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Research Article

Focal Surfaces Connected with VFF for Type 1-PAF in $\mathbb{E}^3$

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Abstract
In this paper, we obtain vortex filament flow (VFF) surfaces for three classes according to Type 1-PAF in Euclidean 3-space $\mathbb{E}^3$. Additionally, we obtain the first and second fundamental forms, Gauss and mean curvatures of focal surfaces generated by vortex filament flow surfaces in Euclidean 3-space $\mathbb{E}^3$. Furthermore , we present focal surfaces generated by vortex filament flow for three classes according Type 1-PAF in Euclidean 3-space $\mathbb{E}^3$. Finaly, we provide examples related to focal surfaces generated by vortex filament flow surfaces for three classes according Type 1-PAF in Euclidean 3-space $\mathbb{E}^3$.

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1. Introduction

In recent years, the connection between space curves and solitons have attracted significant attention from mathematicians due to their extensive applications in physical fields. Özen and Tosun [1] introduced the first type of position-adapted frame $\{T,J,Z\}$ (Type 1-PAF) via the Frenet frame $\{T,N,B\}$ of the curve α with the arc length s as the following:

$$Z = \frac{< -\alpha, N > N + < \alpha, B > B}{\sqrt{< \alpha, N >^2 + < \alpha, B >^2}} \ , \ J = Z \times T. \tag{1}$$

Also, they presented derivative formulas of $\{T,J,Z\}$ with continious coefficients functions η_1, η_2, η_3 as follows:

$$\begin{bmatrix} T_s \\ J_s \\ Z_s \end{bmatrix} = \begin{bmatrix} 0 & \eta_1(s) & \eta_2(s) \\ -\eta_1(s) & 0 & \eta_3(s) \\ -\eta_2(s) & -\eta_3(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ J(s) \\ Z(s) \end{bmatrix}. \tag{2}$$

Özen et al. [2] presented the second type of position-adapted frame (Type 2-PAF). Solouma [3] showed the characterization of Smarandache trajectory curves of constant mass point particles as they move along the trajectory curve via PAF. Ertuğ [4] investigated the change in the electric field in Minkowski 3-space according to the first type of position-adapted frame and the position adapted frame on regular surfaces. Additionally, Ertuğ and Yoon [5] presented the change in the electric field for the Type 2 and the Type 3-PAFs.

Hasimoto [6] was the first to investigate the connection between the motion of a vortex filament and elasticity. In 1972, Hasimoto explored the relationship of the isolated thin vortex filament motion in an incompressible non-viscous flow with the nonlinear Schrödinger (NLS) equation [7]. These vortex motions correspond to soliton wave solutions of the NLS equation. The surface associated with the NLS equation is called vortex filament surface (VFF) [8].

Lamb presented the first class of the curve evolution for the Frenet frame in the Euclidean 3- space [9]. The relationship between the time evolution of a spin vector and the nonlinear Schrödinger equation was provided by Lakshmanan [10].

In this paper, we will study on focal surfaces, which are significant in physics, quantum and optics. Focal curves and surfaces used in eye treatments achieve optimal results by focusing light rays at a common point. Knowing the distance from the surface to these focal points is essential. Hagen et al. [11, 12] first utilized focal surfaces, known as congruences of lines, in the field of imaging. Hagen and Hahmann [13] investigated generalized focal surfaces. Honda and Takahashi [14] studied the focal surfaces of an immersion with a frame. Güler [15] investigated the geometry of the offset focal surfaces of a given parametric surface. Al-Dayel et al. [16] provided tubular surfaces according to the B-Darboux frame and the second type Bishop frame, examining their focal surfaces.

In Section 2, we obtain focal surfaces generated by vortex filament flow for three classes according to Type 1-PAF. In the final section, we present the conclusions.

2. Focal surfaces generated by VFF for Type 1-PAF in $\mathbb{E}^3$

In this section, we study focal surfaces generated by VFF for Type 1-PAF in $\mathbb{E}^3$.

2.1 Focal surfaces generated by VFF with the first class for Type 1-PAF in $\mathbb{E}^3$

Let h be a moving space curve, and $h = h(s,u)$ be the position vector of a point on the moving space curve where u is time and s is arc-length parameter. The VFF equation is denoted as [17]:

$$h_u = h_s \times h_{ss}. \tag{3}$$

The time evolution of the $\{T, J, Z\}$ frame for the first class according to the Type 1-PAF is given as follows:

$$\begin{aligned} T_u &= yJ + wZ, \\ J_u &= -yT + \gamma_1 Z, \\ Z_u &= -wT - \gamma_1 J. \end{aligned} \tag{4}$$

Here γ_1, y and w are differentiable functions for the Type 1-PAF. From equations (2) and (4), the following equations are obtained:

$$\begin{aligned} T_{su} &= -(\eta_1 y + \eta_2 w)T + (\eta_{1u} - \eta_2\gamma_1)J + (\eta_1\gamma_1 + \eta_{2u})Z, \\ T_{us} &= -(y\eta_1 + w\eta_2)T + (y_s - w\eta_3)J + (y\eta_3 + w_s)Z, \\ J_{su} &= -(\eta_{1u} + \eta_3 w)T - (\eta_1 y + \eta_3\gamma_1)J + (\eta_{3u} - \eta_1 w)Z, \\ J_{us} &= -(y_s + \gamma_1\eta_1)T - (y\eta_1 + \gamma_1\eta_3)J + (\gamma_{1s} - y\eta_2)Z. \end{aligned} \tag{5}$$

From $T_{us} = T_{su}, \quad J_{us} = J_{su}$, we obtain:

$$\begin{aligned} \eta_{1u} &= y_s + \eta_2\gamma_1 - w\eta_3, \\ \eta_{2u} &= y\eta_3 - \eta_1\gamma_1 + w_s, \\ \eta_{3u} &= \eta_1 w - y\eta_2 + \gamma_{1s}. \end{aligned} \tag{6}$$

The velocity of the curve for the Type 1-PAF is given by:

$$h_u = f_1 T + d_1 J + q_1 Z. \tag{7}$$

Via equation (7), it is given by:

$$h_{us} = (f_{1s} - d_1 y - q_1\eta_1)T + (f_1\eta_1 + d_{1s} - q_1\eta_3)J + (f_1\eta_2 + q_{1s} + d_1\gamma_1)Z. \tag{8}$$

If $h_s = T$ is substituted in equation (3),

$$h_u = \eta_1 Z - \eta_2 J. \tag{9}$$

From $h_{su} = h_{us}$, it can be derived:

$$\begin{aligned} f_{1s} &= d_1 y + q_1\eta_1, \\ d_{1s} &= y + q_1\eta_3 - f_1\eta_1, \\ q_{1s} &= w - f_1\eta_2 - d_1\gamma_1. \end{aligned} \tag{10}$$

From equations (7) and (9),

$$f_1 = 0, \qquad d_1 = -\eta_2, \qquad q_1 = \eta_1. \tag{11}$$

If equation (11) is rewritten in equation (10), it is given by:

$$y = -(\eta + \eta_3\eta_1), \qquad w = \eta_{1s} - \eta_2\eta_3. \tag{12}$$

From equation (12),

$$w_s = \eta_{1ss} - (\eta_2\eta_3)_s, \qquad y_s = -(\eta_{2ss} + (\eta_3\eta_1)_s). \tag{13}$$

If equations (12) and (13) are rewritten in equation (6), we obtain:

$$\begin{aligned} \eta_{1u} &= \eta_2\gamma_1 - \eta_{2ss} - (\eta_3\eta_1)_s - \eta_3(\eta_{1s} - \eta_2\eta_3), \\ \eta_{2u} &= \eta_{1ss} - \eta_3(\eta_{2s} + \eta_3\eta_1) - \eta_1\gamma_1 - (\eta_2\eta_3)_s, \\ \eta_{3u} &= \gamma_{1s} + \eta_1\eta_{1s} + \eta_2\eta_{2s}. \end{aligned} \tag{14}$$

If equations (12) is rewritten in equation (4), the time evolution of the frame $\{T, J, Z\}$ is derived:

$$\begin{aligned} T_u &= -(\eta_{2s} + \eta_3\eta_1)J + (\eta_{1s} - \eta_2\eta_3)Z, \\ J_u &= (\eta_{2s} + \eta_3\eta_1)T + \gamma_1 Z, \\ Z_u &= (\eta_2\eta_3 - \eta_{1s})T - \gamma_1 J. \end{aligned} \tag{15}$$

The unit normal vector field of the VFF surface with respect to the Type 1-PAF for the first class is given by:

$$\mathbb{N}_1 = -\frac{\eta_1 J + \eta_2 Z}{c}, \qquad c = \sqrt{\eta_1^2 + \eta_2^2}. \tag{16}$$

Respectively, the first and second fundamental forms with respect to the Type 1-PAF for the first class associated with VFF are obtained:

$$\begin{aligned} I &= ds^2 + c^2 du^2, \\ II &= \tfrac{1}{c}(c^2 ds^2 + 2(\eta_3(c^2 + \eta_1\eta_{2s} + \eta_{1s}\eta_2)dsdu + \gamma_1(c^2 - \eta_1\eta_{2u} - \eta_{1u}\eta_2)du^2). \end{aligned} \tag{17}$$

Here, the coefficients of the first and second fundamental forms for the Type 1-PAF of the first class are

$$E = 1, \qquad F = 0, \qquad G = c^2, \tag{18}$$

$$l = -c, \qquad m = \frac{\eta_3 c^2 + \eta_1\eta_{2s} - \eta_{1s}\eta_2}{c}, \qquad n = \frac{\gamma_1 c^2 + \eta_1\eta_{2u} - \eta_2\eta_{1u}}{c}. \tag{19}$$

Respectively, the Gauss and mean curvatures of the VFF surface for the Type 1-PAF for the first class with the aid of equations (18), (19) are given by

$$K_1 = \frac{-c^2(\gamma_1(\eta_1^2 + \eta_2^2) + \eta_1\eta_{2u} - \eta_2\eta_{1u}) + (\eta_1\eta_{2s} + \eta_3 c^2 + \eta_{1s}\eta_2)^2}{c^4}, \tag{20}$$

$$H_1 = \frac{\gamma_1 c^2 - \eta_1\eta_{2u} - \eta_2\eta_{1u} + c^4}{2c^3}. \tag{21}$$

Respectively, the principal curvatures k_1 and k_2 of the Type 1-PAF for the first class associated with the VFF with the aid of equations (20), (21) are obtained:

$$\begin{aligned} k_1 = &\frac{\gamma_1 c^2 - \eta_1\eta_{2u} - \eta_2\eta_{1u} + c^4}{2c^3} \\ &+ \sqrt{\left(\frac{\gamma_1 c^2 - \eta_1\eta_{2u} - \eta_2\eta_{1u} + c^4}{2c^3}\right)^2 + \frac{c^2(\gamma_1 c^2 + \eta_1\eta_{2u} - \eta_2\eta_{1u}) - (\eta_1\eta_{2s} + \eta_3 c^2 + \eta_{1s}\eta_2)^2}{c^4}} \end{aligned} \tag{22}$$

$$k_2 = \frac{\gamma_1 c^2 - \eta_1\eta_{2u} - \eta_2\eta_{1u} + c^4}{2c^3} - \sqrt{\left(\frac{\gamma_1 c^2 - \eta_1\eta_{2u} - \eta_2\eta_{1u} + c^4}{2c^3}\right)^2 + \frac{c^2(\gamma_1 c^2 + \eta_1\eta_{2u} - \eta_2\eta_{1u}) - (\eta_1\eta_{2s} + \eta_3 c^2 + \eta_{1s}\eta_2)^2}{c^4}}. \tag{23}$$

Let

$$\Upsilon_{1;i}(s,u) = h(s,u) + k_i^{-1}(s,u)\mathbb{N}_1(s,u), \quad i = 1,2 \tag{24}$$

be the focal surfaces of the Type 1-PAF for the first class be generated by the VFF. From equation (22), (23) and (24), we obtain

$$\Upsilon_{1;is} = \frac{c}{k_i}T + \frac{\eta_1 k_i^2 c^2 + (\eta_{1s} + \eta_2\eta_3)k_i c + \eta_1(k_i c_s - k_{is}c)}{k_i^2 c^2}J + \frac{\eta_3 k_i^2 c^2 - (\eta_{2s} + \eta_1\eta_3)k_i c + \eta_2(k_i c_s - k_{is}c)}{k_i^2 c^2}Z, \tag{25}$$

$$\Upsilon_{1;iu} = \frac{\eta_{2s}\eta_1 - c^2\eta_3 - \eta_{1s}\eta_2}{k_i c}T + \frac{(\gamma_1\eta_2 - \eta_{2u})k_i c + (\eta_1\eta_3 - \eta_{2s})k_i^2 c^2 + \eta_1(k_i c_u - k_{iu}c)}{k_i^2 c^2}J + \frac{(\eta_{1s-}\eta_2\eta_3)k_i^2 c^2 - (\eta_{3u} + \gamma_1\eta_1)k_i c + \eta_2(k_i c_u - k_{iu}c)}{k_i^2 c^2}Z. \tag{26}$$

The normal vector field of the focal surface generated by the VFF with respect to the Type 1-PAF for the first class in $\mathbb{E}^3$ is given by:

$$\tilde{\mathbb{N}}_1 = \frac{\lambda_1 T + \lambda_2 J + \lambda_3 Z}{\mathring{\lambda}}, \qquad \mathring{\lambda} = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \neq 0. \tag{27}$$

Here,

$$\lambda_1 = \mathring{b}_1\mathring{f}_1 - \mathring{c}_1\mathring{e}_1, \qquad \lambda_2 = \mathring{c}_1\mathring{d}_1 - \mathring{a}_1\mathring{f}_1, \qquad \lambda_3 = \mathring{a}_1\mathring{e}_1 - \mathring{b}_1\mathring{d}_1. \tag{28}$$

$$\begin{aligned}
\mathring{a}_1 &= \frac{c}{k_i}, \\
\mathring{b}_1 &= \frac{\eta_1 k_i^2 c^2 + (\eta_{1s} + \eta_2\eta_3)k_i c + \eta_1(k_{is}c - k_i c_s)}{k_i^2 c^2}, \\
\mathring{c}_1 &= \frac{\eta_3 k_i^2 c^2 - (\eta_{2s} + \eta_1\eta_3)k_i c + \eta_2(k_i c_s - k_{is}c_2)}{k_i^2 c^2}, \\
\mathring{d}_1 &= \frac{\eta_{2s}\eta_1 - c^2\eta_3 - \eta_{1s}\eta_2}{k_i c}, \\
\mathring{e}_1 &= \frac{-(\eta_{2s} + \eta_1\eta_3)k_i^2 c^2 + (\gamma_1\eta_2 - \eta_{2u})k_i c + \eta_1(k_i c_u + k_{iu}c)}{k_i^2 c^2}, \\
\mathring{f}_1 &= \frac{(\eta_{1s} - \eta_2\eta_3)k_i^2 c^2 - (\eta_{3u} + \gamma_1\eta_1)k_i c + \eta_2(k_i c_u - k_{iu}c)}{k_i^2 c^2}.
\end{aligned} \tag{29}$$

The first fundamental form of the focal surface generated by the VFF with respect to the Type 1-PAF for the first class in $\mathbb{E}^3$ is given by:

$$\begin{aligned}
\tilde{E} &= \langle \Upsilon_{1;is}, \Upsilon_{1;is} \rangle = \mathring{a}_1^2 + \mathring{b}_1^2 + \mathring{c}_1^2, \\
\tilde{F} &= \langle \Upsilon_{1;is}, \Upsilon_{1;iu} \rangle = \mathring{a}_1\mathring{d}_1 + \mathring{b}_1\mathring{e}_1 + \mathring{c}_1\mathring{f}_1, \\
\tilde{G} &= \langle \Upsilon_{1;iu}, \Upsilon_{1;iu} \rangle = \mathring{e}_1^2 + \mathring{d}_1^2 + \mathring{f}_1^2,
\end{aligned} \tag{30}$$

The second fundamental form of the focal surface generated by VFF with respect to the Type 1-PAF for the first class in $\mathbb{E}^3$ is given by:

$$\begin{aligned}
\tilde{l} &= <\Upsilon_{1;iss}, \tilde{\mathbb{N}}_1> = \frac{\mu_1\lambda_1 + \mu_2\lambda_2 + \mu_3\lambda_3}{\mathring{\lambda}}, \\
\tilde{m} &= <\Upsilon_{1;isu}, \tilde{\mathbb{N}}_1> = \frac{\varpi_1\lambda_1 + \varpi_2\lambda_2 + \varpi_3\lambda_3}{\mathring{\lambda}}, \\
\tilde{n} &= <\Upsilon_{1;iuu}, \tilde{\mathbb{N}}_1> = \frac{\rho_1\lambda_1 + \rho_2\lambda_2 + \rho_3\lambda_3}{\mathring{\lambda}}.
\end{aligned} \tag{31}$$

Here,

$$\Upsilon_{1;iss} = (\mathring{a}_{1s} - \mathring{b}_1\eta_1 - \mathring{c}_1\eta_2)T + (\mathring{a}_1\eta_1 + \mathring{b}_{1s} - \mathring{c}_1\eta_3)J + (\mathring{a}_1\eta_2 + \mathring{b}_1\eta_3 + \mathring{c}_{1s})Z, \tag{32}$$

$$\begin{aligned}\Upsilon_{1;iuu} = {} & (\mathring{d}_{1u} + \mathring{e}_1(\eta_{2s} + \eta_3\eta_1) + \mathring{f}_1(-\eta_{1s} + \eta_2\eta_3))T + (\mathring{e}_{1u} - \mathring{d}_1(\eta_{2s} + \eta_3\eta_1) - \mathring{f}_1\gamma_1)J \\ & + ((\eta_{1s} - \eta_2\eta_3)\eta_2 + \mathring{e}_1\gamma_1 + \mathring{f}_{1u})Z,\end{aligned} \tag{33}$$

$$\begin{aligned}\Upsilon_{1;isu} = {} & (\mathring{a}_{1u} + \mathring{b}_1(\eta_{2s} + \eta_3\eta_1) + \mathring{c}_1(-\eta_{1s} + \eta_2\eta_3))T + (\mathring{a}_{1u} + \mathring{b}_1(\eta_{2s} + \eta_3\eta_1) + \mathring{c}_1(-\eta_{1s} + \eta_2\eta_3))J \\ & + (\mathring{a}_1(\eta_{1s} - \eta_2\eta_3) + \mathring{b}_1\gamma_1 + \mathring{c}_u)Z.\end{aligned} \tag{34}$$

$$\begin{aligned}\mu_1 &= \mathring{a}_{1s} - \mathring{b}_1\eta_1 - \mathring{c}_1\eta_2, \\ \mu_2 &= \mathring{a}_1\eta_1 + \mathring{b}_{1s} - \mathring{c}_1\eta_3, \\ \mu_3 &= \mathring{a}_1\eta_2 + \mathring{b}_1\eta_3 + \mathring{c}_{1s}, \\ \varpi_1 &= \mathring{a}_{1u} + \mathring{b}_1(\eta_{2s} + \eta_3\eta_1) + \mathring{c}_1(\eta_2\eta_3 - \eta_{1s}), \\ \varpi_2 &= (\mathring{a}_{1u} + \mathring{b}_1(\eta_{2s} + \eta_3\eta_1) + \mathring{c}_1(\eta_2\eta_3 - \eta_{1s}), \\ \varpi_3 &= \mathring{a}_1(\eta_{1s} - \eta_2\eta_3) + \mathring{b}_1\gamma_1 + \mathring{c}_u, \\ \rho_1 &= \mathring{d}_{1u} + \mathring{e}_1(\eta_{2s} + \eta_3\eta_1) + \mathring{f}_1(\eta_2\eta_3 - \eta_{1s}), \\ \rho_2 &= \mathring{e}_{1u} - \mathring{d}_1(\eta_{2s} + \eta_3\eta_1) - \mathring{f}_1\gamma_1, \\ \rho_3 &= (\eta_{1s} - \eta_2\eta_3)\eta_2 + \mathring{e}_1\gamma_1 + \mathring{f}_{1u}.\end{aligned} \tag{35}$$

From equations (27), (30) and (31), the Gauss and mean curvatures of focal surfaces for the Type 1-PAF of the first class are given by:

$$\tilde{K}_1 = \frac{(\mu_1\lambda_1+\mu_2\lambda_2+\mu_3\lambda_3)(\rho_1\lambda_1+\rho_2\lambda_2+\rho_3\lambda_3) - (\varpi_1\lambda_1+\varpi_2\lambda_2+\varpi_3\lambda_3)^2}{\mathring{\lambda}(\mathring{\chi}_1(a_1\mathring{d}_1+b_1e_1+\mathring{c}_1f_1) - \mathring{\xi}_1^2)}, \tag{36}$$

and

$$\tilde{H}_1 = \frac{\mathring{\chi}_1(\rho_1\lambda_1+\rho_2\lambda_2+\rho_3\lambda_3) + (\mathring{a}_1\mathring{d}_1+\mathring{b}_1\mathring{e}_1+\mathring{c}_1\mathring{f}_1)(\mu_1\lambda_1+\mu_2\lambda_2+\mu_3\lambda_3) - 2\mathring{\xi}_1(\varpi_1\lambda_1+\varpi_2\lambda_2+\varpi_3\lambda_3)}{2\mathring{\lambda}(\mathring{\chi}_1(\mathring{a}_1\mathring{d}_1+\mathring{b}_1\mathring{e}_1+\mathring{c}_1\mathring{f}_1) - \mathring{\xi}_1^2)}. \tag{37}$$

$$\mathring{\chi}_1 = \mathring{a}_1^2 + \mathring{b}_1^2 + \mathring{c}_1^2, \qquad \mathring{\xi}_1 = \mathring{e}_1^2 + \mathring{d}_1^2 + \mathring{f}_1^2. \tag{38}$$

2.2 Focal surfaces generated by VFF with the second class for Type 1-PAF in $\mathbb{E}^3$

The time evolution of the $\{T, J, Z\}$ frame for the second class, according to the Type 1-PAF is given as follows:

$$\begin{aligned}T_u &= -jJ + \gamma_2 Z, \\ J_u &= jT + w_2 Z, \\ Z_u &= -\gamma_2 T - w_2 J.\end{aligned} \tag{39}$$

Here γ_2, j and w_2 are differentiable functions for the Type 1-PAF. From equations (2) and (39), the followings are obtained:

$$\begin{aligned}T_{su} &= -(\eta_1 y + \eta_2\gamma_2)T + (\eta_{1u} - \eta_2 w_2)J + (\eta_1 w_2 + \eta_{2u})Z, \\ T_{us} &= -(j\eta_1 + \gamma_2\eta_2)T + (j_s - \gamma_2\eta_3)J + (\gamma_{2s} - j\eta_3)Z, \\ J_{su} &= -(\eta_{1u} + \eta_3\gamma_2)T + (\eta_1 j - \eta_3 w)J + (\eta_{3u} - \eta_1\gamma_2)Z, \\ J_{us} &= (j_s - w_2\eta_2)T + (j\eta_1 - w_2\eta_3)J + (j\eta_2 + w_{2s})Z.\end{aligned} \tag{40}$$

From $T_{us} = T_{su}$, $J_{us} = J_{su}$, we obtain:

$$\begin{aligned}\eta_{1u} &= \eta_2 w_2 - j_s - \gamma_2\eta_3, \\ \eta_{2u} &= y\eta_3 - \eta_1\gamma_1 + w_{2s}, \\ \eta_{3u} &= \eta_1 w_2 - y\eta_2 + \gamma_{1s}.\end{aligned} \tag{41}$$

The velocity of the curve for the Type 1-PAF is given by:

$$h_u = f_2T + d_2J + q_2Z. \tag{42}$$

Via equation (42) it is given by:

$$h_{us} = (f_{2s} - d_2y - q_2\eta_1)T + (f_2\eta_1 + d_{2s} - q_2\eta_3)J + (f_2\eta_2 + q_{2s} + d_2\gamma_2)Z. \tag{43}$$

If $h_s = J$ is substituted in equation (3),

$$h_u = \eta_3T + \eta_1Z. \tag{44}$$

From $h_{su} = h_{us}$, it can be derived:

$$\begin{aligned} f_{2s} &= d_{2s}y - q_2\eta_1, \\ d_{2s} &= j + q_2\eta_3 - f_2\eta_1, \\ q_{2s} &= w_2 - f_2\eta_2 - d_2\gamma_2. \end{aligned} \tag{45}$$

From equations (42) and (44)

$$f_2 = \eta_3, \qquad d_2 = 0, \qquad q_2 = \eta_1. \tag{46}$$

If equation (46) is rewritten in equation (45), it is given by:

$$j = \eta_{3s} - \eta_2\eta_1, \qquad w_2 = \eta_{1s} + \eta_2\eta_3. \tag{47}$$

From equation (47)

$$j_s = \eta_{3ss} - (\eta_2\eta_1)_s, \qquad w_{2s} = \eta_{1ss} + (\eta_2\eta_3)_s. \tag{48}$$

If equations (47) and (48) are rewritten in equation (41), we obtain:

$$\begin{aligned} \eta_{1u} &= \eta_3\gamma_2 + \eta_3\eta_{1s}\eta_2) - (\eta_{3ss} + (\eta_2\eta_1)_s, \\ \eta_{2u} &= \gamma_{2s} - \eta_3(\eta_{3s} - \eta_1\eta_2) + \eta_1(\eta_2\eta_3 - \eta_{1s}), \\ \eta_{3u} &= \eta_2(\eta_{3s} - \eta_2\eta_1) + \eta_1\gamma_2 + \eta_{1ss} + (\eta_2\eta_2)_s. \end{aligned} \tag{49}$$

If equations (47) is rewritten in equation (39), the time evolution of the frame $\{T, J, Z\}$ is derived:

$$\begin{aligned} T_u &= (\eta_{3s} + \eta_2\eta_1)J + \gamma_2Z, \\ J_u &= (\eta_{3s} - \eta_2\eta_1)T + (\eta_{1s} + \eta_2\eta_3)Z, \\ Z_u &= -\gamma_2T - (\eta_{1s} + \eta_2\eta_3)J. \end{aligned} \tag{50}$$

The unit normal vector field of VFF surface with respect to the Type 1-PAF for the second class is given by:

$$\mathbb{N}_2 = \frac{\eta_1T - \eta_3Z}{\Theta}, \qquad \Theta = \sqrt{\eta_1^2 + \eta_3^2}. \tag{51}$$

Respectively,the first and second fundamental forms with respect to te Type 1-PAF for the second class associated with the VFF are obtained:

$$\begin{aligned} I &= ds^2 + \Theta^2du^2, \\ II &= \tfrac{1}{\Theta}(\Theta^2ds^2 + 2(\eta_1(\eta_{3s} - \eta_1\eta_2) - \eta_3(\eta_{1s} + \eta_2\eta_3))dsdu - (\gamma_2\Theta^2 + \eta_1\eta_{3u} + \eta_{1u}\eta_3)du^2). \end{aligned} \tag{52}$$

Here, the coefficients of the first and second fundamental forms for the Type 1-PAF of the second class are

$$E = 1, \qquad F = 0, \qquad G = \eta_1^2 + \eta_3^2 = \Theta^2. \tag{53}$$

$$l = -\Theta, \qquad m = \frac{-\eta_2\Theta^2 + \eta_1\eta_{3s} - \eta_{1s}\eta_3}{\Theta}, \qquad n = \frac{\gamma_2\Theta^2 - \eta_1\eta_{3u} + \eta_{1u}\eta_3}{\Theta}. \tag{54}$$

Respectively, the Gauss and mean curvatures of the VFF surface for the Type 1-PAF for the second class, with the aid of equations (53), (54) are given by:

$$K_2 = \frac{\Theta^2(\gamma_2\Theta^2-\eta_1\eta_{3u}+\eta_{1u}\eta_3)-(\eta_2\Theta^2-\eta_1\eta_{3s}+\eta_{1s}\eta_3)^2}{\Theta^4}, \tag{55}$$

$$H_2 = \frac{(\eta_1\eta_{3u}-\gamma_2\Theta^2-\eta_{1u}\eta_3)+\Theta^4}{2\Theta^3}. \tag{56}$$

Respectively, the principal curvatures k_1 and k_2 of the Type 1-PAF for the second class associated with the VFF equation with the aid of equations (55), (56) are given by:

$$\begin{aligned} k_1 = {} & \frac{(\eta_1\eta_{3u}-\gamma_2\Theta^2-\eta_{1u}\eta_3)+\Theta^4}{2\Theta^3} \\ & +\sqrt{(\frac{(\eta_1\eta_{3u}-\gamma_2\Theta^2-\eta_{1u}\eta_3)+\Theta^4}{2\Theta^3})^2-\frac{\Theta^2(-\gamma_2\Theta^2+\eta_1\eta_{3u}-\eta_{1u}\eta_3)+(\eta_2\Theta^2-\eta_1\eta_{3s}+\eta_{1s}\eta_3)^2}{\Theta^4}}, \end{aligned} \tag{57}$$

$$\begin{aligned} k_2 = {} & \frac{(\eta_1\eta_{3u}-\gamma_2\Theta^2-\eta_{1u}\eta_3)+\Theta^4}{2\Theta^3} \\ & -\sqrt{(\frac{(\eta_1\eta_{3u}-\gamma_2\Theta^2-\eta_{1u}\eta_3)+\Theta^4}{2\Theta^3})^2-\frac{\Theta^2(-\gamma_2\Theta^2+\eta_1\eta_{3u}-\eta_{1u}\eta_3)+(\eta_2\Theta^2-\eta_1\eta_{3s}+\eta_{1s}\eta_3)^2}{\Theta^4}}. \end{aligned} \tag{58}$$

Let

$$\Upsilon_{2;i}(s,u) = h(s,u)+k_i^{-1}(s,u)\mathbb{N}_2(s,u), \quad i=1,2 \tag{59}$$

be the focal surfaces of the Type 1-PAF for the second class be generated by the VFF. From equation (57), (58) and (59), we obtain:

$$\Upsilon_{2;is} = \frac{\eta_1k_i^2\Theta^2+(\eta_{1s}+\eta_2\eta_3)k_i\Theta-\eta_1(k_{is}\Theta-k_i\Theta_s)}{k_i^2\Theta^2}T+\frac{\Theta}{k_i}J+\frac{\eta_3k_i^2\Theta^2+(\eta_{3s}+\eta_1\eta_2)k_i\Theta+\eta_3(k_{is}\Theta-k_i\Theta_s)}{k_i^2\Theta^2}Z, \tag{60}$$

$$\begin{aligned} \Upsilon_{2;iu} = {} & \frac{(\eta_{3s}-\eta_1\eta_2)k_i^2\Theta^2+(\eta_{1u}+\gamma_2\eta_3)k_i\Theta+\eta_1(k_ic_{2u}-k_{iu}\Theta)}{k_i^2\Theta^2}T+\frac{\Theta^2\eta_{2s}+\eta_{3s}\eta_1+\eta_3\eta_{1s}}{k_i\Theta}J \\ & +\frac{(\eta_{1s}-\eta_2\eta_3)k_i^2\Theta^2+(\gamma_2\eta_1-\eta_{3u})k_i\Theta-\eta3(k_{iu}\Theta-k_i\Theta_u)}{k_i^2\Theta^2}Z. \end{aligned} \tag{61}$$

The unit normal vector field of the focal surface generated by the VFF with respect to the Type 1-PAF for the second class in $\mathbb{E}^3$ is given by:

$$\tilde{\mathbb{N}}_2 = \frac{\lambda_4T+\lambda_5J+\lambda_6Z}{\sqrt{\lambda_4^2+\lambda_5^2+\lambda_6^2}}, \qquad \mathring{\lambda}_2=\sqrt{\lambda_4^2+\lambda_5^2+\lambda_6^2}\neq 0. \tag{62}$$

Here

$$\lambda_4=\mathring{b}_2\mathring{f}_2-\mathring{c}_2\mathring{e}_2, \qquad \lambda_5=\mathring{c}_2\mathring{d}_2-\mathring{a}_2\mathring{f}_2, \qquad \lambda_6=\mathring{a}_2\mathring{e}_2-\mathring{b}_2\mathring{d}_2, \tag{63}$$

$$
\begin{aligned}
\mathring{a}_2 &= \frac{\eta_1 k_i^2\Theta^2+(\eta_{1s}+\eta_2\eta_3)k_i\Theta-\eta_1(k_{is}\Theta-k_i\Theta_s)}{k_i^2\Theta^2},\\
\mathring{b}_2 &= \frac{\Theta}{k_i},\\
\mathring{c}_2 &= \frac{\eta_3 k_i^2\Theta^2+(\eta_{3s}+\eta_1\eta_2)k_i\Theta+\eta_3(k_{is}\Theta-k_i\Theta_s)}{k_i^2\Theta^2},\\
\mathring{d}_2 &= \frac{(\eta_{3s}-\eta_1\eta_2)k_i^2\Theta^2+(\eta_{1u}+\gamma_2\eta_3)k_i\Theta-\eta_1(k_{iu}\Theta-k_i\Theta_u}{k_i^2\Theta^2},\\
\mathring{e}_2 &= \frac{\Theta^2\eta_{2s}+\eta_{3s}+\eta_3\eta_{1s}}{k_i\Theta},\\
\mathring{f}_2 &= \frac{(\eta_{1s-}\eta_2\eta_3)k_i^2\Theta^2+(-\eta_{3u}+\gamma_2\eta_1)k_i\Theta+\eta 3(k_i\Theta_u-k_{iu}\Theta)}{k_i^2\Theta^2}.
\end{aligned} \tag{64}
$$

The first fundamental form of the focal surface generated by the VFF with respect to the Type 1-PAF for the second class in $\mathbb{E}^3$ is given by:

$$
\begin{aligned}
\tilde{E} &= \langle \Upsilon_{2;is}, \Upsilon_{2;is}\rangle = \mathring{a}_2^2+\mathring{b}_2^2+\mathring{c}_2^2,\\
\tilde{F} &= \langle \Upsilon_{2;is}, \Upsilon_{2;iu}\rangle = \mathring{a}_2\mathring{d}_2+\mathring{b}_2\mathring{e}_2+\mathring{c}_2\mathring{f}_2,\\
\tilde{G} &= \langle \Upsilon_{2;iu}, \Upsilon_{2;iu}\rangle = \mathring{d}_2^2+\mathring{e}_2^2+\mathring{f}_2^2.
\end{aligned} \tag{65}
$$

The second fundamental form of the focal surface generated by the VFF with respect to the Type 1-PAF for the second class in $\mathbb{E}^3$ is given by:

$$
\begin{aligned}
\tilde{l} &= <\Upsilon_{2;iss}, \tilde{\mathbb{N}}_2> = \frac{\mu_4\lambda_4+\mu_5\lambda_5+\mu_6\lambda_6}{\mathring{\lambda}_2},\\
\tilde{m} &= <\Upsilon_{2;isu}, \tilde{\mathbb{N}}_2> = \frac{\varpi_4\lambda_4+\varpi_5\lambda_5+\varpi_6\lambda_6}{\mathring{\lambda}_2},\\
\tilde{n} &= <\Upsilon_{2;iuu}, \tilde{\mathbb{N}}_2> = \frac{\rho_4\lambda_4+\rho_5\lambda_5+\rho_6\lambda_6}{\mathring{\lambda}_2}.
\end{aligned} \tag{66}
$$

Here

$$
\Upsilon_{2;iss} = (\mathring{a}_{2s}-\mathring{b}_2\eta_1-\mathring{c}_2\eta_2)T+(\mathring{a}_2\eta_1+\mathring{b}_{2s}-\mathring{c}_2\eta_3)J+(\mathring{a}_2\eta_2+\mathring{b}_2\eta_3+\mathring{c}_{2s})Z, \tag{67}
$$

$$
\begin{aligned}
\Upsilon_{2;iuu} = {} & (\mathring{d}_{2u}+\mathring{e}_2(\eta_{3s}-\eta_2\eta_1)-\mathring{f}_2\gamma_2)T+(\mathring{d}_2(\eta_{3s}+\eta_2\eta_1)+\mathring{e}_{2u}-\mathring{f}_2(\eta_{1s}+\eta_2\eta_3))J\\
& +(\mathring{d}_2\gamma_2+(\eta_{1s}+\eta_2\eta_3)\eta_2+\mathring{e}_2(\eta_{1s}+\eta_2\eta_3)+\mathring{f}_{2u})Z,
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
\Upsilon_{2;isu} = {} & (\mathring{a}_{2u}+\mathring{b}_2(\eta_{3s}+\eta_2\eta_1)-\mathring{c}_2\gamma_2)T+(\mathring{a}_2(\eta_{3s}+\eta_2\eta_1)+\mathring{b}_{2u}+\mathring{c}_2(\eta_{1s}+\eta_2\eta_3))J\\
& +(\mathring{a}_2\gamma_2+\mathring{b}_2(\eta_{1s}+\eta_2\eta_3)+\mathring{c}_{2u})Z.
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
\mu_4 &= \mathring{a}_{2s}-\mathring{b}_2\eta_1-\mathring{c}_2\eta_2,\\
\mu_5 &= \mathring{a}_2\eta_1+\mathring{b}_{2s}-\mathring{c}_2\eta_3,\\
\mu_6 &= \mathring{a}_2\eta_2+\mathring{b}_2\eta_3+\mathring{c}_{2s},\\
\varpi_4 &= \mathring{d}_{2u}+\mathring{e}_2(\eta_{3s}-\eta_2\eta_1)-\mathring{f}_2\gamma_2,\\
\varpi_5 &= \mathring{d}_2(\eta_{3s}+\eta_2\eta_1)+\mathring{e}_{2u}-\mathring{f}_2(\eta_{1s}+\eta_2\eta_3),\\
\varpi_6 &= \mathring{d}_2\gamma_2+(\eta_{1s}+\eta_2\eta_3)\eta_2+\mathring{e}_2(\eta_{1s}+\eta_2\eta_3)+\mathring{f}_{2u},\\
\rho_4 &= \mathring{a}_{2u}+\mathring{b}_2(\eta_{3s}+\eta_2\eta_1)-\mathring{c}_2\gamma_2,\\
\rho_5 &= \mathring{a}_2(\eta_{3s}+\eta_2\eta_1)+\mathring{b}_{2u}+\mathring{c}_2(\eta_{1s}+\eta_2\eta_3),\\
\rho_6 &= \mathring{a}_2\gamma_2+\mathring{b}_2(\eta_{1s}+\eta_2\eta_3)+\mathring{c}_{2u}.
\end{aligned} \tag{70}
$$

From equations (62), (65) and (66), the Gauss and mean curvatures of focal surfaces for the Type 1-PAF of the second class are given:

$$\tilde{K}_2 = \frac{(\mu_4\lambda_4+\mu_5\lambda_5+\mu_6\lambda_6)(\rho_4\lambda_4+\rho_5\lambda_5+\rho_6\lambda_6)-(\varpi_4\lambda_4+\varpi_5\lambda_5+\varpi_6\lambda_6)^2}{\mathring{\lambda}_2\mathring{\chi}_2(\mathring{a}_2\mathring{d}_2+\mathring{b}_2\mathring{e}_2+\mathring{c}_2\mathring{f}_2)-\mathring{\xi}_2^2)}, \tag{71}$$

$$\tilde{H}_2 = \frac{(\mathring{\chi}_2(\rho_4\lambda_4+\rho_5\lambda_5+\rho_6\lambda_6)+(\mathring{a}_2\mathring{d}_2+\mathring{b}_2\mathring{e}_2+\mathring{c}_2\mathring{f}_2)(\mu_4\lambda_4+\mu_5\lambda_5+\mu_6\lambda_6)-2\mathring{\xi}_2(\varpi_4\lambda_4+\varpi_5\lambda_5+\varpi_6\lambda_6)}{2\mathring{\lambda}_2\mathring{\chi}_2(\mathring{a}_2\mathring{d}_2+\mathring{b}_2\mathring{e}_2+\mathring{c}_2\mathring{f}_2)-\mathring{\xi}_2^2)}, \tag{72}$$

where

$$\mathring{\chi}_2 = \mathring{a}_2^2+\mathring{b}_2^2+\mathring{c}_2^2, \qquad \mathring{\xi}_2 = \mathring{e}_2^2+\mathring{d}_2^2+\mathring{f}_2^2. \tag{73}$$

2.3 Focal surfaces generated by VFF with the third class for Type 1-PAF in $\mathbb{E}^3$

The time evolution of the $\{T,J,Z\}$ frame for the third class according to the Type 1-PAF is given as follows:

$$\begin{aligned} T_u &= \gamma_3 J - j_2 Z, \\ J_u &= -\gamma_3 T - vZ, \\ Z_u &= j_2 T + vJ. \end{aligned} \tag{74}$$

Here γ_3, j_2 and v are differentiable functions for the Type 1-PAF. From equations (2) and (74) the followings are obtained:

$$\begin{aligned} T_{su} &= (\eta_2 j_2 - \eta_1\gamma_3)T + (\eta_{1u}+\eta_2 v)J + (\eta_{2u}-\eta_1 v)Z, \\ T_{us} &= -(\gamma_3\eta_1 + j_2\eta_2)T + (\gamma_{3s}+j_2\eta_3)J + (\gamma_3\eta_3 - j_{2s})Z, \\ Z_{su} &= (\eta_3\gamma_3 - \eta_{2u})T - (\eta_2\gamma_3+\eta_{3u})J + (\eta_3 v + \eta_2 j)Z, \\ Z_{us} &= (j_{2s} - v\eta_1)T + (j_2\eta_1 + v_s)J + (j_2\eta_2 + v\eta_3)Z. \end{aligned} \tag{75}$$

From $T_{us} = T_{su}, \quad Z_{us} = Z_{su}$, we obtain:

$$\begin{aligned} \eta_{1u} &= \gamma_{3s} - \eta_2 v + j_2\eta_3, \\ \eta_{2u} &= \eta_1 v + \gamma_3\eta_3 - j_{2s}, \\ \eta_{3u} &= -(\eta_2\gamma_3 + j_2\eta_1 + v_s). \end{aligned} \tag{76}$$

The velocity of the curve for the Type 1-PAF is given by:

$$h_u = f_3 T + d_3 J + q_3 Z. \tag{77}$$

Via equation (77) it is given by:

$$h_{us} = (f_{3s} - d_3\eta_1 - q_3\eta_2)T + (f_3\eta_1 + d_{3s} - q_3\eta_3)J + (f_3\eta_2 + q_{3s} + d_3\eta_3)Z. \tag{78}$$

If $h_s = Z$ is substituted in equation (3),

$$h_u = \eta_3 T - \eta_2 J. \tag{79}$$

From $h_{su} = h_{us}$, it can be derived:

$$\begin{aligned} f_{3s} &= d_3\eta_1 + q_3\eta_2 + j_2, \\ d_{3s} &= f_3\eta_1 - v - q_3\eta_3, \\ q_{3s} &= -(f_3\eta_2 + d_3\gamma_3). \end{aligned} \tag{80}$$

From equations (77) and (79)

$$f_3 = \eta_3, \qquad d_3 = -\eta_2, \qquad q_3 = 0. \tag{81}$$

If equation (81) is rewritten in equation (80), it is given by:

$$j_2 = \eta_{3s} + \eta_2\eta_1, \qquad v = \eta_{2s} + \eta_1\eta_3. \tag{82}$$

From equation (82)

$$j_{2s}=\eta_{3ss}+(\eta_2\eta_1)_s,\qquad v_s=\eta_{2ss}+(\eta_1\eta_3)_s. \tag{83}$$

If equations (82), and (83) are rewritten in equation (76), we obtain:

$$\begin{array}{rcl}\eta_{1u} &=& \gamma_{3s}-\eta_2(\eta_{2s}+\eta_1\eta_3)+\eta_3(\eta_{3s}+\eta_2\eta_1),\\ \eta_{2u} &=& \eta_1\gamma_3-(\eta_{3ss}+(\eta_2\eta_1)_s)-\eta_1(\eta_{2s}+\eta_1\eta_3),\\ \eta_{3u} &=& -(\eta_{2ss}+(\eta_1\eta_3)_s)-\eta_1(\eta_{3s}+\eta_2\eta_1)-\eta_2\gamma_3.\end{array} \tag{84}$$

If equations (82) is rewritten in equation (74), the time evolution of the frame $\{T,J,Z\}$ is derived:

$$\begin{array}{rcl}T_u &=& \gamma_3 J-(\eta_{3s}+\eta_2\eta_1)Z,\\ J_u &=& -\gamma_3 T-(\eta_{2s}+\eta_1\eta_3)Z,\\ Z_u &=& (\eta_{3s}+\eta_2\eta_1)T+(\eta_{2s}+\eta_1\eta_3)J.\end{array} \tag{85}$$

The unit normal field of VFF surface with respect to the Type 1-PAF for the third class associated is given by:

$$\mathbb{N}_3=\frac{\eta_2 T+\eta_3 J}{\ell},\qquad \ell=\sqrt{\eta_2^2+\eta_3^2}. \tag{86}$$

Respectively, the first and second fundamental forms with respect to the Type 1-PAF for the third class associated with the VFF are obtained:

$$\begin{array}{rcl}I &=& ds^2+\ell^2du^2,\\ II &=& \frac{1}{\ell}(\ell^2ds^2+2(\eta_{3s}\eta_2+\eta_1\ell^2+\eta_3\eta_{2s})dsdu+(\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u})du^2).\end{array} \tag{87}$$

Here, the coefficients of first and second fundamental forms for the Type 1-PAF with the third class are

$$E=1,\qquad F=0,\qquad G=\eta_2^2+\eta_3^2=\ell^2. \tag{88}$$

$$l=-\ell,\qquad m=\frac{\eta_{3s}\eta_2+\eta_1\ell^2+\eta_3\eta_{2s}}{\ell^2},\qquad n=\frac{\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u}}{\ell}. \tag{89}$$

Respectively, the Gauss and mean curvatures of the VFF surface for the Type 1-PAF of the third class with the aid of equations (88) and (89) are given by:

$$K_3 = \frac{-\ell^2(\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u})+(\eta_{3s}\eta_2+\eta_1\ell^2+\eta_3\eta_{2s})^2}{\ell^4}, \tag{90}$$

$$H_3 = -\frac{\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u}+\ell^4}{2\ell^3}. \tag{91}$$

Respectively, the principal curvatures k_1 and k_2 of the Type 1-PAF for the third class associated with VFF with the aid of equations (90) and (91) are given by:

$$\begin{array}{rcl}k_1 &=& -\frac{\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u}+\ell^4}{2\ell^3}\\ && +\sqrt{(-\frac{\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u}+\ell^4}{2\ell^3})^2-\frac{\ell^2(\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u})+(\eta_{3s}\eta_2+\eta_1\ell^2+\eta_3\eta_{2s})^2}{\ell^4}},\end{array} \tag{92}$$

$$\begin{array}{rcl}k_2 &=& -\frac{\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u}+\ell^4}{2\ell^3}\\ && -\sqrt{(-\frac{\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u}+\ell^4}{2\ell^3})^2-\frac{\ell^2(\gamma_3\ell^2+\eta_2\eta_{3u}-\eta_3\eta_{2u})+(\eta_{3s}\eta_2+\eta_1\ell^2+\eta_3\eta_{2s})^2}{\ell^4}}.\end{array} \tag{93}$$

Let

$$\Upsilon_{3;i}(s,u) = h(s,u)+k_i^{-1}(s,u)\mathbb{N}_3(s,u), \quad i=1,2 \tag{94}$$

be the focal surfaces of the Type 1-PAF for the third class be generated by VFF. From equation (92), (93) and (94), we obtain:

$$\Upsilon_{3;is} = \frac{(\eta_{2s}-\eta_1\eta_3)k_i\ell+\eta_2(k_{is}\ell+k_i\ell_s)-\eta_2k_i^2\ell^2}{k_i^2\ell^2}T+\frac{(\eta_{3s}+\eta_1\eta_2)k_i\ell+\eta_3(k_{is}\ell+k_i\ell_s)-\eta_3k_i^2\ell^2}{k_i^2\ell^2}J+\frac{\ell}{k_i}Z, \tag{95}$$

$$\begin{aligned}\Upsilon_{3;iu} = {} & \frac{(\eta_{3s}+\eta_1\eta_2)k_i^2\ell^2+(\eta_{2u}-\gamma_3\eta_3)k_i\ell+\eta_2(k_{iu}\ell+k_i\ell_u)}{k_i^2\ell^2}T+\frac{(\eta_{2s+}\eta_1\eta_3)k_i^2\ell^2+(\gamma_3\eta_2-\eta_{3u})k_i\ell+\eta 3(k_{iu}\ell+k_i\ell_u)}{k_i^2\ell^2}J\\ & -\frac{\ell^2\eta_1+\eta_{3s}\eta_2-\eta_3\eta_{2s}}{k_i\ell}Z.\end{aligned} \tag{96}$$

The unit normal vector field of the focal surface generated VFF with respect to the Type 1-PAF for the third class in $\mathbb{E}^3$ is given by:

$$\tilde{\mathbb{N}}_3 = \frac{\lambda_7T+\lambda_8J+\lambda_9Z}{\sqrt{\lambda_7^2+\lambda_8^2+\lambda_9^2}}, \qquad \mathring{\lambda}_3=\sqrt{\lambda_7^2+\lambda_8^2+\lambda_9^2}\neq 0, \tag{97}$$

Here

$$\lambda_7=\mathring{b}_3\mathring{d}_3-\mathring{c}_3\mathring{e}_3, \qquad \lambda_8=\mathring{c}_3\mathring{d}_3-\mathring{a}_3\mathring{f}_3, \qquad \lambda_9=\mathring{a}_3\mathring{e}_3-\mathring{b}_3\mathring{d}_3. \tag{98}$$

$$\begin{aligned}\mathring{a}_3 &= \frac{(\eta_{2s}-\eta_1\eta_3)k_i\ell+\eta_2(k_{is}\ell+k_i\ell_s)-\eta_2k_i^2\ell^2}{k_i^2\ell^2},\\ \mathring{b}_3 &= \frac{(\eta_{3s}+\eta_1\eta_2)k_i\ell+\eta_3(k_{is}\ell+k_i\ell_s)-\eta_3k_i^2\ell^2}{k_i^2\ell^2},\\ \mathring{c}_3 &= \frac{\ell}{k_i},\\ \mathring{d}_3 &= \frac{(\eta_{3s}+\eta_1\eta_2)k_i^2\ell^2+(\eta_{2u}-\gamma_3\eta_3)k_i\ell+\eta_2(k_{iu}\ell+k_i\ell_u)}{k_i^2\ell^2},\\ \mathring{e}_3 &= \frac{(\eta_{2s+}\eta_1\eta_3)k_i^2\ell^2+(-\eta_{3u}+\gamma_3\eta_2)k_i\ell+\eta 3(k_{iu}\ell+k_i\ell_u)}{k_i^2\ell^2},\\ \mathring{f}_3 &= \frac{-\ell\eta_1-\eta_{3s}\eta_2+\eta_3\eta_{2s}}{k_i\ell}.\end{aligned} \tag{99}$$

The first fundamental form of the focal surface generated by VFF with respect to the Type 1-PAF for the third class in $\mathbb{E}^3$ is given by:

$$\begin{aligned}\tilde{E} &= \langle\Upsilon_{3;is},\Upsilon_{3;is}\rangle = \mathring{a}_3^2+\mathring{b}_3^2+\mathring{c}_3^2,\\ \tilde{F} &= \langle\Upsilon_{3;is},\Upsilon_{3;iu}\rangle = \mathring{a}_3\mathring{d}_3+\mathring{b}_3\mathring{e}_3+\mathring{c}_3\mathring{f}_3,\\ \tilde{G} &= \langle\Upsilon_{3;iu},\Upsilon_{3;iu}\rangle = \mathring{d}_3^2+\mathring{e}_3^2+\mathring{f}_3^2.\end{aligned} \tag{100}$$

The second fundamental form of the focal surface generated by VFF with respect to the Type 1-PAF for the third class in $\mathbb{E}^3$ is given by:

$$\begin{aligned}\tilde{l} &= <\Upsilon_{3;iss},\tilde{\mathbb{N}}_3>=\frac{\mu_7\lambda_7+\mu_8\lambda_8+\mu_9\lambda_9}{\mathring{\lambda}_3},\\ \tilde{m} &= <\Upsilon_{3;iss},\tilde{\mathbb{N}}_3>=\frac{\varpi_7\lambda_7+\varpi_8\lambda_8+\varpi_9\lambda_9}{\mathring{\lambda}_3},\\ \tilde{n} &= <\Upsilon_{3;iuu},\tilde{\mathbb{N}}_3>=\frac{\rho_7\lambda_7+\rho_8\lambda_8+\rho_9\lambda_9}{\mathring{\lambda}_3}.\end{aligned} \tag{101}$$

Here

$$\Upsilon_{3;iss} = (\mathring{a}_{6s} - \mathring{b}_3\eta_1 - \mathring{c}_3\eta_2)T + (\mathring{a}_3\eta_1 + \mathring{b}_{3s} - \mathring{c}_3\eta_3)J + (\mathring{a}_3\eta_2 + \mathring{b}_3\eta_3 + \mathring{c}_{3s})Z, \tag{102}$$

$$\begin{aligned}\Upsilon_{3;iuu} = & (\mathring{d}_{3u} - \mathring{e}_3\gamma_3 + \mathring{f}_3(\eta_{3s} + \eta_1\eta_2))T + (\mathring{d}_3\gamma_3 + \mathring{e}_{3u} + \mathring{f}_3(\eta_{3s} + \eta_1\eta_2))J \\ & + (\mathring{d}_3(\eta_{3s} + \eta_1\eta_2) - (\eta_{2s} + \eta_1\eta_3)\mathring{e}_3 + \mathring{f}_{3u})Z,\end{aligned} \tag{103}$$

$$\begin{aligned}\Upsilon_{3;isu} = & (\mathring{a}_{3u} - \mathring{b}_3\gamma_3 + \mathring{c}_3(\eta_{3s} + \eta_1\eta_3))T + (\mathring{a}_3\gamma_3 + \mathring{b}_3(\eta_{2s} + \eta_3\eta_1) + \mathring{c}_3(\eta_{2s} + \eta_1\eta_3))J \\ & + (-\mathring{a}_3(\eta_{3s} + \eta_1\eta_2) + \mathring{b}_3(\eta_{2s} + \eta_1\eta_3) + \mathring{c}_{3u})Z.\end{aligned} \tag{104}$$

$$\begin{aligned}
\mu_7 &= \mathring{a}_{6s} - \mathring{b}_3\eta_1 - \mathring{c}_3\eta_2, \\
\mu_8 &= \mathring{a}_3\eta_1 + \mathring{b}_{6s} - \mathring{c}_3\eta_3, \\
\mu_9 &= \mathring{a}_3\eta_2 + \mathring{b}_3\eta_3 + \mathring{c}_{3s}, \\
\varpi_7 &= \mathring{d}_{3u} - \mathring{e}_3\gamma_3 + \mathring{f}_3(\eta_{3s} + \eta_1\eta_2), \\
\varpi_8 &= \mathring{d}_3\gamma_3 + \mathring{e}_{3u} + \mathring{f}_3(\eta_{3s} + \eta_1\eta_2), \\
\varpi_9 &= \mathring{d}_3(\eta_{3s} + \eta_1\eta_2) - (\eta_{2s} + \eta_1\eta_3)\mathring{e}_3 + \mathring{f}_{3u}, \\
\rho_7 &= \mathring{a}_{3u} - \mathring{b}_3\gamma_3 + \mathring{c}_3(\eta_{3s} + \eta_1\eta_3), \\
\rho_8 &= \mathring{a}_3\gamma_3 + \mathring{b}_3(\eta_{2s} + \eta_3\eta_1) + \mathring{c}_3(\eta_{2s} + \eta_1\eta_3), \\
\rho_9 &= \mathring{c}_{3u} - \mathring{a}_3(\eta_{3s} + \eta_1\eta_2) + \mathring{b}_3(\eta_{2s} + \eta_1\eta_3).
\end{aligned} \tag{105}$$

From equations (97), (100) and (101), the Gauss and mean curvatures of focal surfaces for the Type 1-PAF for the third class are given by:

$$\tilde{K}_3 = \frac{(\mu_7\lambda_7+\mu_8\lambda_8+\mu_9\lambda_9)(\rho_7\lambda_7+\rho_8\lambda_8+\rho_9\lambda_9) - (\varpi_7\lambda_7+\varpi_8\lambda_8+\varpi_9\lambda_9)^2}{\mathring{\lambda}_3\mathring{\chi}_3(\mathring{a}_3\mathring{d}_3+\mathring{b}_3\mathring{e}_3+\mathring{c}_3\mathring{f}_3) - \mathring{\xi}_3^2)}, \tag{106}$$

$$\tilde{H}_3 = \frac{\mathring{\chi}_3(\rho_7\lambda_7 + \rho_8\lambda_8 + \rho_9\lambda_9) + (\mathring{a}_3\mathring{d}_3 + \mathring{b}_3\mathring{e}_3 + \mathring{c}_3\mathring{f}_3)(\mu_7\lambda_7 + \mu_8\lambda_8 + \mu_9\lambda_9) - 2\mathring{\xi}_3(\varpi_7\lambda_7 + \varpi_8\lambda_8 + \varpi_9\lambda_9)}{2\mathring{\lambda}_3(\mathring{\chi}_3(\mathring{a}_3\mathring{d}_3+\mathring{b}_3\mathring{e}_3+\mathring{c}_3\mathring{f}_3)-\mathring{\xi}_3^2)}. \tag{107}$$

where

$$\mathring{\chi}_3 = \mathring{a}_3^2 + \mathring{b}_3^2 + \mathring{c}_3^2, \qquad \mathring{\xi}_3 = \mathring{e}_3^2 + \mathring{d}_3^2 + \mathring{f}_3^2. \tag{108}$$

Example 1. *Consider the VFF surface with the following parametrization:*

$$h(s,u) = ((s) - 2\tanh(s), -2\sec h(s)\cos(u), 2\sec h(s)\sin(u)).$$

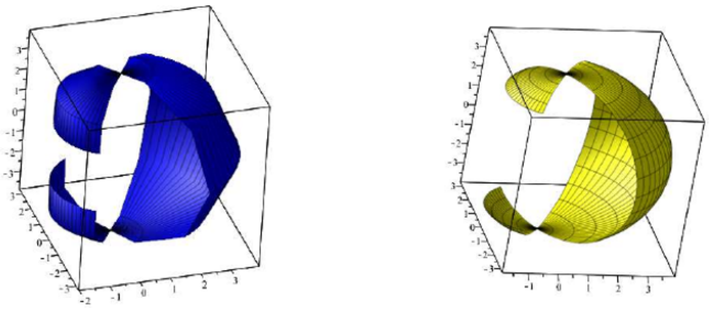
Here, the curvature and torsion are given by: $\eta(s) = 2\sec h(s)$ *and* $\tau(s) = 0$.

3. Conclusion

In this paper, we studied time evolutions of the Type 1-Position Adapted Frame (Type 1-PAF) connected with the Vortex Filament Flow (VFF) and constructed focal surfaces in three distinct classes within Euclidean 3-space. Beginning with the first class, we derived the time evolution equations of Type 1-PAF and obtained the unit normal vector field of VFF surface. This allowed us to study the first and second fundamental forms, from which we explicitly obtained the Gaussian curvature, mean curvature, and principal curvatures of the VFF surface.

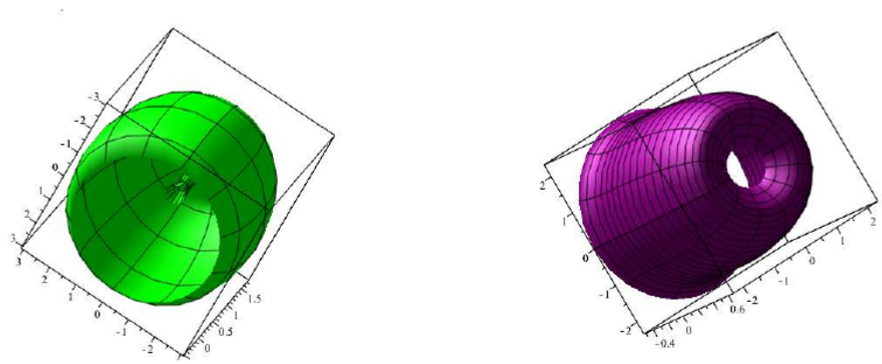
Later, we constructed focal surfaces generated by the VFF surface with respect to the first class of the Type 1-PAF. These focal surfaces were defined by means of the inverse of the principal curvatures. Later, we computed the first and second fundamental forms, the Gaussian curvature, mean curvature, and principal curvatures of the surface of the first focal surface for the Type 1-PAF in Euclidean 3-space. Then, we extended this methodology to the second and third classes, applying the same geometric approach to obtain the respective focal surfaces under Type 1-PAF.

To verify and visualize the geometric behavior of the surfaces, we presented examples via Maple 18.



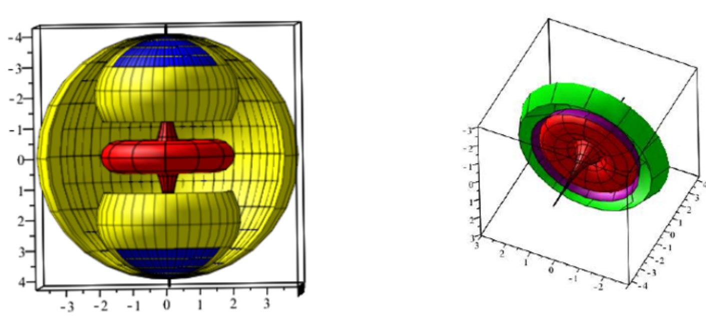
(a) (b)

Fig. 1. (a) 1. Focal surface generated by VFF for the first class (b) 2. Focal surface generated by VFF for the first class



(a) (b)

Fig. 2. (a) 1. Focal surface generated by VFF for the third class (b) 2. Focal surface generated by VFF for the third class



(a) (b)

Fig. 3. (a) Focal surfaces generated by VFF surface (red) for the first class (b) Focal surfaces generated by VFF surface (red) for the third class [18]

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