, we have

$$y_{2k} = y_{2k+1}. \tag{4}$$

If $y_{2k+1} \neq y_{2k+2}$, then $\omega_1(y_{2k+1}, y_{2k+2}) > 0$.

Now,

$$\begin{aligned}
\Omega(y_{2k+2}, y_{2k+1}) &= \max\Big\{\omega_1(Px_{2k+2}, Rx_{2k+2}), \omega_1(Qx_{2k+1}, Sx_{2k+1}), \omega_1(Qx_{2k+1}, Rx_{2k+2}), \\
&\quad \omega_2(Px_{2k+2}, Sx_{2k+1}), \omega_2(Px_{2k+2}, Qx_{2k+1})\Big\} \\
&= \max\Big\{\omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k}, y_{2k+2}), \\
&\quad \omega_2(y_{2k+1}, y_{2k+1}), \omega_2(y_{2k+1}, y_{2k})\Big\} \\
&= \max\Big\{\omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k}, y_{2k+2})\Big\} \\
&= \max\Big\{\omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k+1}, y_{2k+2})\Big\} \\
&= \omega_1(y_{2k+1}, y_{2k+2}).
\end{aligned}$$

Hence, $\Omega(y_{2k+2}, y_{2k+1}) = \omega_1(y_{2k+1}, y_{2k+2})$.

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+2}, y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+2}, Sx_{2k+1})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k+2}, x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+2}, x_{2k+1})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\Omega(y_{2k+1}, y_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k+1}, y_{2k+2})\}} \lambda(t)dt \\
&< \int_0^{\gamma\{\Omega(y_{2k+2}, y_{2k+1})\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction.

Hence,

$$y_{2k} = y_{2k+1} \Rightarrow y_{2k+1} = y_{2k+2}. \tag{5}$$

If, n is odd, i.e., for $n = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, we have

$$y_{2k+1} = y_{2k+2} \tag{6}$$

and

$$\Omega(x_{2k+2}, x_{2k+3}) = \omega_1(x_{2k+2}, x_{2k+3}). \tag{7}$$

Now,

$$\begin{aligned}
\Omega(x_{2k+2},x_{2k+3}) &= \max\Big\{\omega_1(Px_{2k+2},Rx_{2k+2}),\omega_1(Qx_{2k+3},Sx_{2k+3}),\omega_1(Qx_{2k+3},Rx_{2k+2}),\\
&\quad \omega_2(Px_{2k+2},Sx_{2k+3}),\omega_2(Px_{2k+2},Qx_{2k+3})\Big\}\\
&= \max\Big\{\omega_1(y_{2k+1},y_{2k+2}),\omega_1(y_{2k+2},y_{2k+3}),\omega_1(y_{2k+2},y_{2k+2}),\omega_2(y_{2k+1},y_{2k+3}),\\
&\quad \omega_2(y_{2k+1},y_{2k+2})\Big\}\\
&\le \max\Big\{\omega_1(y_{2k+2},y_{2k+3}),\omega_2(y_{2k+1},y_{2k+3})\Big\}\\
&\le \max\Big\{\omega_1(y_{2k+2},y_{2k+3}),\frac{\omega_1(y_{2k+1},y_{2k+2})+\omega_1(y_{2k+2},y_{2k+3})}{2}\Big\}\\
&= \omega_1(y_{2k+2},y_{2k+3}).
\end{aligned}$$

Hence, $\Omega(x_{2k+2},x_{2k+3})=\omega_1(y_{2k+2},y_{2k+3})$.

Now,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+2},y_{2k+3})\}}\lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+2},Sx_{2k+3})\}}\lambda(t)dt\\
&\le \int_0^{\upsilon\{\Omega(x_{2k+2},x_{2k+3})\}}\lambda(t)dt-\int_0^{\phi\{\Omega(x_{2k+2},x_{2k+3})\}}\lambda(t)dt\\
&= \int_0^{\upsilon\{\Omega(y_{2k+2},y_{2k+3})\}}\lambda(t)dt-\int_0^{\phi\{\Omega(y_{2k+2},y_{2k+3})\}}\lambda(t)dt\\
&< \int_0^{\gamma\{\Omega(y_{2k+2},y_{2k+3})\}}\lambda(t)dt
\end{aligned}$$

which is a contradiction.

Hence,

$$y_{2k+1}=y_{2k+2}\Rightarrow y_{2k+2}=y_{2k+3}.$$

If we continue thus process, then we obtain $y_n=y_{n+1}\Rightarrow y_n=y_{n+k}$ for $k=1,2,\cdots$. Therefore $\{y_n\}$ is a constant sequence and hence ω-Cauchy sequence in F.

Case-2: Let $y_n\neq y_{n+1}$ for all $n\in\mathbb{N}$. If n is even, i.e., $n=2k$, $k\in\mathbb{N}$, we have

$$\begin{aligned}
\Omega(x_{2k},x_{2k+1}) &= \max\Big\{\omega_1(Px_{2k},Rx_{2k}),\omega_1(Qx_{2k+1},Sx_{2k+1}),\omega_1(Qx_{2k+1},Rx_{2k}),\omega_2(Px_{2k},Sx_{2k+1}),\\
&\quad \omega_2(Px_{2k},Qx_{2k+1})\Big\}\\
&= \max\Big\{\omega_1(y_{2k-1},y_{2k}),\omega_1(y_{2k},y_{2k+1}),\omega_1(y_{2k},y_{2k}),\omega_2(y_{2k-1},y_{2k+1}),\omega_2(y_{2k-1},y_{2k})\Big\}\\
&\le \max\Big\{\omega_1(y_{2k-1},y_{2k}),\omega_1(y_{2k},y_{2k+1}),\frac{\omega_1(y_{2k-1},y_{2k})+\omega_1(y_{2k},y_{2k+1})}{2},\omega_2(y_{2k-1},y_{2k})\Big\}
\end{aligned}$$

If $\Omega(x_{2k}, x_{2k+1}) = \omega(y_{2k}, y_{2k+1})$, then

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}} \lambda(t)dt \\ &\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt \\ &= \int_0^{\upsilon\{\Omega(y_{2k},y_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k},y_{2k+1})\}} \lambda(t)dt \\ &< \int_0^{\gamma\{\Omega(y_{2k},y_{2k+1})\}} \lambda(t)dt\end{aligned}$$

which is a contradiction. Hence,

$$\Omega(x_{2k}, x_{2k+1}) = \omega_1(y_{2k-1}, y_{2k}). \tag{8}$$

Now from (1)

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}} \lambda(t)dt \\ &\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt \\ &= \int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k-1},y_{2k})\}} \lambda(t)dt \\ &< \int_0^{\gamma_1\{\Omega(y_{2k-1},y_{2k})\}} \lambda(t)dt\end{aligned}$$

Since γ_1 is strictly increasing, so we have

$$\omega_1(y_{2k}, y_{2k+1}) < \omega_1(y_{2k-1}, y_{2k}). \tag{9}$$

If n is odd, i.e., $n = 2k+1$, $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\Omega(x_{2k+1}, x_{2k+2}) &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(Qx_{2k+2}, Sx_{2k+2}), \omega_1(Qx_{2k+2}, Rx_{2k+1}), \\ &\quad \omega_2(Px_{2k+1}, Sx_{2k+2}), \omega_2(Px_{2k+1}, Qx_{2k+2})\Big\} \\ &= \max\Big\{\omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k+1}, y_{2k+2}), \omega_1(y_{2k+1}, y_{2k+1}), \\ &\quad \omega_2(y_{2k}, y_{2k+2}), \omega_2(y_{2k}, y_{2k+1})\Big\} \\ &\leq \max\Big\{\omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k+1}, y_{2k+2}), \\ &\quad \frac{\omega_1(y_{2k}, y_{2k+1}) + \omega_1(y_{2k+1}, y_{2k+2})}{2}, \omega_2(y_{2k}, y_{2k+1})\Big\}.\end{aligned}$$

If $\Omega(x_{2k+1}, x_{2k+2}) = \omega_1(y_{2k+1}, y_{2k+2})$, then

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+1},Sx_{2k+2})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k+1},x_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+1},x_{2k+2})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt \\
&< \int_0^{\gamma\{\omega_1(y_{2k+1},y_{2k+2})\}} \lambda(t)dt
\end{aligned}$$

which is a contradiction.

Hence,

$$\Omega(x_{2k+1}, x_{2k+2}) = \omega_1(x_{2k}, x_{2k+1}) \tag{10}$$

Now, from (1)

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}} \lambda(t)dt \\
&= \int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(y_{2k-1},y_{2k})\}} \lambda(t)dt \\
&< \int_0^{\gamma_1\{\omega_1(y_{2k-1},y_{2k})\}} \lambda(t)dt.
\end{aligned}$$

Since γ_1 is strictly increasing, so we have

$$\omega_1(y_{2k}, y_{2k+1}) < \omega_1(y_{2k-1}, y_{2k}). \tag{11}$$

From (9) and (11) we conclude that

$$\omega_1(y_n, y_{n+1}) < \omega_1(y_{n-1}, y_n) foralln = 1, 2, 3, \cdots. \tag{12}$$

Therefore $\{\omega_1(y_n, y_{n+1})\}$ is monotone decreasing and bounded below, so convergent.

Let

$$\lim_{n\to\infty} \omega_1(y_n, y_{n+1}) = l \quad and \quad \lim_{n\to\infty} \int_0^{\gamma_1\{\omega_1(y_n,y_{n+1})\}} \lambda(t)dt = l^*,$$

where l and $l^* \geq 0$.

Claim: $l = 0$.

If not, then $l > 0$. Then $\lim_{k\to\infty} \omega_1(y_{2k}, y_{2k+1}) = l$ and $\lim_{n\to\infty} \int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}})\lambda(t)dt = l^*$.

Now, from (1) we have

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}}\lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k},Sx_{2k+1})\}}\lambda(t)dt\\ &\le \int_0^{\upsilon\{\Omega(x_{2k},x_{2k+1})\}}\lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k},x_{2k+1})\}}\lambda(t)dt\\ &= \int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt - \int_0^{\phi\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt\end{aligned}$$

Taking limit as $k \to \infty$ in the above inequalities, we have

$$\lim_{k\to\infty}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}}\lambda(t)dt \le \lim_{k\to\infty}\int_0^{\upsilon\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt - \lim_{k\to\infty}\inf\int_0^{\phi\{\omega(y_{2k-1},y_{2k})\}}\lambda(t)dt$$

This implies

$$\lim_{k\to\infty}\int_0^{\gamma_1\{\omega_1(y_{2k},y_{2k+1})\}}\lambda(t)dt \le \lim_{k\to\infty}\int_0^{\gamma_1\{\omega_1(y_{2k-1},y_{2k})\}}\lambda(t)dt,$$

i.e., $l^* \le l^*$ which is a contradiction. Hence, $l = 0$, i.e.,

$$\lim_{n\to\infty}\omega_1(y_n, y_{n+1}) = 0. \tag{13}$$

Since (X,ω) be a convex modular space and F is a ω-complete subset of X satisfying Δ_2-type condition and $\lim_{n\to\infty}\omega_1(y_n, y_{n+1}) = 0$, so by Lemma 2.14 $\{y_n\}$ is ω-Cauchy in F. Since F is ω-complete, there exists $z \in F$ such that $\lim_{n\to\infty}\omega(y_n, z) = 0$.

Therefore

$$\lim_{n\to\infty}\omega_1(y_{2n}, z) = \lim_{n\to\infty}\omega_1(Rx_{2n}, z) = \lim_{n\to\infty}\omega_1(Qx_{2n+1}, z) = 0$$

and

$$\lim_{n\to\infty}\omega_1(y_{2n+1}, z) = \lim_{n\to\infty}\omega_1(Sx_{2n+1}, z) = \lim_{n\to\infty}\omega_1(Px_{2n+2}, z) = 0.$$

Since the mappings P and R are compatible, so $\lim_{k\to\infty} RPx_{2k} = \lim_{k\to\infty} PRx_{2k}$.

We assume that $Pz = z$. If not, then we will arrives a contradiction.

$$\begin{aligned}\Omega(Px_{2k}, x_{2k+1}) &= \max\Big\{\omega_1(P^2x_{2k}, RPx_{2k}), \omega_1(Qx_{2k+1}, Sx_{2k+1}), \omega_1(Qx_{2k+1}, RPx_{2k}),\\ &\quad \omega_2(P^2x_{2k}, Sx_{2k+1}), \omega_2(P^2x_{2k}, Qx_{2k+1})\Big\}\\ &= \max\Big\{\omega_1(P^2x_{2k}, PRx_{2k}), \omega_1(Qx_{2k+1}, Sx_{2k+1}), \omega_1(Qx_{2k+1}, PRx_{2k}),\\ &\quad \omega_2(P^2x_{2k}, Sx_{2k+1}), \omega_2(P^2x_{2k}, Qx_{2k+1})\Big\}\\ &= \max\Big\{\omega_1(Py_{2k-1}, Py_{2k}), \omega_1(y_{2k}, y_{2k+1}), \omega_1(y_{2k}, Py_{2k}),\\ &\quad \omega_2(Py_{2k-1}, y_{2k+1}), \omega_2(Py_{2k-1}, y_{2k})\Big\}.\end{aligned}$$

Taking limit as $k \to \infty$,

$$\begin{aligned}\lim_{k\to\infty} \Omega(Px_{2k}, x_{2k+1}) &= \max\Big\{\omega_1(Pz, Pz), \omega_1(z, z), \omega_1(z, Pz), \omega_2(Pz, z), \omega_2(Pz, z)\Big\}\\ &= \omega_1(Pz, z)\end{aligned}$$

Now,

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(Py_{2k}, y_{2k+1})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(PRx_{2k}), Sx_{2k+1}\}} \lambda(t)dt\\ &= \int_0^{\gamma_1\{\omega_1(R(Px_{2k}), Sx_{2k+1})\}} \lambda(t)dt\\ &\le \int_0^{\upsilon\{\Omega(Px_{2k}, x_{2k+1})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(Px_{2k}, x_{2k+1})\}} \lambda(t)dt\\ \Rightarrow \int_0^{\gamma_1\{\omega_1(Pz, z)\}} \lambda(t)dt &< \int_0^{\gamma_1\{\omega_1(Pz, z)\}} \lambda(t)dt\end{aligned}$$

which is a contradiction. Hence, $Pz = z$.

We assume that $Rz = z$. If not, then we will arrive a contradiction.

$$\begin{aligned}\Omega(z, x_{2k}) &= \max\Big\{\omega_1(Pz, Rz), \omega_1(Qx_{2k}, Sx_{2k}), \omega_1(Qx_{2k}, Rz),\\ &\quad \omega_2(Pz, z), \omega_2(Pz, Qx_{2k})\Big\}.\end{aligned}$$

Taking limit as $k \to \infty$,

$$\begin{aligned}\lim_{k\to\infty} \Omega(z, x_{2k}) &= \max\Big\{\omega_1(z, Rz), \omega_1(z, z), \omega_1(z, Rz),\\ &\quad \omega_2(Pz, z), \omega_2(Pz, z)\Big\}\\ &= \omega_1(z, Rz).\end{aligned}$$

Now,

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(Rz,y_{2k})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rz,Sx_{2k}\}} \lambda(t)dt \\ &\leq \int_0^{\upsilon\{\Omega(z,x_{2k})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(z,x_{2k})\}} \lambda(t)dt \\ \Rightarrow \int_0^{\gamma_1\{\omega_1(Pz,z)\}} \lambda(t)dt &< \int_0^{\gamma_1\{\omega_1(Pz,z)\}} \lambda(t)dt\end{aligned}$$

which is a contradiction. Hence, $Rz = z$.

We assume that $Qz = z$. If not, then we will arrive a contradiction. Since the mappings S and Q are compatible, so $SQx_{2k+2} = QSx_{2k+2}$.

$$\begin{aligned}\Omega(x_{2k+1}, Qx_{2k+2}) &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(Q^2x_{2k+2}, SQx_{2k+2}), \omega_1(Q^2x_{2k+2}, Rx_{2k+1}), \\ &\quad \omega_2(Px_{2k+1}, SQx_{2k+2}), \omega_2(Px_{2k+1}, Q^2x_{2k+2})\Big\} \\ &= \max\Big\{\omega_1(y_{2k}, y_{2k+1}), \omega_1(Q^2x_{2k+2}, QSx_{2k+2}), \omega_1(Q^2x_{2k+2}, Rx_{2k+1}), \\ &\quad \omega_2(Px_{2k+1}, QSx_{2k+2}), \omega_2(Px_{2k+1}, Q^2x_{2k+2})\Big\}.\end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\begin{aligned}\lim_{k\to\infty} \Omega(x_{2k+1}, Qx_{2k+2}) &= \max\Big\{\omega_1(z,z), \omega_1(Qz,Qz), \omega_1(Qz,z), \omega_2(z,Qz), \omega_2(z,Qz)\Big\} \\ &= \omega_1(z,Qz).\end{aligned}$$

Now,

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k+1},Qy_{2k+2})\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+1},Q(Sx_{2k+2}))\}} \lambda(t)dt \\ &\leq \int_0^{\upsilon\{\Omega(x_{2k+1},Qx_{2k+2})\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+1},Qx_{2k+2})\}} \lambda(t)dt \\ &= \int_0^{\upsilon\{\Omega(x_{2k+1},Qx_{2k+2})\}} \lambda(t)dt.\end{aligned}$$

In both side taking limit as $k \to \infty$, we have

$$\int_0^{\gamma_1\{\omega_1(z,Qz)\}} \lambda(t)dt < \int_0^{\gamma_1\{\omega_1(z,Qz)\}} \lambda(t)dt$$

which is a contradiction. Hence $Qz = z$.

We assume that $Sz = z$. If not, then we will arrive a contradiction.

$$\begin{aligned}\Omega(x_{2k+1}, z) &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(Qz, Sz), \omega_1(Qz, Rx_{2k+1}),\\ &\quad \omega_2(Px_{2k+1}, Sz), \omega_2(Px_{2k+1}, Qz)\Big\}\\ &= \max\Big\{\omega_1(Px_{2k+1}, Rx_{2k+1}), \omega_1(z, Sz), \omega_1(z, Rx_{2k+1}),\\ &\quad \omega_2(Px_{2k+1}, Sz), \omega_2(Px_{2k+1}, z)\Big\}\end{aligned}$$

Taking limit as $k \to \infty$

$$\begin{aligned}\lim_{k\to\infty} \Omega(x_{2k+1}, z) &= \max\Big\{\omega_1(z, z), \omega_1(z, Sz), \omega_1(z, z), \omega_2(z, Sz), \omega_2(z, z)\Big\}\\ &= \omega_1(z, Sz).\end{aligned}$$

Now,

$$\begin{aligned}\int_0^{\gamma_1\{\omega_1(y_{2k+1}, Sz)\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rx_{2k+1}, Sz\}} \lambda(t)dt\\ &\leq \int_0^{\upsilon\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt.\end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \int_0^{\gamma_1\{\omega_1(y_{2k+1}, Sz)\}} \lambda(t)dt \leq \lim_{k\to\infty} \int_0^{\upsilon\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt - \liminf_{k\to\infty} \int_0^{\phi\{\Omega(x_{2k+1}, z)\}} \lambda(t)dt$$

$$\Rightarrow \int_0^{\gamma_1\{\omega_1(z, Sz)\}} \lambda(t)dt < \int_0^{\gamma_1\{\omega_1(z, Sz)\}} \lambda(t)dt$$

which is a contradiction. Hence, $Sz = z$.

Therefore z is a common fixed point of P, Q, R and S.

To prove the uniqueness, we assume that $w(\neq z)$ is also a fixed point of P, Q, R and S. Then $Pz = Qz = Rz = Sz = z$ and $Pw = Qw = Rw = Sw = w$.

Now,

$$\begin{aligned}\Omega(z, w) &= \max\Big\{\omega_1(Pz, Rz), \omega_1(Qw, Sw), \omega_1(Qw, Rz), \omega_2(Pz, Sw), \omega_2(Pz, Qw)\Big\}\\ &= \max\Big\{\omega_1(z, z), \omega_1(w, w), \omega_1(w, z), \omega_2(z, w), \omega_2(z, w)\Big\}\\ &= \omega_1(w, z).\end{aligned}$$

From (1) we have,

$$\begin{aligned}
\int_0^{\gamma_1\{\omega_1(z,w)\}} \lambda(t)dt &= \int_0^{\gamma_1\{\omega_1(Rz,Sw\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\Omega(z,w)\}} \lambda(t)dt - \int_0^{\phi\{\Omega(z,w)\}} \lambda(t)dt \\
&\leq \int_0^{\upsilon\{\omega(z,w)\}} \lambda(t)dt - \int_0^{\phi\{\omega(z,w)\}} \lambda(t)dt \\
&< \int_0^{\gamma\{\omega(z,w)\}} \lambda(t)dt
\end{aligned}$$

which contradicts our hypothesis. Hence, $z = w$. This completes the proof.

4. Conclusion

This study contributes to the ongoing development of fixed point theory by establishing new results for four mappings that satisfy integral-type contraction conditions within the framework of convex modular metric spaces. These findings not only extend classical results like the Banach Contraction Principle but also build upon recent advancements in modular metric and function spaces. Given the rich structure and flexibility of modular frameworks, there remains substantial potential for further exploration in this direction. Future work may focus on broader classes of contractive conditions, additional structural generalizations, and and examine practical applications across diverse mathematical and applied contexts within the modular framework.

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Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Jayanta Das]: Thought and designed the research/problem; Contributed to research method and evaluation of data; Collected the data; Wrote the manuscript. (60%).

Author [Ashoke Das]: Collected the data; Contributed to completing the research and solving the problem. (40%).

Conflicts of Interest

The authors declare no conflict of interest.

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