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COMPARISON OF ESTIMATION METHODS FOR THE INVERTED EXPONENTIATED PARETO DISTRIBUTION

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ABSTRACT

The inverted exponentiated exponential densities family is known for its flexibility and applicability in the field of reliability. This study evaluates the performance of different estimation methods for the inverted exponentiated Pareto (IEP) distribution, which is a special case of this family of distributions. In this study, the point and interval estimates of the parameters for the IEP distribution are obtained using Maximum Likelihood (ML), Maximum Product of Spacings (MPS), Cramer von Mises (CvM), and Anderson Darling (AD) methods. A Monte Carlo simulation is conducted to compare the efficiency of these estimation methods, while real data applications from different fields are utilized to demonstrate practical performance. The fitting performance of the methods is assessed using metrics such as root mean squared error, coefficient of determination, Anderson Darling, and the Kolmogorov-Smirnov test. Simulation results indicate that the MPS method generally outperforms the ML and CvM methods, whereas real data applications reveal that the CvM method provides the best parameter estimates, followed by MPS.

Keywords: AD, Inverted exponentiated Pareto, Inverted exponentiated, MPS, CVM, MLE.

1 INTRODUCTION

Numerous probability distributions with flexible characteristics have been extensively studied. Lifetime distributions are key in characterizing reliability and life characteristics in engineering and practical applications. Recently, Ghitany et al. [1] introduced the inverted exponentiated exponential distribution (IEED) family, which became a versatile option for analyzing various data. Distributions such as the inverted exponentiated exponential (IEE), inverted exponentiated Rayleigh (IER), and inverted exponentiated Pareto (IEP) are part of this distribution family. The hazard rate function for this class is non-monotonic, allowing for flexibility in fitting different types of data. Previous studies examined this family and found it a promising alternative for modeling lifetime data, outperforming some commonly used distributions (see: [2,3]). The IEP distribution is one of the special cases of this family of distributions. The previous studies [4] highlighted the versatility of the IEP distribution, demonstrating its suitability for modeling diverse datasets that exhibit decreasing or non-monotone hazard rate behavior. The probability density function (pdf) and cumulative distribution function (cdf) for the IEP distribution are given as

$$f(x;\alpha,\beta) = \alpha\beta x^{-(\beta+1)}(1+x)^{-(\beta+1)}\left[1 - \left(\frac{1+x}{x}\right)^{(-\beta)}\right]^{\alpha-1}, \qquad x,\alpha,\beta > 0 \tag{1}$$

and

$$F(x;\alpha,\beta) = 1 - \left[1 - \left(\frac{1+x}{x}\right)^{-\beta}\right]^{\alpha}, \qquad x,\alpha,\beta > 0 \tag{2}$$

respectively. Here, α and β are model parameters. In Figure 1, in addition to cdf, different shapes for probability density and hazard rate functions are plotted for different values of parameters.

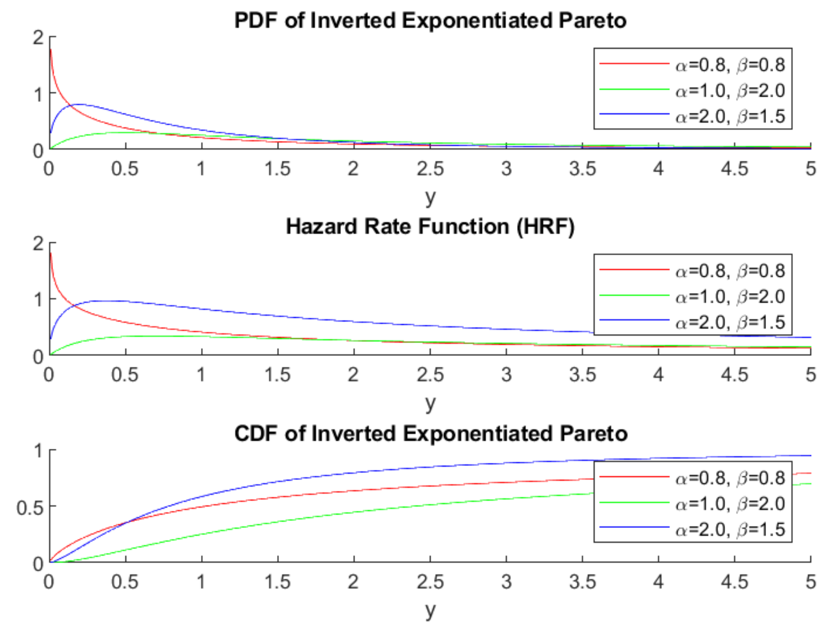

Figure 1. The pdf, hrf, and cdf of the IEP distribution chosen parameters.

Many studies have been conducted on the parameters, reliability, and hazard estimation of the IEED family. For example, system reliability was examined using classical and Bayesian approaches for the IEED family of distributions [5, 6]. Estimation for IER distribution was studied by Maurya et al. and Hashem et al. [7, 8] on censored data. In addition, the study by Rastogi [9] involves estimating the unknown parameters of an IER distribution under Type II progressive censoring, along with the estimation of reliability and hazard functions, using the Expectation–Maximization (EM) algorithm for maximum likelihood estimation (MLE). Similarly, Maurya et al. [4] obtained maximum likelihood (ML) estimates of an IEP distribution under progressive censoring. [10] estimated parameters for the inverse exponential distribution, using maximum likelihood and least squares methods. [8] proposed a pivotal inference approach for estimating the two parameters of the inverse exponentiated Rayleigh distribution using progressive censored data. According to their work, point and interval estimators are derived via the pivotal quantity method.[11] examined the impact of pressure on micro splat splashing diameters using stress-strength reliability analysis. They utilized ML and Bayesian estimators, along with confidence intervals for the IER distribution. [12] discusses confidence set estimation for the generalized inverted exponential distribution based on k-record values. Using pivotal quantities, exact balanced confidence intervals and regions are constructed, with criteria proposed to select the optimal candidates. [13] addressed parameter estimation for a competing risks model with latent failure times following a general family of inverted exponentiated exponential distributions using ML and Bayesian methods with generalized progressive hybrid censored data. Most recently, [14] conducted a study on the estimation of reliability in a multi-component system for the IEP distribution. On the parameter estimation side, as previously mentioned, MLE and Bayesian methods generally stand out in the related literature due to their widespread application and proven effectiveness for IEP distribution (see [4,14]).

Although the MLE method is the most effective under regularity conditions, in some cases, alternative methods can provide successful estimations, as the characteristics of the data can make certain methods more suitable than others. The minimum distance estimators are recognized for their robustness to unusual observations [15]. Also, the Maximum Product of Spacings (MPS) method is a viable alternative to MLE and can offer an improved performance in specific scenarios. This study employs the MPS, Cramér-von Mises (CvM), Anderson Darling (AD), and the MLE method in estimating the parameters of the IEP distribution. Moreover, while studies on the IEED family of distributions have primarily focused on reliability estimation using data from its typical reliability applications, this work also considers a precipitation dataset for its application. To the extent of the author's knowledge, the CvM and AD estimations have not been utilized before for estimating unknown parameters of the IEP distribution previously. Here a Monte Carlo simulation study across different parameter values and sample sizes is conducted, and the observed Fisher information matrix is computed. Also, applications to real-world data from different fields are presented.

The structure of the study is outlined as follows: Section 2 describes the data used in this study and provides an overview of the estimation methods along with objective functions. Next, a Monte Carlo simulation study is carried out using the AD, CvM, MPS, and MLE methods. In the subsequent section, real data applications are presented, followed by concluding remarks summarizing the findings.

2 MATERIAL AND METHOD

2.1 Data

The first application involves a real dataset provided by [16], comprising thirty consecutive measurements of March precipitation (in inches) recorded in Minneapolis/St. Paul. The second dataset consists of 63 service times (measured in thousand hours) for aircraft windshields, as documented by [17].

First dataset :0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Second dataset: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

2.2 Methods

Let $X_1, X_2, \dots, X_n$ be a random sample following the IEP distribution, and $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ are the ordered observations.

2.2.1 The ML Estimation

The MLEs of the parameters α and β, denoted as $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ are obtained by $(\hat{\alpha}_{ML}, \hat{\beta}_{ML}) = \arg\max log\, L\,(\alpha, \beta; \boldsymbol{x})$. As ML estimation has been previously detailed in earlier studies in the literature e.g. [4], the specific derivation is not repeated. Iterative methods are employed to estimate the parameters of the IEP distribution. The loglikelihood function for the IEP distribution is given in Eq.3.

$$log\, L(\alpha, \beta; x) = \begin{array}{c} n\, log(\alpha) + n\, log(\beta) + (\beta - 1)\sum_{i=1}^{n} log(x_i) - \\ (\beta + 1)\sum_{i=1}^{n} log(1 + x_i) + \\ \sum_{i}^{n} (\alpha - 1)\, log\left(1 - \left(\frac{(1 + x_i)}{x_i}\right)^{-\beta}\right) \end{array} \quad (3)$$

The asymptotic variance-covariance matrix for the MLEs of parameters α and β is represented by the information matrix (Eq.4.).

$$I(\alpha, \beta) = -E\left[\frac{\partial^2 LogL}{\partial\alpha\partial\beta}\right] \quad (4)$$

The use of the observed asymptotic variance-covariance matrix, rather than the exact expectations in the above expressions, is a typical implementation due to the difficulty of obtaining the expectations of the components of this matrix. Here the observed asymptotic variance-covariance matrix is used as well. The asymptotic variance-covariance matrix of the parameters can be obtained as

$$\begin{bmatrix} Var(\hat{\alpha}\,) & cov(\hat{\alpha}\hat{\beta}) \\ cov(\hat{\alpha}\hat{\beta}) & Var(\hat{\beta}) \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 logL}{\partial\alpha^2} & -\frac{\partial^2 logL}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 logL}{\partial\alpha\partial\beta} & -\frac{\partial^2 logL}{\partial\beta^2} \end{bmatrix}^{-1} \quad (5)$$

Components of this matrix are,

$\frac{\partial^2 logL}{\partial\alpha^2} = -\frac{n}{\alpha^2}$, $\frac{\partial^2 logL}{\partial\beta^2} = \frac{\sigma_1}{\left(\frac{x_i+1}{x_i}\right)^{\beta}\sigma_2} - \frac{n}{\beta^2} - \frac{\sigma_1}{\left(\frac{x_i+1}{x_i}\right)^{2\beta}{\sigma_2}^2}$ where $\sigma_1 = \log(\frac{x_i+1}{x_i})^2\,(a-1)$ and $\sigma_2 = \frac{1}{\left(\frac{x_i+1}{x_i}\right)^{\beta}} - 1$ and $\frac{\partial^2 logL}{\partial\alpha\partial\beta} = -\frac{\log(\frac{x_i+1}{x_i})}{\left(\frac{x_i+1}{x_i}\right)^{\beta}\left(\frac{1}{\left(\frac{x_i+1}{x_i}\right)^{\beta}} - 1\right)}$.

Under regularity conditions, the asymptotic properties of the MLE method promise that the asymptotic distribution of $\hat{\alpha}$ and $\hat{\beta}$ are normal (see Maurya et al., 2018). Consequently, using the asymptotic distribution of the MLEs, the approximate confidence intervals for unknown parameters can be obtained using $\hat{\alpha} \mp z_{\theta/2}\sqrt{V(\hat{\alpha})}$ and $\hat{\beta} \mp z_{\theta/2}\sqrt{V(\hat{\beta})}$ when $z_{\theta/2}$ is the $[100(1-\theta/2)]^{th}$ percentile of standard normal distribution.

2.2.2 The CvM Estimation

The CvM estimation is a minimum distance estimation method that minimizes the Cramér–von Mises criterion to obtain parameter estimates by assessing the difference between the empirical and theoretical distribution functions. The CvM estimates of the parameters α and β, denoted as $\hat{\alpha}_{CvM}$ and $\hat{\beta}_{CvM}$ are obtained by $(\hat{\alpha}_{CvM}, \hat{\beta}_{CvM}) = \arg\min CvM\ (\alpha, \beta; \boldsymbol{x})$ where

$$CvM(\alpha,\beta;x) = \sum_{i=1}^{n}\left[1 - \left(1 - \left(\frac{(1 + x_{(i)})}{x_{(i)}}\right)^{-\beta}\right)^{\alpha} - \frac{(2i-1)}{(2n)}\right]^2 \tag{6}$$

The estimators of the parameters α and β can be obtained by solving the following nonlinear equations.

$$\frac{\partial CvM}{\partial \alpha} = 2\log\left(1 - \left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{-\beta}\right)\left(\left(1 - \left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{-\beta}\right)^{\alpha}\right)$$
$$\left(\frac{2i-1}{2n} + \left(1 - \left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{-\beta}\right)^{\alpha} - 1\right) = 0 \tag{7}$$

$$\frac{\partial CvM}{\partial \beta} = 2a\,log\left(\frac{x_{(i)}+1}{x_{(i)}}\right)\left(\left(1 - \left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{-\beta}\right)\right)^{a-1}\frac{\left(\frac{2i-1}{2n} + \left(\left(1 - \left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{-\beta}\right)\right)^{a} - 1\right)}{\left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{\beta}} = 0 \tag{8}$$

Since these derivations involve nonlinear equations, iterative methods are employed to estimate the parameters of the IEP distribution using the CvM method.

2.2.3 The MPS Estimation

The MPS estimation is a method that maximizes the product of the spacings between ordered sample points to obtain parameter estimates. The MPS estimators of the parameters α and β, denoted as $\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$ are obtained by $(\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}) = \arg\max\ D\ (\alpha, \beta; x)$ where

The objective function for the MPS method is,

$$D = \sum_{i=0}^{n} log\left[F\left(X_{(i+1)}\right) - F\left(X_{(i)}\right)\right] \tag{9}$$

Here, $X(i)$ and F(·) are ordered observations, and the cdf is for the IEP distribution. The partial derivatives of the MPS objective function with respect to the parameters are

$$\frac{\partial MPS}{\partial \alpha} = \frac{\dfrac{a\ log\left(\frac{x_{(i)}+1}{x_{(i)}}\right)\sigma_4{}^{a-1}}{\left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{\beta}} - \dfrac{a\ log\left(\frac{x_{(i+1)}+1}{x_{(i+1)}}\right)\sigma_3{}^{a-1}}{\left(\frac{x_{(i+1)}+1}{x_{(i+1)}}\right)^{\beta}}}{\sigma_4{}^{a} - \sigma_3{}^{a}} = 0 \tag{10}$$

and

$$\frac{\partial MPS}{\partial \beta} = \frac{\dfrac{a\ log\left(\frac{x_{(i)}+1}{x_{(i)}}\right)\sigma_4{}^{a-1}}{\left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{\beta}} - \dfrac{a\ log\left(\frac{x_{(i+1)}+1}{x_{(i+1)}}\right)\sigma_3{}^{a-1}}{\left(\frac{x_{(i+1)}+1}{x_{(i+1)}}\right)^{\beta}}}{\sigma_4{}^{a} - \sigma_3{}^{a}} = 0\,, \tag{11}$$

where

$\sigma_3 = 1 - \frac{1}{\left(\frac{x_{(i+1)}+1}{x_{(i+1)}}\right)^{\beta}}$ and $\sigma_4 = 1 - \frac{1}{\left(\frac{x_{(i)}+1}{x_{(i)}}\right)^{\beta}}$.

Since these derivations involve nonlinear equations, iterative methods are employed to estimate the parameters of the IEP distribution using the MPS method.

2.2.4 The AD Estimation

The AD estimators of the parameters α and β, denoted as $\hat{\alpha}_{AD}$ and $\hat{\beta}_{AD}$, are obtained by $(\hat{\alpha}_{AD}, \hat{\beta}_{AD}) = \arg\min AD\ (\alpha, \beta; \boldsymbol{x})$ where

$$AD = -n - \frac{1}{n\sum_{i=1}^{n}(2i-1)\left\{logF\left(x_{(i)}\right) + log\left(1 - F\left(x_{(n+i-1)}\right)\right)\right\}} \tag{12}$$

Here, iterative methods are considered to estimate the parameters α and β.

2.3 Evaluating Criteria

The fitting performance is assessed through multiple metrics, including root mean squared error (RMSE), coefficient of determination (R^2), and the Kolmogorov-Smirnov (KS) and Anderson Darling (AndDar) test statistic and p-values. Formulas of criteria used in evaluating results are given below.

$$RMSE = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\hat{F}(X_{(i)}) - \frac{i}{n+1}\right)^2\right]^{\frac{1}{2}} \tag{13}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left(\hat{F}(X_{(i)}) - \frac{i}{n+1}\right)^2}{\sum_{i=1}^{n}\left(\hat{F}(X_{(i)}) - \bar{\hat{F}}(X_{(i)})\right)^2} \tag{14}$$

$$KS = max\left|\hat{F}(X_{(i)}) - \frac{i}{n+1}\right| \tag{15}$$

$$AndDar^2 = -n - S_n\,, \tag{16}$$

where $S_n = \frac{2i-1}{n}(\log(F(x_i)) + \log(1 - F(x_{n+1-i})))$

Here, $\hat{F}(X_{(i)})$ is the estimated cdf, X(i) is the i-th order statistics, k is the number of the parameters, and n is the number of observations.

3 SIMULATION STUDY

This section focuses on a Monte Carlo simulation study conducted to evaluate the efficiency of estimation methods under various conditions. The study involves 1000 simulation runs, using sample sizes of n=10, 20, 50, 100, and 300. Parameter values are set as α=1, 0.5, and 3; β=1, 2, and 3. Estimates are obtained through the "genetic algorithm" function available in the Matlab R2021a optimization toolbox. The performance of the ML, MPS, CvM, and AD methods is assessed based on the well-known mean, variance, and Mean Squared Error (MSE) criteria.

Table 1. The Simulation Results for α and β=1.

Method	Mean	Variance	MSE	Mean	Variance	MSE
	n=10		$\hat{\alpha}$		$\hat{\beta}$	
MLE	1.14716891	0.11653618	0.13819487	1.14116204	0.11379887	0.13372560
CvM	1.09369312	0.13512124	0.14389964	1.09598416	0.13239732	0.14161028
MPS	0.89252066	0.11289056	0.12444237	0.87960376	0.11996129	0.13445655
AD	1.22845677	0.77304981	0.82524230	1.15532363	0.38391538	0.40804082
	n=20		$\hat{\alpha}$		$\hat{\beta}$	
MLE	1.1102750	0.08424498	0.09640557	1.09842024	0.07650079	0.08618734
CvM	1.1003792	0.0987992	0.1088752	1.0912126	0.0947396	0.1030593
MPS	0.9188027	0.0718207	0.0784137	0.90658045	0.07261128	0.08133849
AD	1.0926948	0.1835829	0.19217526	1.06753173	0.13072170	0.13528223
	n=50		$\hat{\alpha}$		$\hat{\beta}$	
MLE	1.05831	0.047419	0.050819	1.043302	0.040942	0.042817
CvM	1.06615	0.069658	0.074035	1.046368	0.06054	0.06269
MPS	0.94488	0.034942	0.037980	0.932457	0.033509	0.038071
AD	1.02769	0.049671	0.050439	1.014062	0.045379	0.045577
	n=100		$\hat{\alpha}$		$\hat{\beta}$	
MLE	1.03534	0.021187	0.022436	1.029194	0.018441	0.019293
CvM	1.03744	0.030809	0.032212	1.031819	0.027428	0.028441
MPS	0.96743	0.017637	0.018698	0.962811	0.016303	0.017686
AD	1.022734	0.022982	0.023499	1.018341	0.021035	0.021371
	n=300		$\hat{\alpha}$		$\hat{\beta}$	
MLE	1.01515	0.00601	0.00624	1.01274	0.006189	0.006351
CvM	1.00718	0.008712	0.008763	1.00477	0.008774	0.008797
MPS	0.98035	0.005649	0.006035	0.978706	0.005988	0.006441
AD	1.01006	0.007165	0.007267	1.007574	0.00731	0.007367

When Table 1 is examined, according to all of the sample sizes considered (except for β parameter for n=10 and n=300), the MPS estimations provided more efficient estimations for parameters of the IEP distribution according to the MSE criterion. The ML estimation for the β parameter performed better when n=10 and n=300 than the other estimation methods.

Table 2. The Simulation Results for α=0.5 and β=1.5.

Method	Mean	Variance	MSE	Mean	Variance	MSE
	n=10		$\hat{\alpha}$		$\hat{\beta}$	
MLE	0.577859	0.037183	0.043245	1.659585	0.145069	0.170536
CvM	0.556911	0.039066	0.042305	1.605075	0.172322	0.183362
MPS	0.491265	0.027559	0.027635	1.385413	0.165099	0.178229
AD	0.53109	0.034298	0.035265	1.522759	0.171612	0.17213
	n=20		$\hat{\alpha}$		$\hat{\beta}$	
MLE	0.547037	0.021364	0.023576	1.604913	0.138804	0.149811
CvM	0.531148	0.023325	0.024296	1.575843	0.151472	0.157225
MPS	0.485016	0.014875	0.0151	1.381182	0.137015	0.151132
AD	0.515455	0.019184	0.019423	1.52078	0.151043	0.151475
	n=50		$\hat{\alpha}$		$\hat{\beta}$	

Table 2 (Continued). The Simulation Results for α=0.5 and β=1.5.

Method	Mean	Variance	MSE	Mean	Variance	MSE
MLE	0.522472	0.008162	0.008667	1.583312	0.087704	0.094645
CvM	0.519148	0.0107	0.011066	1.566425	0.113044	0.117456
MPS	0.487298	0.006478	0.00664	1.424244	0.087126	0.092865
AD	0.509835	0.008639	0.008735	1.533871	0.103991	0.105139
	n=100		$\hat{\alpha}$		$\hat{\beta}$	
MLE	0.511018	0.003376	0.003497	1.537007	0.049374	0.050743
CvM	0.509777	0.004268	0.004364	1.534701	0.068247	0.069452
MPS	0.490184	0.002895	0.002991	1.432924	0.045813	0.050313
AD	0.503828	0.003416	0.003431	1.507302	0.054463	0.054516
	n=300		$\hat{\alpha}$		$\hat{\beta}$	
MLE	0.501821	0.001155	0.001158	1.508832	0.020478	0.020556
CvM	0.501935	0.001573	0.001577	1.508298	0.030791	0.03086
MPS	0.493891	0.001076	0.001113	1.465412	0.019429	0.020626
AD	0.499776	0.001288	0.001288	1.499337	0.023764	0.023764

According to Table 2, the MPS estimations provided more efficient estimations for parameters of the IEP distribution for nearly all of the cases according to the MSE criterion when α<β. When sample sizes were n=10 and n=300, the MLE method slightly performed better for the β parameter.

Table 3. The Simulation Results for α=3 and β=2.

Method	Mean	Variance	MSE	Mean	Variance	MSE
	n=10		$\hat{\alpha}$		$\hat{\beta}$	
MLE	3.157901	0.199023	0.223956	2.109823	0.133131	0.145192
CvM	3.117907	0.205831	0.219733	2.092254	0.137451	0.145961
MPS	2.86754	0.161828	0.179373	1.957424	0.130235	0.132047
AD	3.023329	0.198482	0.199026	2.052693	0.133464	0.136241
	n=20		$\hat{\alpha}$		$\hat{\beta}$	
MLE	3.176184	0.179092	0.210133	2.068628	0.094346	0.099056
CvM	3.13639	0.198528	0.21713	2.054232	0.095054	0.097995
MPS	2.874764	0.151692	0.167377	1.925536	0.088318	0.093863
AD	3.073496	0.187776	0.193177	2.027683	0.092983	0.093749
	n=50		$\hat{\alpha}$		$\hat{\beta}$	
MLE	3.101851	0.14072	0.151093	2.029305	0.04506	0.045919
CvM	3.099932	0.15792	0.167906	2.022161	0.045969	0.04646
MPS	2.850305	0.139326	0.161735	1.918465	0.041734	0.048382
AD	3.042292	0.151618	0.153407	2.001181	0.043785	0.043787
	n=100		$\hat{\alpha}$		$\hat{\beta}$	
MLE	3.045159	0.122246	0.124286	2.0097	0.02689	0.026984
CvM	3.032136	0.143489	0.144522	2.004854	0.032214	0.032237
MPS	2.861807	0.112669	0.131766	1.932097	0.025289	0.0299
AD	3.00291	0.133243	0.133251	1.992761	0.028761	0.028813
	n=300		$\hat{\alpha}$		$\hat{\beta}$	

Table 3 (Continued). The Simulation Results for α=3 and β=2.

Method	Mean	Variance	MSE	Mean	Variance	MSE
MLE	3.044297	0.056834	0.058796	2.010898	0.012503	0.012621
CvM	3.048189	0.068014	0.070336	2.012961	0.01488	0.015048
MPS	2.949892	0.056863	0.059374	1.972669	0.012465	0.013212
AD	3.029772	0.059695	0.060581	2.005611	0.012985	0.013017

Finally, for α>β, all of the methods performed similarly for the cases considered. For smaller sample sizes, the MPS estimations are more efficient, and for larger sample sizes, the MLEs stand out more; also, the AD method is competitive when larger sample sizes are considered.

Overall, it can be said that the MSEs of all estimates decline with increasing sample size. This indicates that the examined estimation methods are potentially effective for data-fitting applications. In addition, the MPS estimations stand out for nearly all scenarios considered, and as expected, the ML estimations are improved when the sample size increases.

4 APPLICATION RESULTS AND DISCUSSION

This section presents modeling two different data from environmental and operational reliability areas with the IEP distribution using the ML, MPS, CvM, and AD methods. The fitting performances of the methods are examined through the criteria given in Section 2. In addition, fitted density, quantile-quantile (Q-Q), and probability–probability (P-P) plots are presented. Superior fit is indicated by smaller RMSE, AD, and KS statistics, alongside larger R^2 values and higher p-values from the KS test. In Table 4, estimated parameters and asymptotic confidence intervals (ACI) for the first data set are provided.

Table 4. Estimated Parameters and confidence intervals for the first data set.

Method	$\hat{\alpha}$	$\hat{\beta}$	ACI for $\hat{\alpha}$	ACI for $\hat{\beta}$
ML	7.093474	4.479320	(6.5912, 7.5958)	(4.2895, 4.6692)
MPS	7.593473	4.663268	(7.0912, 8.0958)	(4.4734, 4.8531)
CVM	6.593476	4.4767998	(6.0912, 7.0958)	(4.287, 4.6667)
AD	6.093473	4.0720337	(5.5912, 6.5958)	(3.8822, 4.2619)

According to Table 4, the ML and CvM methods yield relatively narrower intervals, suggesting stable estimation. The comparative results for the estimation methods are presented in Table 5 for precipitation data. Additionally, the fitted densities, Q-Q, and P-P plots for each method are illustrated in Figures 2 and 3. The analysis is carried out using Matlab R2021 and its built-in functions.

Table 5. Evaluating criteria for the first dataset.

Method	AndDar	R^2	RMSE	KS (p-value)
ML	0.160548	0.9896	0.0302	0.0757 (0.99026)
MPS	0.178832	0.9895	0.0307	0.0695 (0.99657)
CVM	0.171681	0.9916	0.0267	0.0663 (0.99819)
AD	0.236604	0.9825	0.0381	0.1068 (0.8480)

According to Table 5, the ML, MPS, AD, and CvM methods performed very closely for modeling precipitation data. However, the CvM method stands out for all criteria by providing the highest R^2 and p-values and the lowest AD, KS, and RMSE values.

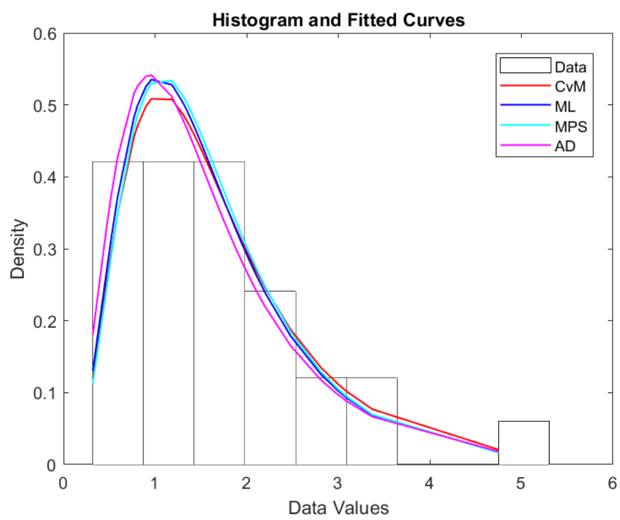

Figure 2. Fitting plots of estimation methods for the first dataset.

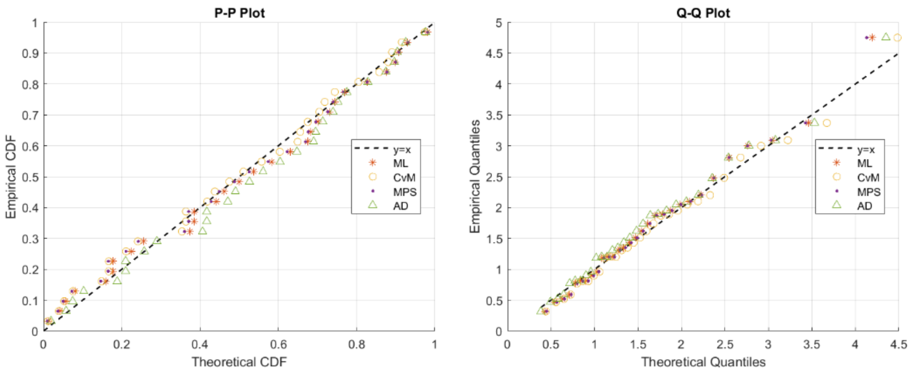

Figure 3. Q--Q and P-P plots for the first data set.

It can be seen from Figure 2 that the CvM estimation method described the first dataset better than the other two methods. The ML, MPS, and AD methods are overfitted at the peak of the distribution compared to the CvM. Considering the sample size of this dataset (n=30), the CvM method seems to provide a more accurate estimation in this case. The Q-Q and P-P plots suggest that all four estimators perform similarly, with slight deviations in the upper quantiles suggesting underestimation at the tails.

In Table 6, estimated parameters and asymptotic confidence intervals for the second data set are provided.

Table 6. Estimated parameters and confidence intervals for the second data set.

Method	$\widehat{\alpha}$	$\widehat{\beta}$	ACI α	ACI β
ML	3.09890	3.25060	(2.8304, 3.3674)	(3.0608, 3.4405)
MPS	3.59890	3.7506	(3.3304, 3.8674)	(3.5607, 3.9405)
CVM	2.59893	3.30898	(2.3304, 2.8674)	(3.1192, 3.4989)
AD	2.0989	2.75060	(1.8304, 2.3674)	(2.5608, 2.9405)

According to Table 6, the ML, MPS, and CVM methods exhibited narrower confidence intervals, and the AD has the widest confidence intervals, implying greater variability. The comparative results for the estimation methods are presented in Table 7 for aircraft windshield data. Moreover, the fitted density, Q-Q, and P-P plots are provided for each method in Figures 4 and 5.

Table 7. Evaluating criteria for the second dataset.

Method	AndDar	R^2	RMSE	KS(p-value)
ML	2.7410	0.903899	0.08550	0.185989(0.022169)
MPS	2.2232	0.9486991	0.06403	0.152975(0.094129)
CVM	2.2841	0.944583	0.06114	0.129424(0.221862)
AD	2.5650	0.908621	0.075221	0.1372821 (0.1694)

Table 7 reveals that the MPS and CvM methods exhibit similar performance in modeling the aircraft windshield data and overweighing the ML and AD estimations. Among these, the MPS method achieved the highest R^2 while the CvM method presented the lowest KS and RMSE metrics.

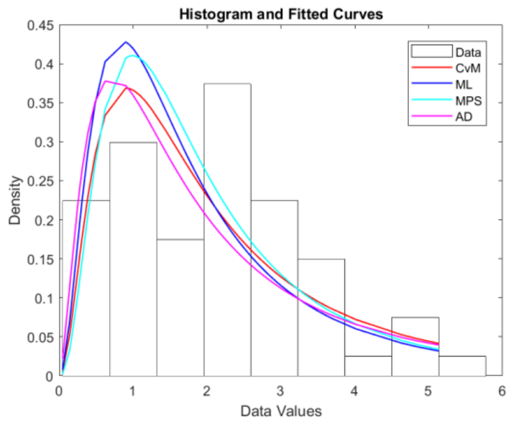

Figure 4. Fitting plots of estimation methods for the second dataset.

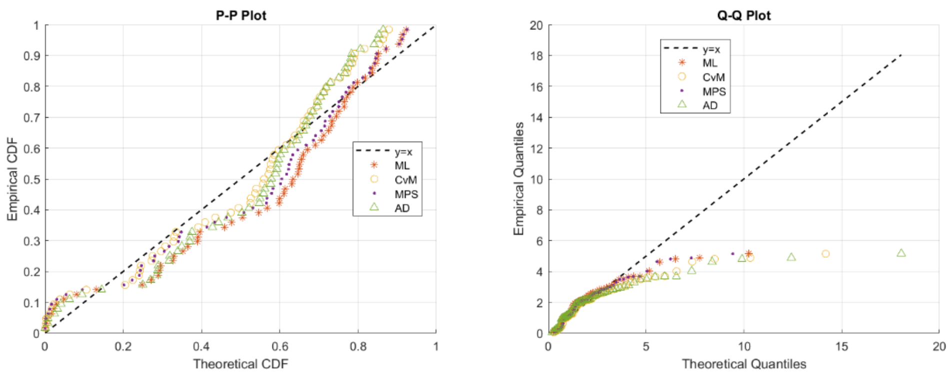

Figure 5. The Q-Q and P-P plots for the second data set.

According to Figure 4, although MPS described the distribution much better than the other methods, the ML method fitted data well in general but overfitted slightly at the peak. The CVM method is performed well in fitting the distribution as well. When the Q-Q and P-P plots are examined, it can be seen that similar outcomes are obtained as in fitted density plots. Since P-P plots emphasize deviations in distribution tails and Q-Q plots detect differences in distribution shape, the Q-Q and P-P plots indicate that the CVM and MPS provide a closer fit to the theoretical distribution, and the ML and MPS reflect the tails of distribution much better.

In the applications, the MPS and CvM estimation methods demonstrated greater accuracy in estimating the parameters of the IEP distribution compared to the AD and ML methods for the data considered. For the first dataset, the CvM method proved to be the most effective. For the second dataset, both CvM and MPS outperformed ML and AD in fitting the parameters.

Although there is a growing body of research on the IEED family of distributions, studies remain limited and typically focus on maximum likelihood and Bayesian estimation. This work extends beyond traditional reliability studies by applying the approach to environmental data, highlighting the potential of the IEP distribution for modeling data across diverse fields. In addition, it is seen that considering the MPS method can lead to greater efficiency for considered data and simulations, even for small samples. Overall, the MPS and CVM methods can be strong alternatives to the ML method in estimating the parameters of the IEP distribution.

5 CONCLUSION AND SUGGESTIONS

Given its flexibility and applicability, the family of inverted exponentiated densities is frequently used as a reliable model for fitting a wide range of data. The IEP distribution is a special case of this family of distributions and is appreciated due to its flexibility. Here the ML, MPS, AD, and CvM methods are employed to estimate the parameters of the IEP distribution and evaluate their modeling performance through simulations and real data applications from different fields. The Monte Carlo simulation results reveal that the MPS method outperformed the AD, ML, and CvM for nearly all the cases considered. In real data applications, the CvM method emerges as the best method, with its closest competition being the MPS method. This study explores the application of MPS, AD, and CvM estimation methods for the IEP distribution, in addition to the traditional MLE approach, and demonstrates their effectiveness using two real data sets. These findings provide valuable insights into the practical use of different estimation techniques for IEP distribution.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the author.

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