Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Sayı:10 Mayıs 2006 ISSN - 1302 - 3055

WINKLER ZEMİNİ ÜZERİNDEKİ ENİNE YÜKLEMEYE MARUZ TEMEL KİRİŞLERİNİN DİNAMİK ANALİZİ

Y. YEŞİLCE*

* Dokuz Eylül Üniversitesi Mühendislik Fakültesi, İnşaat Mühendisliği Bölümü, Tınaztepe Kampusü, 35160, Buca, İzmir, Türkiye
yusuf.yesilce@deu.edu.tr

ÖZET

Bu çalışmada, elastik zemine oturan, enine kuvvet ve momentlere maruz bir kirişin dinamik analizi incelenmiştir. Temel modelinde Winkler hipotezi baz alınmıştır. Kiriş üzerindeki tekil etkiler, diferansiyel denklemi kullanabilmek amacıyla, Dirac dağılım teorisi vasıtasıyla yayılı yüklere dönüştürülmüştür. Çalışmanın kullanışını göstermek amacıyla açıklayıcı bir örnek sunulmuş ve bir kısım hesaplanan değerler tablolarda ve şekillerde verilmiştir.

Anahtar Kelimeler: *Dinamik Analiz, Dirac Dağılım Teorisi, Enine Yükleme, Winkler Zemini*

DYNAMIC ANALYSIS OF FOUNDATION BEAMS SUBJECTED TO TRANSVERSE LOADING ON WINKLER SOIL

Y. YEŞİLCE*

ABSTRACT

In this study, the dynamic analysis of a free beam subjected to transverse forces and moments on an elastic soil are investigated. The foundation model is based on the Winkler hypothesis. Concentrated disturbances on beams are transformed to distributed loads in order to be able to use the governing differential equation established for distributed loads by using Dirac distribution theory. An illustrative example is presented in order to demonstrate the use of the study and some of the obtained results are given in tables and figures.

Key Words: *Dirac Distribution Theory, Dynamic Analysis, Transverse Loading, Winkler Soil*

1. Introduction

There are various methods used in the analysis of continuous foundations as a beam resting on elastic soils. The most important two of them are the subgrade modulus method pertaining to the theory of the first order and the method of modulus of elasticity based on a second order theory. The former presents a model in which the soil is assumed as dense liquid while the latter offers an elastic solid model.

In the subgrade modulus method, proposed by Winkler, it is assumed that the deflection at any point of the beam on elastic soil is proportional to the pressure applied at that point and is independent of pressure acting at nearby points of the beam [9]. In other words, in this method the beam is considered as if it is resting on infinitely long independent elastic springs with subgrade modulus. In the elastic solid model, the effects of the neighbouring points to the point in question are taken into account by Boussinesque's load-deformation relation in an isotropic elastic semi-space. In this case, the soil is characterized by its elastic properties, namely, elastic modulus and Poisson's ratio. However, the solution of the differential equation established for this model may present certain computational difficulties and approximate methods may be needed to involve for the solution.

However, both models do not represent the real soil exactly. It behaves neither as a dense liquid nor as an elastic solid. With a more realistic hypothesis, some researchers developed two-parameter models for the elastic soil [3, 4, 6, 8, 5]. In comparison with the single parameter model, i.e. Winkler model, these two-parameter foundation models represent the foundation characteristics more accurately. Vallabhan and Daloglu had developed relations in which subgrade modulus varies with depth which is equivalent to the two-parameter Vlasov -Leontiev solution and can be used in classical Winkler model [2].

In this paper, the subgrade modulus method is used, which is also preferred in practice for static problems due to its simplicity of mathematical formulation. One of the most important drawbacks of this method is difficulties in determining the modulus of subgrade reaction of the soil. The variation of contact pressure over the bearing area requires the variation of subgrade modulus as well; subgrade modulus depends not only on the physical characteristics of the soil but also on the foundation dimensions, the rigidity of the foundation, the distribution of loading on the superstructure and the thickness of the compressible layer which causes settlement. Therefore accurate determination of deflection of the foundation and stresses on the superstructure can only be possible by using these factors [7].

In addition to all of these, it is known that the subgrade modulus values for dynamic loading are different from those for static loading. Based on these facts, values for subgrade modulus should be determined by field tests conducted for different types of soils, different loading conditions and different loading areas. However in practice, except for very important structures, subgrade modulus values are taken from tables prepared for different soil types. In the subgrade modulus method, the rigidity of the superstructure, the stress distribution under the foundation base and lateral movement of the base soil are left out from the mathematical model.

2. MATHEMATICAL FORMULATION OF THE MODEL

(Figure 1.1-a) shows a foundation beam with flexural rigidity EI(x), coefficient of viscous damping c(x) per unit length, base width b(x), cross-sectional area A(x), mass density ρ and mass [m(x) = ρ.A(x)] per unit length on a soil with subgrade modulus K_0. The beam is subjected to distributed external load f(x,t) which may vary with position *x* and time *t.* The forces on a differential element of length dx are shown in (Figure 1.1-b), where V(x,t) is the transverse shear force, M(x,t) is the bending moment, y(x,t) is the transverse displacement, $\left[c(x)\cdot\partial y/\partial t\right]$ is the viscous damping force, $[m(x)\cdot\partial^2 y/\partial t^2]$ is the inertia force and $[k(x)\cdot y = K_0\cdot b(x)\cdot y]$ is the elastic response of the soil.

In the analysis the effects of shear and axial deformations and rotational inertia are ignored. The governing differential equation for the transverse vibration of a beam on elastic soil shown in (Figure 1.1) can be written as :

$$\frac{\partial^2}{\partial x^2}(EI(x)\cdot\frac{\partial^2 y}{\partial x^2}) + m(x)\cdot\frac{\partial^2 y}{\partial t^2} + c(x)\cdot\frac{\partial y}{\partial t} + k(x)\cdot y = f(x,t) \tag{1}$$

The solution of this partial differential equation under the boundary and initial conditions yields the response $\left[y(x,t)\right]$ of the beam in position x and at time t. Once the deflection is determined, the slope, bending moment and the shear can be calculated by taking the first, second and third derivative of the solution (response) function with respect to x, respectively.

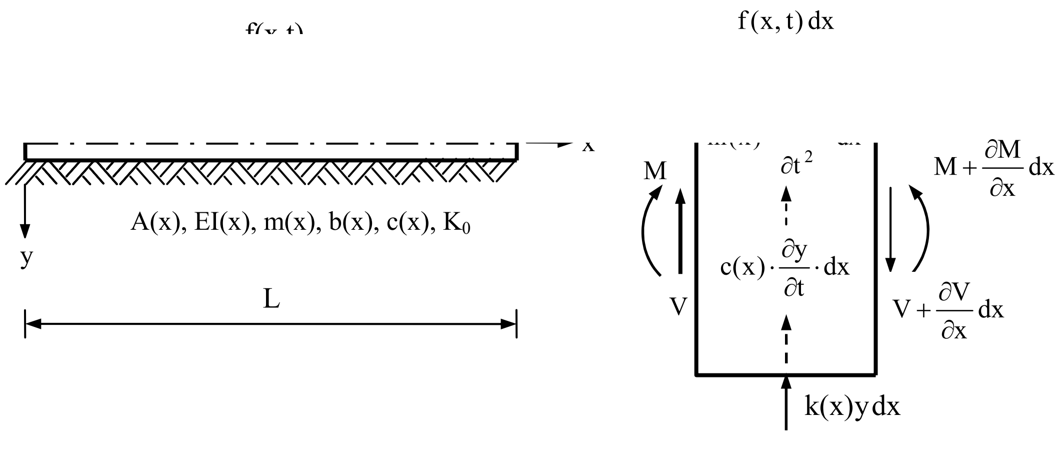


Figure 1.1-a The foundation beam on Winkler soil
-b The forces on a differential element

2.1. Natural Frequencies and Mode Shapes

For free vibration, $[f(x,t)=0]$, and with the assumption that the damping coefficient and the section characteristics are constant along the beam, the natural frequencies and modes can be obtained by the solution of the following homogeneous partial differential equation with constant coefficients.

$$EI\cdot\frac{\partial^4 y}{\partial x^4}+m\cdot\frac{\partial^2 y}{\partial t^2}+c\cdot\frac{\partial y}{\partial t}+k\cdot y=0 \tag{2}$$

The solution function of Eq. (2) can be written by the method of separation of variables as:

$$y(x,t)=X(x)\cdot T(t) \tag{3}$$

where; $X(x)$ is the characteristic shape function and $T(t)$ is a time function. The substitution of Eq. (3) into Eq. (2) leads to:

$$\frac{EI\cdot X^{IV}}{X}=-\frac{m\cdot\ddot{T}+c\cdot\dot{T}+k\cdot T}{T} \tag{4}$$

where; Roman indices denote derivatives with respect to x and over dots indicate derivatives with respect to time. Since the left hand side of Eq. (4) is a function only of x while the right hand side is a function of t only, Eq. (4) is true only if each side is equal to the same constant. Designating this constant by p and setting both sides equal to it yields

$$EI\cdot X^{IV}-p\cdot X=0 \tag{5}$$

and

$$m\cdot\ddot{T}+c\cdot\dot{T}+(k+p)\cdot T=0 \tag{6}$$

The solution of Eq. (5) is can be written as:

$$X(x)=C_1\cdot\sin(\lambda x)+C_2\cdot\cos(\lambda x)+C_3\cdot\sinh(\lambda x)+C_4\cdot\cosh(\lambda x) \tag{7}$$

where;

$$\lambda=\sqrt[4]{\frac{p}{EI}} \tag{8}$$

The four integration constants in Eq. (7) are determined via the boundary conditions.
In the case of under damped motion, the solution of Eq. (6) is can be written as:

$$T(t) = e^{-\zeta\omega t}\left[T_0 \cdot \cos(\omega_D t) + \frac{\dot{T}_0 + \zeta \cdot \omega \cdot T_0}{\omega_D} \cdot \sin(\omega_D t)\right] \tag{9}$$

where; T_0 and $\dot{T}_0$ are parameters which depend on initial conditions and ω_D is the damped natural frequency of the system which is given by:

$$\omega_D = \omega \cdot \sqrt{1-\zeta^2} \tag{10}$$

where; ω is the undamped natural frequency, namely

$$\omega = \sqrt{\frac{k+p}{m}} \tag{11}$$

and

$$\zeta = \frac{c}{2m\omega} \tag{12}$$

which is called as damping ratio.

2.2 Shape Function of the Free Beam

The four integration constants in the general solution of the characteristic shape function given in Eq. (7) are determined by the boundary conditions of the free beam shown in (Figure 1.1). The boundary conditions for such a beam are as follows:

$$\begin{aligned}
&\text{at } x=0 \quad \begin{cases} M(0,t)=0 & \text{or} \quad X''(0)=0 \\ V(0,t)=0 & \text{or} \quad X'''(0)=0 \end{cases} \\
&\text{at } x=L \quad \begin{cases} M(L,t)=0 & \text{or} \quad X''(L)=0 \\ V(L,t)=0 & \text{or} \quad X'''(L)=0 \end{cases}
\end{aligned} \tag{13}$$

Eqs. (13) gives a set of equations with constant coefficients. The determinant of the coefficients must be equal to zero for nontrivial solution. The expansion of this determinant leads to:

$$\cosh(\lambda L) \cdot \cos(\lambda L) = 1 \tag{14}$$

which is the frequency equation for the free beam. The numerical solution of this transcendental equation gives with a good approximation the following relationship.

$$\lambda L = (n + 1/2) \cdot \pi \qquad n = 1,2,3,... \tag{15}$$

For the first mode, that is n = 1, Eq. (15) yields a value of 4.71 for (λL) while the exact value is appr. 4.73. For upper modes the difference is getting smaller. In the computations, the exact solutions of Eq. (14) must be used for the first several modes (say 5 modes), while Eq. (15) may be utilized for higher modes. After finding the values of (λL), the natural frequencies can be obtained from Eqs. (8) and (11) as follows:

$$\omega_n = \sqrt{\frac{\lambda_n^4 \cdot EI + k}{m}} \tag{16}$$

The solution to the set of homogeneous equations (13) is parameter-dependent. However, normal modes are determined to a relative magnitude, therefore the constant arose in the solution may be taken unity. Hence, the characteristic shape function for the n-th mode is obtained as:

$$X_n(x) = \sin(\lambda_n x) + \sinh(\lambda_n x) - \beta_n \cdot [\cos(\lambda_n x) + \cosh(\lambda_n x)] \tag{17}$$

where;

$$\beta_n = \frac{\sinh(\lambda_n L) - \sin(\lambda_n L)}{\cosh(\lambda_n L) - \cos(\lambda_n L)} \tag{18}$$

From Eq. (3) the displacement function for the n-th mode is given by:

$$y_n(x,t) = X_n(x) \cdot T_n(t) \tag{19}$$

The general solution to the equation of motion, namely the total deflection is obtained by superimposing all modes as follows:

$$y(x,t) = \sum_{n=1}^{\infty} [X_n(x) \cdot T_n(t)] \tag{20}$$

3. FORCED VIBRATION

This paper deals with the transverse vibration of continuous beams on elastic soils subjected to dynamic disturbances due to concentrated loads and moments. However, the right hand side of Eq. (1) has been established for distributed loads. For this reason, concentrated loads will be transformed to distributed loads by the theory of generalized functions (distributions). The technique used in this study is to expand the Dirac distribution into a series of an orthogonal function family.

3.1. Foundation Beams under Concentrated Forces

The concentrated force $F(s,t)$ on any position s of the beam may be transformed to distributed load by Dirac distribution as:

$$f(x,t) = F_{0s} \cdot \phi_s(t) \cdot \delta_s(x) \tag{21}$$

where; the load function is of the form:

$$F(s,t) = F_{0s} \cdot \phi_s(t) \tag{22}$$

and

$$\delta_s(x) = \delta(x-s) = \sum_n \left[A_n X_n(x) \right] \tag{23}$$

In these equations F_{0s} is the amplitude of the force located at a point s, $\phi_s(t)$ is the time function of the force, $\delta_s(x)$ is Dirac distribution function centered at position *s*. This distribution is expanded into a series of shape functions. By taking the inner product of Eq. (23) with X_m, and utilizing the properties of distributivity, homogeneity and orthogonality, A_n can be obtained as:

$$A_n = \frac{\langle \delta_s, X_n \rangle}{\langle X_n, X_n \rangle} \tag{24}$$

The inner product of shape function of free beam for the same mode is :

$$\langle X_n, X_n \rangle = \|X_n\|^2 = L \cdot \beta_n^2 \tag{25}$$

and from the definition of Dirac distribution [1].

$$\langle \delta_s, X_n \rangle = X_n(s) = X_{ns} \tag{26}$$

the constant A_n is obtained as:

$$A_n = \frac{X_{ns}}{L \cdot \beta_n^2} \tag{27}$$

Substituting A_n into Eq. (23) yields:

$$\delta_s(x) = \sum_n \frac{X_{ns}}{L \cdot \beta_n^2} \cdot X_n(x) \tag{28}$$

From Eqs. (21) and (28), the concentrated load $F(s,t)$ located at position s is transformed to distributed load as:

$$f(x,t) = \frac{F_{0s}}{L} \cdot \phi_s(t) \cdot \sum_n \frac{1}{\beta_n^2} \cdot X_{ns} \cdot X_n(x) \tag{29}$$

It follows from this that the general differential equation for foundation beam under concentrated loads may be expressed as:

$$EI \cdot y^{IV} + m \cdot \ddot{y} + c \cdot \dot{y} + k \cdot y = \frac{F_{0s}}{L} \cdot \phi_s(t) \cdot \sum_n \frac{1}{\beta_n^2} \cdot X_{ns} \cdot X_n(x) \tag{30}$$

By rearranging Eqs. (5), (8) and (16), and by the method of separation of variables, Eq. (30) reduces to

$$\ddot{T}_n + \frac{c}{m} \cdot \dot{T}_n + \omega_n^2 \cdot T_n = \frac{F_{0s} \cdot \phi_s(t) \cdot X_{ns}}{m \cdot L \cdot \beta_n^2} \tag{31}$$

The solution of the differential equation (31) is determined by Duhamel integral as follows:

$$\begin{aligned} T_n(t) = e^{-\zeta\omega t}\Big[T_{n0} \cdot \cos(\omega_{Dn} t) + \frac{\dot{T}_{n0} + \zeta \cdot \omega \cdot T_{n0}}{\omega_{Dn}} \cdot \sin(\omega_{Dn} t)\Big] \\ + \frac{X_{ns} \cdot F_{0s}}{L \cdot m \cdot \omega_{Dn} \cdot \beta_n^2} \int_0^t \phi_s(\tau) \cdot e^{-\zeta\omega(t-\tau)} \cdot \sin(\omega_{Dn}(t-\tau)) \cdot d\tau \end{aligned} \tag{32}$$

where; T_{n0} and $\dot{T}_{n0}$ are parameters depending only on the initial conditions.

3.2. Foundation Beams under Concentrated Moment Loads

Since Eq. (1) is arranged for distributed loads, the concentrated moment $M(s,t)$ on any position s of the beam, positive in clockwise direction, may be transformed to distributed load by Dirac distribution as:

$$V(x,t) = \frac{\partial M(x,t)}{\partial x} = M_{0s} \cdot \psi_s(t) \cdot \delta_s(x) \tag{33}$$

and hence;

$$f(x,t) = -\frac{\partial V(x,t)}{\partial x} = -M_{0s} \cdot \psi_s(t) \cdot \delta'_s(x) \tag{34}$$

where; the moment function is of the form:

$$M(s,t) = M_{0s} \cdot \psi_s(t) \cdot H_s(x) \tag{35}$$

In these equations M_{0s} is the amplitude of the moment located at a point s, $\psi_s(t)$ is the time function of the force, $\delta_s(x)$ is Dirac distribution function centered at position *s*, and $H_s(x)$ is Heaviside function. The first derivative of Dirac distribution may be expanded into a series of shape functions as:

$$\delta'_s(x) = \delta'_s(x-s) = \sum_n \left[D_n \cdot X_n(x)\right] \tag{36}$$

By taking the inner product of Eq. (36) with X_m and taking into consideration the modal orthogonality together with the solution in Eq. (25), D_n can be obtained as:

$$D_n = \frac{\langle \delta'_s, X_n \rangle}{L\beta_n^2} \tag{37}$$

The inner product of *k-th* derivative of Dirac distribution with any function *g(x)* is given as:

$$\int_0^L \delta^{(k)}(x-s) \cdot g(x) = \int_0^L \delta_s^{(k)}(x) \cdot g(x+s) = (-1)^k g^{(k)}(s) \tag{38}$$

from last two equations the constant D_n is found as:

$$D_n = -\frac{X'_{ns}}{L \cdot \beta_n^2} \tag{39}$$

which leads to:

$$\delta'_s(x) = -\sum_n \frac{X'_{ns}}{L \cdot \beta_n^2} \cdot X_n(x) \tag{40}$$

Consequently, from Eqs. (1), (34) and (40), the general differential equation for foundation beam on elastic soil under concentrated moment loads is obtained as:

$$EI \cdot y^{IV} + m \cdot \ddot{y} + c \cdot \dot{y} + k \cdot y = \frac{M_{0s}}{L} \cdot \psi_s(t) \cdot \sum_n \frac{1}{\beta_n^2} \cdot X'_{ns} \cdot X_n(x) \tag{41}$$

which, by the method of separation of variables reduces to:

$$\ddot{T}_n + \frac{c}{m}\cdot\dot{T}_n + \omega_n^2\cdot T_n = \frac{M_{0s}\cdot\psi_s(t)\cdot X'_{ns}}{m\cdot L\cdot\beta_n^2} \tag{42}$$

The solution of the differential equation (42) is determined by Duhamel integral as follows:

$$\begin{aligned} T_n(t) = e^{-\zeta\omega t}[T_{n0}\cdot\cos(\omega_{Dn}t) + \frac{\dot{T}_{n0}+\zeta\cdot\omega\cdot T_{n0}}{\omega_{Dn}}\cdot\sin(\omega_{Dn}t)] \\ + \frac{X'_{ns}\cdot M_{0s}}{L\cdot m\cdot\omega_{Dn}\cdot\beta_n^2}\int_0^t \psi_s(\tau)\cdot e^{-\zeta\omega(t-\tau)}\cdot\sin(\omega_{Dn}(t-\tau))\cdot d\tau \end{aligned} \tag{43}$$

If more than one load acts on the system the generic equation to be solved may be written by superposition as:

$$\ddot{T}_n + \frac{c}{m}\cdot\dot{T}_n + \omega_n^2\cdot T_n = \frac{1}{m\cdot L\cdot\beta_n^2}\left[\sum_{s=1}^{i}\left(F_{0s}\cdot X_{ns}\cdot\phi_s(t)\right) + \sum_{s=1}^{j}\left(M_{0s}\cdot X'_{ns}\cdot\psi_s(t)\right)\right] \tag{44}$$

where; i is the number of concentrated force and j is the number of concentrated moment acting on the beam.

4. INITIAL CONDITIONS AND INTERNAL FORCES

The values of displacement and velocity functions for the beam at $t = 0$ have to be transformed to the time function and the first derivative of the time function with respect to *x*, namely the initial conditions for the time function. Let the displacement and velocity functions at initial time be $u(x)$ and $v(x)$, respectively, that is

$$y(x,0) = u(x) \tag{45}$$

$$\dot{y}(x,0) = v(x) \tag{46}$$

Hence, from Eq. (20)

$$u(x) = \sum_n\left[X_n(x)\cdot T_n(0)\right] \tag{47}$$

and

$$v(x) = \sum_n\left[X_n(x)\cdot\dot{T}_n(0)\right] \tag{48}$$

By taking the inner product of last two equations with X_m in view of the modal orthogonality and Eq. (25), $T_n(0)$ and $\dot{T}_n(0)$ can be obtained as:

$$T_n(0) = T_{n0} = \frac{\langle u, X_n \rangle}{L \cdot \beta_n^2} \tag{49}$$

$$\dot{T}_n(0) = \dot{T}_{n0} = \frac{\langle v, X_n \rangle}{L \cdot \beta_n^2} \tag{50}$$

The inner product of shape function given in Eq. (17) with an arbitrary constant yields zero, namely

$$\langle 1, X_n \rangle = \int_0^L X_n(x) \cdot dx = 0 \tag{51}$$

Therefore, the parameters T_{n0} and $\dot{T}_{n0}$ take the value of zero for constant displacement and velocity and shape functions do not represent the initial conditions. For this reason, in case of constant $u(x)$, this value has to be superimposed with the values $y(x,t)$ calculated from Eq. (20).

After determining the displacements caused by external loads acting on the foundation beam, the slope $\theta(x,t)$, bending moment $M(x,t)$ and shear force $V(x,t)$ at any given position x and time t may be evaluated by the following well-known relationships and Eq. (20) :

$$\theta(x,t) = \frac{\partial}{\partial x} y(x,t) = \sum_n \left[X_n'(x) \cdot T_n(t)\right] \tag{52}$$

$$M(x,t) = -EI \cdot \frac{\partial^2}{\partial x^2} y(x,t) = -EI \cdot \sum_n \left[X_n''(x) \cdot T_n(t)\right] \tag{53}$$

$$V(x,t) = -EI \cdot \frac{\partial^3}{\partial x^3} y(x,t) = -EI \cdot \sum_n \left[X_n'''(x) \cdot T_n(t)\right] \tag{54}$$

As mentioned before, if the initial displacement function is constant, namely

$$u(x) = u_c = \text{constant} \tag{55}$$

because of the property given in Eq. (51), it follows that

$$y(x,t) = \sum_n [X_n(x) \cdot T_n(t)] + u_c \tag{56}$$

For constant displacement and velocity functions

$$\begin{aligned} \theta(x,0) &= 0 \\ M(x,0) &= 0 \\ V(x,0) &= 0 \end{aligned} \tag{57}$$

therefore, the values found in Eqs. (52), (53) and (54) would not change.

5. NUMERICAL STUDY

As an application of the method, the foundation beam shown in (Figure 5.1) is considered. The beam is prismatic and has the following properties: the flexural rigidity EI = 3000 MNm^2, mass per unit length m = 2 kNs^2/m^2, base width b = 1.2 m. The subgrade modulus of the soil, K_0 on which the foundation rests is 50 MN/m^3. All the excitation frequencies, Ω are taken as 100 rad/sn. The problem is solved without damping, i.e., the damping ratio $\zeta = 0$. The solution of the problem under this data set is referred to as base.

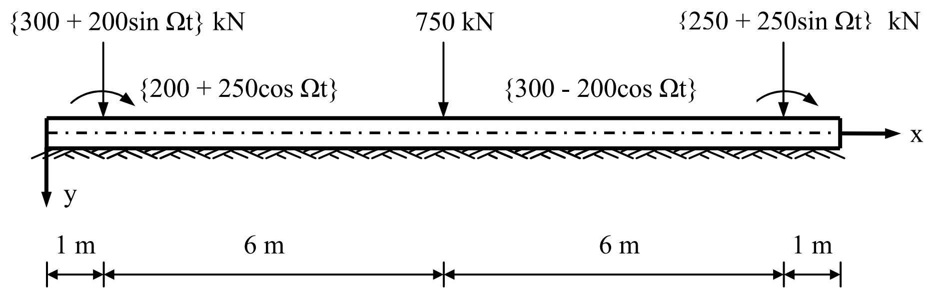


Figure 5.1 The foundation beam on elastic soil subjected to dynamic loading

In order to compare the behaviour of the foundation for different cases, the problem is solved for various data sets. In each set, one parameter is changed only. These parameters are the flexural rigidity, the subgrade modulus, the excitation frequencies and the damping ratio. Three different values of each parameter used are shown in (Table 5.1)

Table 5.1 Values Of The Parameters Used In The Different Solutions

PARAMETER	I	II	III
EI (MNm^2)	1000	10000	50000
K_0 (MN/m^3)	10	100	500
Ω (rad/sn.)	150	200	250
ζ (-)	5 %	10 %	20 %

The damping ratios in (Table 5.1) may be defined from Eq. (12) as:

$$\zeta = \frac{c}{2m\omega_1} \tag{58}$$

where; ω_1 is the first natural frequency of the beam and is equal to 222.587 rad/sn. in the studied case.

The problem is solved by taking into account 200 modes for the interval of time corresponding to a duration of four periods which is approximately 0.1 sn. The first natural period and the first five natural frequencies obtained for different values of parameters are set out in (Table 5.2)

Table 5.2 The First Natural Period (sn.) And The First Five Natural Frequencies (rad/sn.) For Different Values Of Parameters

	T_1 (sn.)	**ω_1 (rad/sn.)**	**ω_2 (rad/sn.)**	**ω_3 (rad/sn.)**	**ω_4 (rad/sn.)**	**ω_5 (rad/sn.)**
BASE	0.02823	222.587	422.509	775.089	1260.815	1873.607
EI_1	0.03288	191.089	281.966	469.313	741.542	1090.933
EI_2	0.02037	308.465	724.600	1390.160	2286.668	3410.475
EI_3	0.01053	596.450	1582.792	3089.130	5101.397	7618.182
$K_{0,1}$	0.03931	159.828	393.082	759.449	1251.261	1867.191
$K_{0,2}$	0.02228	282.037	456.633	794.206	1272.657	1881.596
$K_{0,3}$	0.01112	565.283	669.712	933.147	1363.692	1944.326
D_1	0.02826	222.309	422.362	775.009	1260.766	1873.574
D_2	0.02837	221.472	421.922	774.770	1260.619	1873.474
D_3	0.02881	218.090	420.157	773.810	1260.029	1873.078

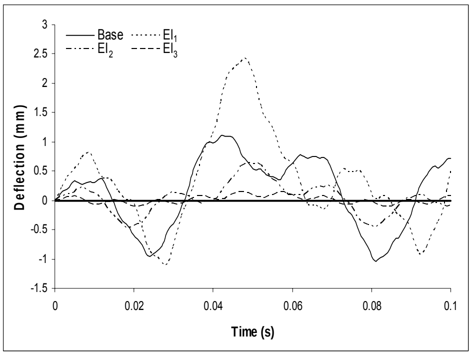
(Table 5.3) gives the extremum values of displacements and bending moments of the foundation beam for different values of parameters. In this table, the positions and the time of occurrences of these extrama are shown, too. Also relative changes (RC) which give comparisons between the results obtained from the base solution and the results obtained by changing the parameters are presented in the table.

Table 5.3 The Extremum Values Of Displacements And Bending Moments Of The Foundation Beam For Different Values Of Parameters

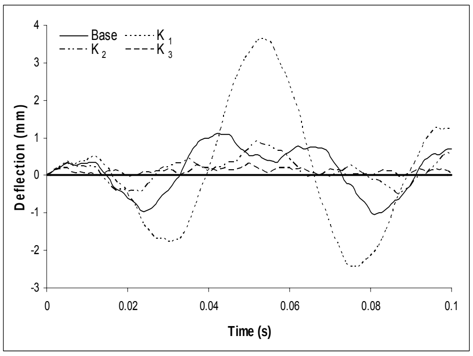
	y_{max}* (mm)			y_{min}† (mm)			M_{max} (kNm)				M_{min} (kNm)			
	Mag. (mm)	RC (%)	Time (10^{-2}s)	Mag (mm)	RC (%)	Time (10^{-2}s)	Mag. (kNm)	RC (%)	Pos (m)	Time (10^{-2}s)	Mag. (kNm)	RC (%)	Pos (m)	Time (10^{-2}s)
Base	2.57	-	8.32	-1.67	-	4.16	1022	-	6.86	4.47	-1014	-	10.04	2.17
EI_1	3.77	47	2.96	-3.21	92	5.37	866	-15	6.99	4.81	-805	-21	12.01	9.77
EI_2	1.07	-58	2.04	-1.10	-34	4.90	1433	40	6.86	4.72	-1235	24	10.04	2.11
EI_3	0.25	-90	2.14	-0.26	-84	5.73	1654	62	7.10	4.74	-1296	28	10.04	2.18
$K_{0,1}$	4.72	84	7.71	-6.00	259	5.27	2136	109	7.00	5.24	-1396	38	9.79	7.23
$K_{0,2}$	1.77	-31	2.03	-1.29	-23	5.17	927	-10	7.04	5.16	-905	-10	11.58	9.06
$K_{0,3}$	0.62	-76	2.26	-0.47	-72	6.02	649	-36	1.09	0.16	-686	-32	11.80	9.05
Ω_1	3.72	45	5.31	-3.55	113	3.78	1441	41	6.97	3.55	-1191	18	9.98	5.20
Ω_2	7.54	193	7.70	-8.45	406	9.20	2827	177	7.09	9.20	-1795	77	4.87	7.63
Ω_3	7.69	199	9.48	-7.41	343	8.17	2405	135	7.04	8.20	-2102	107	8.69	9.53
D_1	2.24	-13	2.29	-1.34	-20	4.19	893	-13	7.04	4.56	-906	-11	10.04	2.17
D_2	2.06	-20	2.30	-1.36	-19	5.17	810	-21	7.03	4.56	-821	-19	10.20	2.16
D_3	1.79	-30	2.33	-1.46	-13	5.47	722	-29	6.97	5.23	-706	-30	11.58	2.32

* maximum displacements occur at x = 14 m in each case; † minimum displacements occur at x = 0 m in each case

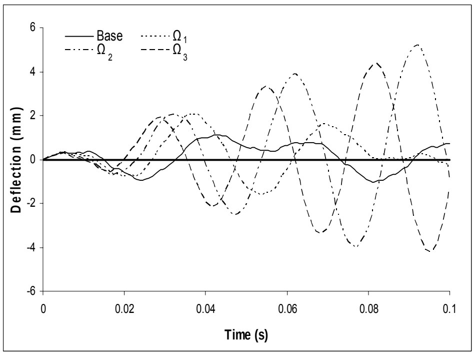
The deflection and bending moment responses of the foundation beam at x = L / 2 obtained for different parameter values mentioned before are presented in (Figure 5.2) and (Figure 5.3), respectively.

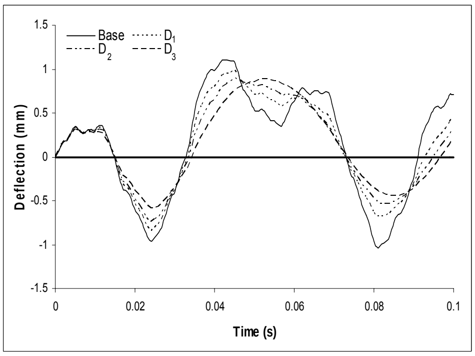


(a)



(b)



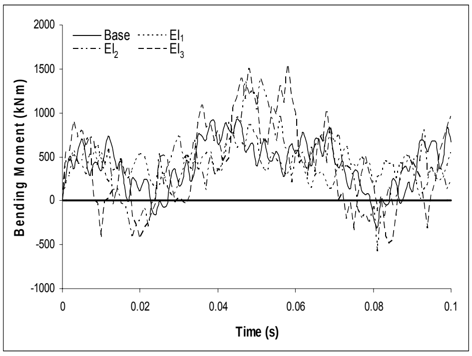


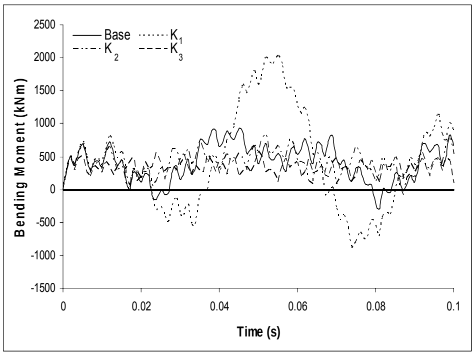
(c)

(d)

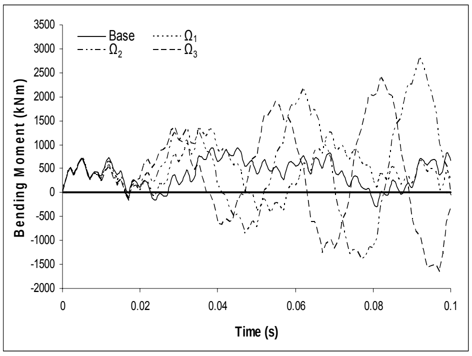
Figure 5.2 Deflection versus time of the beam shown in (Figure 1.1) at x = L / 2 for different parameters:

-a Flexural rigidity
-b Subgrade modulus
-c Excitation frequencies
-d Damping ratio

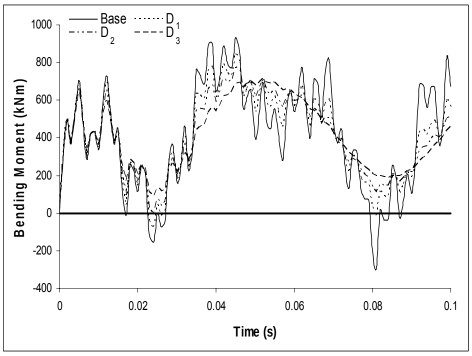




(a) (b)



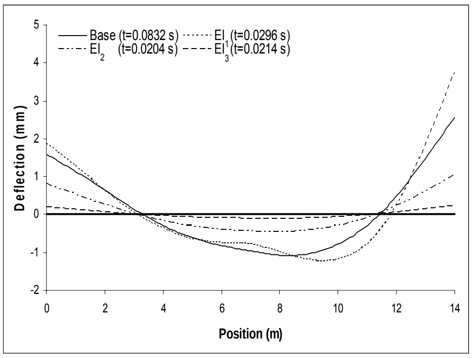
(c)



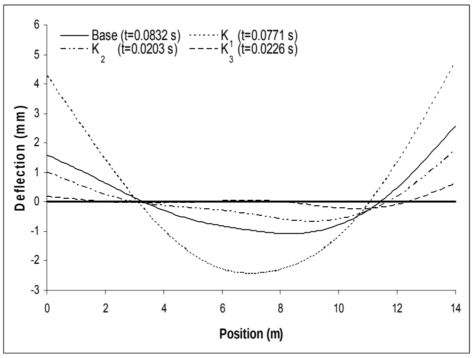
(d)

Figure 5.3 Bending moment versus time of the beam shown in (Figure 1.1) at x = L / 2 for different parameters:
-a Flexural rigidity
-b Subgrade modulus
-c Excitation frequencies
-d Damping ratio

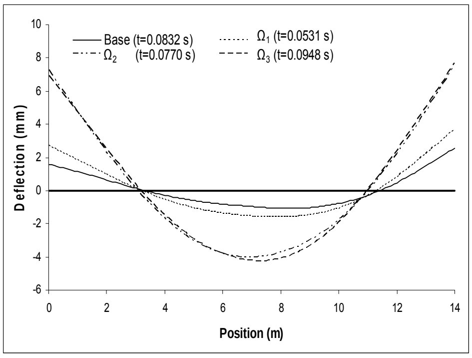
(Figure 5.4) shows the elastic curves of the beam at the time in which the maximum positive deflection occurs for different parameter values. The instances given in (Figure 5.4) can also be seen in (Table 5.3)



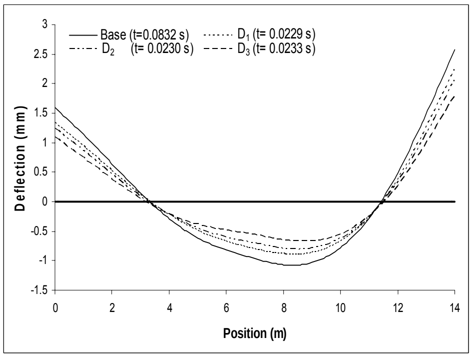
(a)



(b)



(c)



(d)

Figure 5.4 Elastic curve of the beam shown in (Figure 1.1) at the time in which the maximum positive deflection occurs for different parameters:
-a Flexural rigidity
-b Subgrade modulus
-c Excitation frequencies
-d Damping ratio

With the variation of parameters such as flexural rigidity, subgrade modulus, excitation frequency and damping ratio, it is clear from (Table 5.3) and (Figure 5.4) that not only the magnitudes of the maximum deflections and bending moments but also their time of occurrences and their positions change. When (Table 5.3) and (Figure 5.2) through (Figure 5.4) are perused, it can be observed that the influences of the variation of subgrade modulus and flexural rigidity on the dynamic responses are more pronounced compared to the variation of damping ratio and excitation frequency. The damping ratios used in this study are practical values. If higher damping ratios are considered, their influence on responses would be more distinguishable. As the excitation frequencies approach to the natural frequency of the system it is obvious that their responses increase sharply.

6. CONCLUSION

The dynamic responses of a free beam subjected to transverse forces and moments on a Winkler foundation are presented. Since the governing differential equation is established for distributed loads, the concentrated forces and moments on beams have been transformed to distributed loads using Dirac distribution theory. Even though in theory this method is elegant, it turns out to be impractical in some cases. While the method yields reliable results for all dynamic responses (deflection, slope, bending moment and shear force) for concentrated forces, in the case of concentrated moment action, some inconsistencies may appear in shear forces due to the property of distribution functions. For the same reason, it is needed to involve a large number of modes to calculate shear forces for concentrated force and bending moments for concentrated moment loading. Since the distribution functions are expanded into a series of continuous shape functions, the discontinuities in the related internal forces at the points of application of the loads can be noticed only when higher modes are used.

REFERENCES

[1] J. Arsac, *Fourier transforms and the theory of distributions,* Prentice-Hall, Inc., Englewood Cliffs, N.J., (1966).

[2] A.T. Daloglu and C.V.G. Vallabhan, *Values of k for slab on Winkler foundation,* Journal of Geotechnical and Geoenvironmental Engineering, ASCE, 126 (2000), 463-473.

[3] M.M. Filonenko-Borodich, *Some approximate theories of elastic foundation (in Russian)*, Uchenyie Zapiski Moskovskogo Gosudarstvennogo Universitet, Mekhanika, 46 (1946), 3-18.

[4] M. Hetenyi, *Beams on elastic foundation*, The University of Michigan Press, (1946).

[5] A.D. Kerr, *Elastic and visco-elastic foundations models,* Journal of Applied Mechanics, ASME, 31 (1964), 491-498.

[6] P.L. Pasternak*, On a new method of analysis of an elastic foundation by means of two foundations constants (in Russian),* Gasudarstvennoe Izdatelstvo Literaturi po Stroitelstvui Arkhitekure, Moscow, (1954).

[7] K. Terzaghi, *Theoretical soil mechanics*, 14th ed., John Wiley & Sons, Inc., New York, (1966).

[8] V.Z. Vlasov and N. N. Leontiev, *Beams, plates and shells on elastic foundations (in Russian),* Fizmatgiz Publishers, Moscow, (1960).

[9] E. Winkler, *Die Lehre von der Elastizität und Festigkeit,* Prag., (1867).