NOTES ON ESPECIAL CONTINUED FRACTION EXPANSIONS AND REAL QUADRATIC NUMBER FIELDS

Özen ÖZER

Department of Mathematics, Faculty of Science and Arts,
Kırklareli University, 39100, Kırklareli - TURKEY
ozenozer39@gmail.com

Abstract

The primary purpose of this paper is to classify real quadratic fields $Q(\sqrt{d})$ which include the form of specific continued fraction expansion of integral basis element w_d for arbitrary period length $\ell = \ell(d)$ where $d \equiv 2,3(\mathrm{mod}4)$ is a square free positive integers.

Furthermore, the present paper deals with determining new certain parametric formulas of fundamental unit $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ and Yokoi's d-invariants n_d , m_d for such real quadratic fields. All results are also supported by several numerical tabular forms.

Key Words: Quadratic Fields**,** Continued Fractions, Fundamental Units.

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Özet

Bu makalenin asıl amacı, $d \equiv 2,3(\mathrm{mod}4)$ kare çarpansız pozitif tamsayılar olmak üzere keyfi $\ell = \ell(d)$ periyod uzunluğu için tamlık taban elemanı olan w_d nin özel bir sürekli kesre açılımındaki formu içeren $Q(\sqrt{d})$ reel kuadratik sayı cisimlerini sınıflandırmaktır.

Ayrıca bu çalışma, ilgili reel kuadratik sayı cisimleri için Yokoi'nin d-invaryantları olan n_d , m_d ile $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ temel biriminin kesin parametrik formüllerinin belirlenmesi ile ilgilenmektedir. Tüm sonuçlar bir takım nümerik tablolar ile de desteklenmektedir.

Anahtar Kelimeler: Kuadratik Cisimler, Sürekli Kesirler, Temel Birimler.

1. INTRODUCTION

Quadratic fields have applications in different areas of mathematics such as quadratic forms, algebraic geometry, diophantine equations, algebraic number theory, and even cryptography.

The Unit Theorem for real quadratic fields says that every unit in the integer ring of a quadratic field is given in terms of the fundamental unit of the quadratic field. Thus, determining the fundamental units of quadratic fields is of great importance.

Let $k = Q(\sqrt{d})$ be a real quadratic number field where $d > 0$ is a positive square free integer. Integral basis element is denoted by $w_d = \sqrt{d} = [\, a_0; \overline{a_1, a_2, \dots, a_{\ell(d)-1}, 2a_0}\,]$ and $\ell(\mathrm{d})$ is the period length in simple continued fraction expansion of algebraic integer w_d for $d \equiv 2,3(mod4)$. The fundamental unit ε_d of real quadratic number field is also denoted by $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ where $N\left(\varepsilon_d\right) = (-1)^{\ell(\mathrm{d})}$.

Furthermore, Yokoi's invariants are expressed by $m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]$ and $n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$ where $[\![x]\!]$ represents the greatest integer not greater than x. The sequence $\{Y_n\}$ is also special sequence which will be defined in Section 2.

By using coefficients of fundamental unit H.Yokoi defined two significant invariants such as m_d, n_d for class number problem and the solutions of Pell equation in [9].

In [7], Tomita described explicitly form of the fundamental units of the real quadratic fields by using Fibonacci sequence and continued fraction. He also gave some results for the continued fraction expansion of w_d where $d \equiv 1(mod4)$ for $\ell(\mathrm{d}) = 3$ in [6].

Determining of some certain fundamental units $\varepsilon_d = \left(t_d + u_d\sqrt{d}\right)/2 \rangle 1$ of $k = Q(\sqrt{d})$ was studied by R.Sasaki and R.A.Mollin ([4], [1]). Moreover, please see [3],[5] and [8] for more details about continued fraction expansions.

We will investigate the continued fraction expansions which have partial quotients elements as 5s (except the last digit of the period, which is always $2[\![\sqrt{d}]\!]$ for $w_d = \sqrt{d}$) with a given period length. Although there are infinitely many values of d having all 5s in the symmetric part of period of integral basis element, we will classify them according to arbitrary period length.

We will also determine the general forms of fundamental units ε_d and t_d , u_d coefficents of fundamental units $\varepsilon_d = \left(t_d + u_d \sqrt{d}\right)/2 \rangle 1$ in the terms of $\{Y_n\}$ as new formulizations which have been unknown yet for such real quadratic fields. By using results, the fundamental unit, continued fraction expansions and Yokoi's invariants will be calculated more easily for such $Q(\sqrt{d})$.

2. PRELIMINARIES

We need following definitions and lemmas which will be used in our main results for the section 3.

Definition 2.1. $\{Y_i\}$ is said to be a sequence defined by the recurrence relation

$$Y_i = 5Y_{i-1} + Y_{i-2}$$

with seed values $Y_0 = 0$ and $Y_1 = 1$. We can calculate some values of the terms of the sequence as follows:

$Y_2 = 5Y_1 + Y_0 = 5$, $Y_3 = 5Y_2 + Y_1 = 25 + 1 = 26$, $Y_4 = 5Y_3 + Y_2 = 130 + 5 = 135$,

$Y_5 = 5Y_4 + Y_3 = 5.135 + 26 = 701$, $Y_6 = 5Y_5 + Y_4 = 3640$, $Y_7 = 18901$, $Y_8 = 98145$,

$Y_9 = 509626$, $Y_{10} = 2646275$,$Y_{11} = 13741001$, $Y_{12} = 71351280$, $Y_{13} = 370497401$, … This sequence plays an important role in this paper to describe our lemmas and main results.

Lemma 2.1. For a square free positive integer d congruent to 2,3 modulo 4, we put $w_d = \sqrt{d}$, $a_0 = [\![\sqrt{d}]\!]$ into the $w_R = a_0 + w_d$. Then $w_d \notin R(d)$, but $w_R \in R\ (d)$ holds.

Moreover, for the period $l = l(d)$ of w_R, we get

$$w_R = [\ \overline{2a_0, a_1, a_2, \dots,\ a_{l(d)-1}}\ ] \quad \text{and} \quad w_d = [a_0;\ \overline{a_1, a_2, \dots,\ a_{l(d)-1}, 2a_0}\ ].$$

Furthermore, let $w_R = \frac{w_R P_l + P_{l-1}}{w_R Q_l + Q_{l-1}} = [\ 2a_0, a_1, a_2, \dots,\ a_{l(d)-1}, w_R]$ be a modular automorphism of w_R. Then the fundamental unit ε_d of $Q(\sqrt{d})$ is given by the following formula:

$$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1} \rangle 1$$

$$t_d = 2a_0.Q_{l(d)} + 2Q_{l(d)-1} \quad \text{and} \quad u_d = 2Q_{l(d)}$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and $Q_{i+1} = a_i Q_i + Q_{i-1}, (i \geq 1)$.

Proof. Proof is omitted in [6].

Lemma 2.2. Let $d \equiv 2,3(mod4)$ be the square free positive integer and w_d has got partial constant elements repeated 5s in the case of period $l = l(d)$. If $a_0 = [\![\sqrt{d}]\!]$ denote the integer part of $w_d = \sqrt{d}$ for $d \equiv 2,3(mod4)$, then we have continued fraction expansion

$$w_d = \sqrt{d} = [\, a_0; \overline{a_1, a_2, \ldots, a_{\ell(d)-1}, a_{\ell(d)}}\,] = [\, a_0; \overline{5,5, \ldots, 5, 2a_0}\,]$$

for quadratic irrational numbers and $w_R = a_0 + \sqrt{d} = a_0 + [\, a_0; \overline{5,5, \ldots, 5, 2a_0}\,] = [\, \overline{2a_0, 5, , \ldots, 5}\,]$ for reduced quadratic irrational numbers.

Furthermore, $A_k = a_0 Y_{k+1} + Y_k$ and $B_k = Y_{k+1}$ are determined in the continued fraction expansions where $\{A_k\}$ and $\{B_k\}$ are two sequences defined by :

$$A_{-2} = 0 \;\; , A_{-1} = 1 \;\; , \;\; A_k = a_k A_{k-1} + A_{k-2} \qquad (\text{for } 0 \le k \le l-1 \,)$$

$$B_{-2} = 1 \;\; , \; B_{-1} = 0 \;\; , \;\; B_k = a_k B_{k-1} + B_{k-2} \qquad (\text{for } 0 \le k \le l-1 \,)$$

and

$$A_l = 2a_0^2 Y_l + 3a_0 Y_{l-1} + Y_{l-2} \;\; (\text{ for } k = l(d) \,)$$

$$B_l = 2a_0 Y_l + Y_{l-1} \qquad (\text{ for } k = l(d) \,)$$

where $l = l(d)$ is period length of $w_d = \sqrt{d}$. Also, $C_j = {}^{A_j}\!/_{B_j}$ is the j^{th} convergent in the continued fraction expansion of $\sqrt{d}$.

Moreover, in the continued fraction $[b_1, b_2, b_3 \ldots, b_n, \ldots] = [\, 2a_0, 5, 5, \ldots, 5, \ldots]$,

$P_j = 2\, a_0 Y_j + Y_{j-1}$ and $Q_j = Y_j$ are obtained where $\{P_j\}$ and $\{Q_j\}$ are two sequences defined by

$$P_{-1} = 0, P_0 = 1, \; P_{j+1} = b_{j+1} P_j + P_{j-1}$$

$$Q_{-1} = 1, \; Q_0 = 0 \, , \; Q_{j+1} = b_{j+1} Q_j + Q_{j-1}$$

for $j \ge 0$.

Proof. We can prove by using mathematical induction. Using the following table which

k	-2	-1	0	1	2	3	4	5
a_k			a_0	5	5	5	5	…
A_k	0	1	(a_0) $a_0Y_1+Y_0$	$(5a_0+1)$ $a_0Y_2+Y_1$	$(26\,a_0+5)$ $a_0Y_3+Y_2$	$(135\,a_0+26)$ $a_0Y_4+Y_3$	$(701\,a_0+135)$ $a_0Y_5+Y_4$	…
B_k	1	0 Y_0	1 Y_1	5 Y_2	26 Y_3	135 Y_4	701 Y_5	…

Table 2.1.

(Convergent of $[\,a_0; \overline{5,5,\dots,5,2a_0}\,]$ for $l = l(d)$)

includes values of A_k, B_k and a_k, we can easily say that this is true for $k = 0$.

Now, we assume that the result true for $k < i$. Using the defined relations for $\{Y_i\}$ sequence, we obtained ($a_i = 5$ for $1 \le i \le l-1$)

$$A_{k+1} = a_{k+1}A_k + A_{k-1} = 5(a_0Y_{k+1} + Y_k) + (a_0Y_k + Y_{k-1})$$

$$= a_0(5Y_{k+1} + Y_k) + (5Y_k + Y_{k-1})$$

$$= a_0Y_{k+2} + Y_{k+1}$$

$$B_{k+1} = a_{k+1}B_k + B_{k-1} = 5Y_{k+1} + Y_k = Y_{k+2}$$

Moreover, since $a_l = 2a_0$ we get the following result :

$$A_l = 2a_0^2Y_l + 3a_0Y_{l-1} + Y_{l-2}$$

$$B_l = 2a_0Y_l + Y_{l-1} \quad (\text{ for } k = l(d)\ )$$

Furthermore, in the continued fraction $[b_1, b_2, b_3 \dots, b_n, \dots] = [\,2a_0, 5, 5, \dots, 5, \dots]$,we have following table:

j	-1	0	1	2	3	4	5
b_j			$2\,a_0$	5	5	5	…
P_j	0	1	$(2\,a_0)$ $2\,a_0Y_1+Y_0$	$(10\,a_0+1)$ $2\,a_0Y_2+Y_1$	$(52\,a_0+5)$ $2\,a_0Y_3+Y_2$	$(270\,a_0+26)$ $2\,a_0Y_4+Y_3$	…
Q_j	1	0 Y_0	1 Y_1	5 Y_2	26 Y_3	135 Y_4	…

Table 2.2. Convergent of $[\,2a_0, 5, 5, \dots, 5, \dots]$

The Table 2.2 completes proof.

Definition 2.2. Let $c_n = ac_{n-1} + bc_{n-2}$ be the recurence relation of $\{c_n\}$ sequence where a, b are real numbers. The polynomial is called as a characteristic equation is written in the form of

$$r^2 - ar - b = 0$$

The solutions will depend on the nature of the roots of the characteristic equation for recurence relation.

By using the definition, we find characteristic equation as

$$r^2 - 5r - 1 = 0$$

for $\{Y_k\}$ sequence. So, we can write each element of sequence as follows:

$$Y_k = \frac{1}{\sqrt{29}}\left[\left(\frac{5+\sqrt{29}}{2}\right)^k - \left(\frac{5-\sqrt{29}}{2}\right)^k\right]$$

for $k \geq 0$.

Lemma 2.3. Let $\{Y_k\}$ be the sequence defined as in Definition 2.1 and Definition 2.2. Then, we have

$$Y_k > \begin{cases} \frac{2}{5\sqrt{29}+29}\left(\frac{5+\sqrt{29}}{2}\right)^k & ; if\ k\ is\ even\ integer \\ \frac{1}{\sqrt{29}}\left(\frac{5+\sqrt{29}}{2}\right)^k & ; if\ \ k\ is\ odd\ integer \end{cases}$$

for $k \geq 1$.

Proof. As a result of the Lemma 2.2 this proof can be obtained easily.

Remark 2.1. Let $\{Y_n\}$ be the sequence defined as in Definition 2.1. Then, we state the following:

$$Y_n \equiv \begin{cases} 0(mod4) & ; n \equiv 0(mod6) \\ 1(mod4) & ; n \equiv 1,2,5(mod6) \\ 2(mod4) & ; n \equiv 3(mod6) \\ 3(mod4) & ; n \equiv 4(mod6) \end{cases}$$

for n ≥ 0.

3. MAIN THEOREMS AND RESULTS

The followings are our main theorem and results with the notation of the preliminaries.

Main Theorem. Let d be square free positive integer and ℓ be a positive integer satisfying that $3 \nmid \ell, \ell \geq 2$. Suppose that the parametrization of d is

$$d = \left(\frac{5 + (2\delta + 1)Y_\ell}{2}\right)^2 + (2\delta + 1)Y_{\ell-1} + 1$$

where $\delta \geq 0$ is a positive integer. Then following conditions hold:

(1) If $\ell \equiv 1(mod6)$ and δ is even positive integer then $d \equiv 2(mod4)$
(2) If $\ell \equiv 2(mod6)$ and δ is even positive integer then $d \equiv 3(mod4)$
(3) If $\ell \equiv 4(mod6)$ and δ is even positive integer then $d \equiv 3(mod4)$
(4) If $\ell \equiv 5(mod6)$ and δ is odd positive integer then $d \equiv 2(mod4)$

In $Q(\sqrt{d})$ real quadratic fields, we have $w_d = \left[\frac{5+(2\delta+1)Y_\ell}{2}; \overline{\underbrace{5,5,\ldots,5}_{\ell-1}, 5 + (2\delta + 1)Y_\ell}\right]$ with $\ell = \ell(d)$ for $d \equiv 2,3(mod4)$.

Additionally, we get the fundamental unit ε_d and coefficients of fundamental unit t_d, u_d as follows:

$$\varepsilon_d = \left(\frac{5+(2\delta+1)Y_\ell}{2} + \sqrt{d}\right)Y_\ell + Y_{\ell-1},$$

$$t_d = (2\delta + 1)Y_\ell^2 + 5Y_\ell + 2Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

Proof. We say that $d \notin Z_+$ by using Remark 2.1 for all $\ell \equiv 0,3(mod6)$. So, we will assume that $3 \nmid \ell, \ell \geq 2$ in order to get $d \in Z_+$. First of all, we should show that four conditions hold as the followings:

(1) if $\ell \equiv 1\ (mod6)$ holds, then $Y_\ell \equiv 1(mod4)$ and $Y_{\ell-1} \equiv 0(mod4)$ hold. By substituting these values into parametrization of d and considering δ is even positive integer, we obtain $d \equiv 2(mod4)$.

(2) If $\ell \equiv 2(mod6)$ satisfies, then $Y_\ell \equiv 1(mod4)$ and $Y_{\ell-1} \equiv 1(mod4)$ satisfy. By considering δ is even positive substituting these values into parametrization of d and rearranging, we have $d \equiv 3(mod4)$.

(3) If $\ell \equiv 4(mod6)$ and δ is even positive integer, then $Y_\ell \equiv 3(mod4)$ and $Y_{\ell-1} \equiv 2(mod4)$ hold and also by substituting these values into parametrization of d, then $d \equiv 3(mod4)$ holds.

(4) If $\ell \equiv 5(mod6)$ and δ is odd positive integer then we get $Y_\ell \equiv 1(mod4)$ and $Y_{\ell-1} \equiv 3(mod4)$. By substituting these values into parametrization of d and rearranging, we have $d \equiv 2(mod4)$.

So, conditions are satisfied. By using Lemma 2.2 we have

$$w_R = \left(\frac{5+(2\delta+1)Y_\ell}{2}\right) + \left[\frac{5+(2\delta+1)Y_\ell}{2}; \overline{\underbrace{5,5,\ldots,5}_{\ell-1}, 5+(2\delta+1)Y_\ell}\right]$$

$$\Rightarrow \quad w_R = (5+(2\delta+1)Y_\ell) + \cfrac{1}{5+\cfrac{1}{5+\cfrac{1}{\ddots+\cfrac{1}{5+\cfrac{1}{w_R}}}}}$$

$$= (5+(2\delta+1)Y_\ell) + \frac{1}{5} + \cdots + \frac{1}{5} + \frac{1}{w_R}$$

By using Lemma 2.1 and Lemma 2.2, we get

$$w_R = (5+(2\delta+1)Y_\ell) + \frac{Y_{\ell-1}w_R + Y_{\ell-2}}{Y_\ell w_R + Y_{\ell-1}}$$

Using Definition 2.1 and put $Y_{\ell+1} + Y_{\ell-1} = 5Y_\ell + 2Y_{\ell-1}$ equation into the above equality, we obtain

$$w_R^2 - (5+(2\delta+1)Y_\ell)w_R - (1+(2\delta+1)Y_{\ell-1}) = 0$$

This implies that $w_R = \left(\frac{5+(2\delta+1)Y_\ell}{2}\right) + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 2.2, we get

$$\sqrt{d} = \left[\frac{5+(2\delta+1)Y_\ell}{2}; \overline{5,5,\ldots,5,5+(2\delta+1)Y_\ell}\right] \text{ and } \ell = \ell(d).$$

Hence, $w_d = \left[\frac{5+(2\delta+1)Y_\ell}{2}; \overline{\underbrace{5,5,\ldots,5}_{\ell-1}, 5+(2\delta+1)Y_\ell}\right]$ holds.

Now, we can determine ε_d, t_d and u_d using Lemma 2.1 as follows:

$$Q_0 = 0 = Y_0, \quad Q_1 = 1 = Y_1, \quad Q_2 = a_1 . Q_1 + Q_0 \Rightarrow Q_2 = 5 = Y_2,$$

$$Q_3 = a_2 Q_2 + Q_1 = 5Y_2 + Y_1 = Y_3, \quad Q_4 = Y_4, \ \ldots$$

So, this implies that $Q_i = Y_i$ by using mathematical induction for $\forall i \geq 0$. If we substitute these values of sequence into the $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1} \rangle 1$ and rearranged, we get

$$\varepsilon_d = \left(\frac{5+(2\delta+1)Y_\ell}{2} + \sqrt{d}\right) Y_\ell + Y_{\ell-1},$$

$$t_d = (2\delta+1)Y_\ell^2 + 5Y_\ell + 2Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

We can obtain following theorems and conclusions from Main Theorem.

Theorem 3.1. Let d be square free positive integer and ℓ be a positive integer satisfying that $\ell \not\equiv 5(mod6)$, $3 \nmid \ell$ and $\ell \geq 2$. Suppose that parametrization of d is

$$d = \left(\frac{5 + Y_\ell}{2}\right)^2 + Y_{\ell-1} + 1$$

Then ,we have $d \equiv 2{,}3(mod4)$ and $w_d = \left[\frac{5+Y_\ell}{2}; \overline{\underbrace{5,5,\ldots,5}_{\ell-1}, 5 + Y_\ell}\right]$ with $\ell = \ell(d)$.

Additionally, we get the fundamental unit ε_d , coefficients of fundamental unit t_d, u_d and Yokoi's invariant m_d as follows:

$$\varepsilon_d = \left(\frac{5+Y_\ell}{2} + \sqrt{d}\right) Y_\ell + Y_{\ell-1},$$

$$t_d = Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

$$m_d = 3$$

Proof. This theorem is obtained from main theorem by taking $\delta = 0$. Assume that $\ell \not\equiv 5(mod6)$, $3 \nmid \ell$ and $\ell \geq 2$. By using this assumption and Remark 2.1, we obtain that

if $\ell \equiv 1{,}2\ (mod6)$, we have $d \equiv 2{,}3(mod4)$ and if $\ell \equiv 4\ (mod6)$, we get $d \equiv 3(mod4)$.

By using Lemma 2.2 we get

$$w_R = \left(\frac{5 + Y_\ell}{2}\right) + \left[\frac{5 + Y_\ell}{2}; \overline{\underbrace{5,5,\ldots,5}_{\ell-1}, 5 + Y_\ell}\right]$$

$$\Rightarrow\ w_R = (5+Y_\ell) + \cfrac{1}{5+\cfrac{1}{5+\cfrac{1}{\ddots +\cfrac{1}{5+\cfrac{1}{w_R}}}}} = (5+Y_\ell) + \frac{1}{5} + \cdots + \frac{1}{5} + \frac{1}{w_R}$$

By using Lemma 2.1 and Lemma 2.3, we get

$$w_R = (5+Y_\ell) + \frac{Y_{\ell-1}w_R + Y_{\ell-2}}{Y_\ell w_R + Y_{\ell-1}}$$

Using Definition 2.1 and put $Y_{\ell+1} + Y_{\ell-1} = 5Y_\ell + 2Y_{\ell-1}$ equation into the above equality, we obtain

$$w_R^2 - (5+Y_\ell)w_R - (1+Y_{\ell-1}) = 0$$

This implies that $w_R = \left(\frac{5+Y_\ell}{2}\right) + \sqrt{d}$ since $w_R > 0$. If we consider Lemma 2.2 we get

$\sqrt{d} = \left[\frac{5+Y_\ell}{2}; \overline{5,5,\ldots,5,5+Y_\ell}\right]$ and $\ell = \ell(d)$. Hence, $w_d = \left[\frac{5+Y_\ell}{2}; \overline{\underbrace{5,5,\ldots,5}_{\ell-1}, 5+Y_\ell}\right]$ holds.

Now, we can determine ε_d, t_d and u_d using Lemma 2.1. We obtain $Q_i = Y_i$ for $\forall i \geq 0$. If we substitute these values of sequence into the $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1} \rangle 1$ and rearranged, we get

$$\varepsilon_d = \left(\frac{5+Y_\ell}{2} + \sqrt{d}\right) Y_\ell + Y_{\ell-1},$$

$$t_d = Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

Finally, we know that m_d is defined as $m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]$ from H.Yokoi's reference. If we substitue t_d and u_d into the m_d, then we get

$$m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right] = \left[\!\left[\frac{4Y_\ell^2}{Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1}}\right]\!\right]$$

We can't calculate $m_d = \left[\!\left[\frac{u_d^2}{t_d}\right]\!\right]$ due to d is not square free positive integer for $\ell = 2$. From the assumption and by considering Y_ℓ is increasing sequence, we get,

$$4 > 4\left(\frac{Y_\ell^2}{Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1}}\right) > 3{,}846$$

for $\ell \geq 4$. Therefore, we obtain $m_d = \left[\left[\frac{4Y_\ell^2}{Y_\ell^2+Y_{\ell+1}+Y_{\ell-1}}\right]\right] = 3$ for $\ell \geq 4$ due to definition of m_d. This completes the proof of Theorem 3.1.

Corollary 3.1. Let d be the square free positive integer positive integer satisfying the conditions in Theorem 3.1. We state the following Table 3.1 where fundamental unit is ε_d, integral basis elemant is w_d and Yokoi's invariant is m_d for $4 \leq \ell(d) \leq 13$. (In this table, we rule out $\ell(d)$ = 2, 7 since d is not a square free positive integer in these periods).

d	$\ell(d)$	m_d	w_d	ε_d
4927	4	3	$[70;\overline{5,5,5,140}]$	$9476+135\sqrt{4927}$
2408374527	8	3	$[49075;\overline{5,5,5,5,5,5,5,98150}]$	$4816484776+98145\sqrt{2408374527}$
1750699969227	10	3	$[1323140;\overline{5,5,5,5,5,5,5,5,5,2646280}]$	$3501392813126+2646275\sqrt{1750699969227}$
34317082034533490	13	3	$[185248703;\overline{5,5,5,5,5,5,5,5,5,5,5,5,370497406}]$	$68634163071472183+370497401\sqrt{34317082034533490}$

Table 3.1.

Theorem 3.2. Let d be the square free positive integer and ℓ be a positive integer satisfying that $\ell \equiv 5(mod6)$ and $\ell > 1$. We assume that parametrization of d is

$$d = \left(\frac{5+3Y_\ell}{2}\right)^2 + 3Y_{\ell-1} + 1$$

Then ,we get $d \equiv 2(mod4)$ and $w_d = \left[\frac{5+3Y_\ell}{2};\overline{\underbrace{5,5,\dots,5}_{\ell-1},5+3Y_\ell}\right]$ and $\ell = \ell(d)$.

Moreover, we have following equalities :

$$\varepsilon_d = \left(\left(\frac{5+3Y_\ell}{2}\right)Y_\ell + Y_{\ell-1}\right) + Y_\ell\sqrt{d}$$

$$t_d = 3Y_\ell^2 + 5Y_\ell + 2Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

$$m_d = 1$$

for ε_d , t_d, u_d and Yokoi's invariant m_d.

Proof. We can create this theorem by substituting $\delta = 1$ in Main Theorem. If we suppose that $\ell \equiv 5(mod 6)$ and $\ell > 1$,then we have $Y_\ell \equiv 1(mod 4)$ and $Y_{\ell-1} \equiv 3(mod 4)$. Also, if we put these equivalents into $d = \left(\frac{5+3Y_\ell}{2}\right)^2 + 3Y_{\ell-1} + 1$ then we get $d \equiv 2(mod 4)$. By using Lemma 2.2, we have

$$w_R = (5 + 3Y_\ell) + \frac{Y_{\ell-1}w_R + Y_{\ell-2}}{Y_\ell w_R + Y_{\ell-1}}$$

We obtain the proof in a similar way of proof of Theorem 3.1. By using Lemma 2.1, Lemma 2.3 and Definition 2.1, we get $w_R = \left(\frac{5+3Y_\ell}{2}\right) + \sqrt{d}$ since $w_R > 0$.

If we consider Lemma 2.2 we have

$$w_d = \sqrt{d} = \left[\frac{5+3Y_\ell}{2}; \overline{\underbrace{5,5, \dots ,5}_{\ell-1}, 5 + 3Y_\ell}\right] \text{ and } \ell = \ell(d).$$

ε_d, t_d and u_d are determined as follows using Lemma 2.1. It is seen that $Q_i = Y_i$ holds for $\forall i \geq 1$. If we substitute these values of sequence into the $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1} \rangle 1$ and rearranged, we have

$$\varepsilon_d = \left(\left(\frac{3Y_\ell + 5}{2}\right)Y_\ell + Y_{\ell-1}\right) + Y_\ell\sqrt{d}$$

$$t_d = 3Y_\ell^2 + 5Y_\ell + 2Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell.$$

If we substitute t_d and u_d into the m_d and rearranged, then we get

$$m_d = \left[\!\left[\frac{u_d^2}{t_d} \right]\!\right] = \left[\!\left[\frac{4Y_\ell^2}{3Y_\ell^2 + Y_{\ell-1} + Y_{\ell+1}} \right]\!\right]$$

From the assumption and since Y_ℓ is increasing sequence, we have

$$2 > 4\left(\frac{Y_\ell^2}{3Y_\ell^2 + Y_{\ell-1} + Y_{\ell+1}}\right) > 1{,}329$$

where $\ell \equiv 5(mod 6), \ell > 1$. Therefore, we obtain $m_d = \left[\!\left[\frac{4Y_\ell^2}{3Y_\ell^2+Y_{\ell-1}+Y_{\ell+1}} \right]\!\right] = 1$ for $\ell \equiv 5(mod 6)$, $\ell \geq 5$ which completes the proof of Theorem 3.2.

Corollary 3.2. Let d be the square free positive integer satisfying the conditions in Theorem 3.2. We state the following Table 3.2 where fundamental unit is ε_d, integral basis elemant is w_d and Yokoi's invariant is m_d for $1 < \ell(d) \leq 17$.

d	$\ell(d)$	m_d	w_d	ε_d
1111322	5	1	$[1054; \overline{5,5,5,5,2108}]$	738989+ $701\sqrt{1111322}$
424834105080842	11	1	$[20611504; \overline{5,5,\ldots 5,41223008}]$	283222699721779+ $13741001\sqrt{424834105080842}$
163237535004482301880562	17	1	$[404026651354; \overline{5,5,\ldots 5,808053302708}]$	108825023335829727323369+ $269351100901\sqrt{163237535004482301880562}$

Table 3.2.

Theorem 3.3. Let d be square free positive integer and ℓ be a positive integer satisfying that $\ell \not\equiv 5(mod6)$, $3 \nmid \ell$ and $\ell \geq 2$. Suppose that the parametrization of d is

$$d = \left(\frac{5Y_\ell + 5}{2}\right)^2 + 5Y_{\ell-1} + 1$$

Then, we have $d \equiv 2,3(mod4)$ and $w_d = \left[\frac{5Y_\ell+5}{2}; \overline{\underbrace{5,5,\ldots 5}_{\ell-1}, 5 + 5Y_\ell}\right]$ with $\ell = \ell(d)$.

Additionally, we get the fundamental unit ε_d , coefficients of fundamental unit t_d, u_d and Yokoi's invariant n_d as follows:

$$\varepsilon_d = \left(\tfrac{5Y_\ell+5}{2} + \sqrt{d}\right)Y_\ell + Y_{\ell-1},$$

$$t_d = 5Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

$$n_d = 1$$

Proof. We have this theorem for $\delta = 2$ by using Main Theorem. Suppose that $\ell \not\equiv 5(mod6)$, $3 \nmid \ell$ and $\ell \geq 2$. By using this assumption and Remark 2.1 we can find some results as follows:

(i) if $\ell \equiv 1,2\ (mod6)$, then $Y_\ell \equiv 1(mod4)$ as well as either $Y_{\ell-1} \equiv 1(mod4)$ or $Y_{\ell-1} \equiv 0(mod4)$ holds. if $Y_\ell \equiv 1(mod4)$ and $Y_{\ell-1} \equiv 1(mod4)$,then $d \equiv 3(mod4)$ otherwise $d \equiv 2(mod4)$ holds. So, we have $d \equiv 2,3(mod4)$.

(ii) if $\ell \equiv 4\ (mod6)$, then $Y_\ell \equiv 3(mod4)$ and $Y_{\ell-1} \equiv 2(mod4)$ hold. By substituting these values into parametrization of d and rearranging, we have $d \equiv 3(mod4)$.

Hence, $d \equiv 2,3(mod4)$ holds.

We get

$$w_R = (5 + 5Y_\ell) + \frac{Y_{\ell-1}w_R + Y_{\ell-2}}{Y_\ell w_R + Y_{\ell-1}}$$

using Lemma 2.1 and Lemma 2.2 with the properties of continued fraction expansion. Using Definition 2.1 we have

$$w_R^2 - (5 + 5Y_\ell)w_R - (1 + 5Y_{\ell-1}) = 0$$

This implies that $w_R = \left(\frac{5+5S_\ell}{2}\right) + \sqrt{d}$ since $w_R > 0$.

Hence, $w_d = \left[\frac{5Y_\ell+5}{2}; \overline{\underbrace{5,5,\ldots 5}_{\ell-1}, 5 + 5Y_\ell}\right]$ holds.

By using $Q_i = Y_i$ for $\forall i \geq 1$ into the $\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = (a_0 + \sqrt{d})Q_{l(d)} + Q_{l(d)-1} \rangle 1$ and rearranged, we obtain

$$\varepsilon_d = \left(\frac{5Y_\ell+5}{2} + \sqrt{d}\right)Y_\ell + Y_{\ell-1},$$

$$t_d = 5Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1} \quad \text{and} \quad u_d = 2Y_\ell$$

Finally, we know that n_d is defined as $n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right]$. If we substitue t_d and u_d into the n_d, then we get

$$n_d = \left[\!\left[\frac{t_d}{u_d^2}\right]\!\right] = \left[\!\left[\frac{5Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1}}{4Y_\ell^2}\right]\!\right]$$

$$= 1 + \left[\!\left[\frac{1}{4} + \frac{Y_{\ell+1}}{4Y_\ell^2} + \frac{Y_{\ell-1}}{4Y_\ell^2}\right]\!\right]$$

From the assumption, since Y_ℓ is increasing sequence, we calculate following inequality for $\ell \geq 2$

$$0 < \frac{Y_\ell^2 + Y_{\ell+1} + Y_{\ell-1}}{4Y_\ell^2} \leq 0{,}520$$

Hence, we obtain $n_d = 1 + \left[\left[\frac{1}{4} + \frac{Y_{\ell+1}}{4Y_\ell^2} + \frac{Y_{\ell-1}}{4Y_\ell^2} \right]\right] = 1$ for $\ell \geq 2$ due to definition of n_d .This completes the proof of Theorem 3.3.

Corollary 3.3. Let d be the square free positive integer positive integer satisfying the conditions in Theorem 3.3. We state the following Table 3.3 where fundamental unit is ε_d, integral basis elemant is w_d and and Yokoi's invariant is n_d for $2 \leq \ell(d) \leq 13$. (In the following table, we rule out $\ell(d)$ = 4,10 since d is not a square free positive integer in these periods).

d	$\ell(d)$	n_d	w_d	ε_d
231	2	1	$[15; \overline{5,30}]$	$76+5\sqrt{231}$
2233053226	7	1	$[47255; \overline{5,5,5,5,5,5,94510}]$	$893170395+18901\sqrt{2233053226}$
60204077731	8	1	$[245365; \overline{5,5,5,5,5,5,5,490730}]$	$24081366826+98145\sqrt{60204077731}$
857927030911441426	13	1	$[926243505; \overline{5,5,\ldots 5,1852487010}]$	$343170811366981785 +$ $370497401\sqrt{857927030911441426}$

Table 3.3.

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