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Research Article

The New Hahn Sequence Space via (p,q)-Calculus

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Abstract
In this paper, a novel generalized Hahn sequence space, denoted as $h(C(p,q))$, is introduced by utilizing the (p,q)-Cesàro matrix. Fundamental properties of this sequence space, such as its completeness and inclusion relations with other well-known sequence spaces, are explored. The duals of this newly constructed sequence space are also determined, providing insights into its structural and functional characteristics. Furthermore, matrix mapping classes of the form $(h(C(p,q)):\mu)$ are characterized for various classical sequence spaces $\mu \in \{c_0, c, \ell_\infty, \ell_1, h\}$, extending the applicability of the proposed space to broader mathematical contexts. The results obtained contribute to the ongoing development of sequence space theory and its applications in functional analysis.

Keywords: Duals, Hahn sequence space, Matrix mappings, (p,q)-calculus, (p,q)-Cesàro matrix
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1. Introduction

The set containing all sequences of real or complex numbers is symbolized by ω. Each linear subspace of ω is referred to as a sequence space. Any complete metric sequence space Θ with continuous coordinates $f_s:\Theta \longrightarrow \mathbb{C}$, described by $f_s(u)=u_s$, is named as an FK-space for all $u=(u_s)\in\Theta$ and $s\in\mathbb{N}$, where $\mathbb{C}$ represents the complex field and $\mathbb{N}=\{0,1,2,3,\dots\}$. Furthermore, a BK-space is a normed FK-space. Some prominent examples of sequence spaces are c (the space of convergent sequences), c_0 (the space of null sequences), ℓ_∞ (the space of bounded sequences), and ℓ_p (the space of p-summable sequences).

The aforementioned spaces are BK-spaces due to the norms $\|u\|_{\ell_\infty}=\|u\|_c=\|u\|_{c_0}=\sup_{s\in\mathbb{N}}|u_s|$ and $\|u\|_{\ell_p}=\left(\sum_{s=0}^{\infty}|u_s|^p\right)^{1/p}$ for $1\le p<\infty$.

Consider $\mathscr{D}=(d_{rs})_{\mathbb{N}\times\mathbb{N}}$ as an infinite matrix with real or complex elements. It will be denoted by $\mathscr{D}_r=(d_{rs})_{s=0}^{\infty}$ the sequence in the r^{th} row of $\mathscr{D}$ for every $r\in\mathbb{N}$. The $\mathscr{D}$-transform of a sequence $u=(u_s)\in\omega$, denoted by $(\mathscr{D}u)_r$, is described as $\sum_{s=0}^{\infty}d_{rs}u_s$, assuming that the series converges for every $r\in\mathbb{N}$.

Consider the sequence spaces Θ and Λ. A matrix $\mathscr{D}$ is called as a matrix mapping from Θ to Λ, if for all $u\in\Theta$, the image $\mathscr{D}u$ belongs to Λ. The class of all such matrices that defines a mapping from Θ to Λ is denoted by $(\Theta:\Lambda)$. Additionally, the notation $\Theta_{\mathscr{D}}$ is employed to represent the set of all sequences for which the $\mathscr{D}$-transform is contained in Θ, as expressed by:

$$\Theta_{\mathscr{D}}=\{u\in\omega:\mathscr{D}u\in\Theta\}.$$

In this case, $\Theta_{\mathscr{D}} \subset \omega$, too and $\Theta_{\mathscr{D}}$ is named as matrix domain of $\mathscr{D}$.

If $\mathscr{D}u \in c$ for every $u \in c$, the matrix $\mathscr{D}$ is known as conservative matrix. Moreover, the conservative matrix $\mathscr{D}$ that preserves the limit is known as regular.

In the presence of a linear bijection which preserves the norm between Θ and Λ, these spaces are linearly isomorphic spaces and this situation is denoted by $\Theta \cong \Lambda$.

When $u = (u_s) \in \Lambda$, if $v = (v_s) \in \Lambda$ for all vectors v that satisfy the condition $|v_s| \leq |u_s|$ for $s \in \mathbb{N}$, in that case the set $\Lambda \in \omega$ is said to be normal.

Consider that the sequence e^s whose s^{th} term is 1 and remaining terms are 0 and $e = (1,1,1,...)$. For an FK-space Λ, it can be given the following definitions:

1. [1] Λ is a wedge space if $e^s \to 0$ in Λ,
2. [2] Λ is a conservative space if $c \subset \Lambda$,
3. [2] Λ is a semi-conservative space if $\Lambda^{\mathscr{G}} \subset cs$ (equivalently $c_0 \subset \Lambda$) for $\Lambda^{\mathscr{G}} = \{(\mathscr{G}(e^s)) : \mathscr{G} \in \Lambda'\}$, where Λ' denotes the continuous dual of Λ.

Let the acronym ψ represents the set of sequences whose terms are all zero except for a finite number of them. For an FK-space $\Lambda \supset \psi$, the s^{th} section of $u \in \Lambda$ is denoted by $u^{[s]} = \sum_{s=1}^{r} u_s e^s$. If $u^{[s]} \to u$ $(s \to \infty)$ for all $u \in \Lambda$, it is said that the FK-space $\Lambda \supset \psi$ has AK. Moreover, if ψ is dense in Λ, in that case it is said that Λ has AD. It should be noted that if Λ has AK, then Λ has AD.

Studies examining new spaces obtained by the aid of special matrices and necessary basic concepts about sequence spaces can be found in studies [3, 4, 5, 6, 7, 8, 9, 10, 11].

It is known from [12], $[s]_{p,q}$, the (p,q)-integer number s is described as

$$[s]_{p,q} = \begin{cases} \frac{p^s - q^s}{p-q}, & s = 1,2,3,..., \\ 0, & s = 0, \end{cases}$$

for each $s \in \mathbb{N}$ and $0 < q < p \leq 1$.

Moreover, the q-integer number is described by

$$[s]_q = \frac{1-q^s}{1-q}, \ (s = 1,2,3,\ldots), \ \ q \neq 1.$$

Based on the above discussion, by choosing $p = 1$, $[s]_{p,q}$ is reduced to $[s]_q$, and it is understood that $\lim_{q\to 1^-} \lim_{p\to 1^-} [s]_{p,q} = s$. Extensive information about q- and (p,q)-calculus can be obtained from studies [12, 13, 14].

The (p,q)-Cesàro matrix $C(p,q) = \left(c_{rs}^{p,q}\right)$ is described as

$$c_{rs}^{p,q} = \begin{cases} \dfrac{p^{r-s} q^s}{[r+1]_{p,q}}, & (0 \leq s \leq r), \\ 0, & (s > r) \end{cases}$$

for $0 < q < p \leq 1$ [15].

Due to the triangularity of $C(p,q)$, its inverse $C(p,q)^{-1} = \left(\{c_{rs}^{p,q}\}^{-1}\right)$ is expressed uniquely in the form

$$\{c_{rs}^{p,q}\}^{-1} = \begin{cases} (-1)^{r-s} \frac{p^{r-s}[s+1]_{p,q}}{q^r}, & (r-1 \leq s \leq r), \\ 0, & \text{otherwise}. \end{cases}$$

The q-analogue of C_1 (the first order Cesàro mean) is denoted by $C(q)$, while the (p,q)-analogue is denoted by $C(p,q)$. When $p = 1$, it is obvious that $C(p,q)$ simplifies to $C(q)$, which then reduces further to C_1 as $q \to 1$. As a result, $C(p,q)$ is a generalization of the matrices $C(q)$ and C_1.

The space bv, described as the domain of the forward difference operator Δ on ℓ_1, is in the form

$$bv = \left\{ u = (u_s) \in \omega : \sum_{s=1}^{\infty} |u_s - u_{s+1}| < \infty \right\}.$$

Furthermore, bv is a BK space with the norm

$$\|u\|_{bv} = \sum_{s=1}^{\infty} |u_s - u_{s+1}| \qquad (\forall u = (u_s) \in bv).$$

The Hahn sequence space h described in [16] is expressed by

$$h = \left\{ u = (u_s) \in \omega : \sum_{s=1}^{\infty} s|u_s - u_{s+1}| < \infty \right\} \cap c_0$$

and it is a BK-space with

$$\|u\| = \sum_{s=1}^{\infty} s|u_s - u_{s+1}| + \sup_s |u_s| \text{ for all } u = (u_s) \in h.$$

Furthermore, Rao [17] obtained that h is a BK space with AK with

$$\|u\|_h = \sum_{s=1}^{\infty} s|u_s - u_{s+1}| \text{ for all } u = (u_s) \in h.$$

After that, Goes [18] described the generalized Hahn space h^d expressed by

$$h^d = \{u = (u_s) \in \omega : \sum_{s=1}^{\infty} |d_s||u_s - u_{s+1}| < \infty\} \cap c_0$$

for $d = (d_s) \in \omega$ and $d_s \neq 0$.

A more general form of the Hahn sequence space is presented in [19] by

$$h_d = \{u = (u_s) \in \omega : \sum_{s=1}^{\infty} d_s|u_s - u_{s+1}| < \infty\} \cap c_0$$

for an unbounded and monotonically increasing sequence $d = (d_s)$ of positive real numbers. Studies examining Hahn sequence spaces and the necessary basic concepts about this field can be found in studies [17, 18, 19, 20, 21, 22, 23, 24, 25].

In this study, primarily, a new BK-space is described as the domain of $C(p,q)$ in the Hahn sequence space h, as an application of (p,q)-calculus to sequence spaces. After that, in order to specify the position of the mentioned space between the others, inclusion relations are incorporated, some algebraic and topological properties are examined, and its duals are calculated. At the end, some matrix transformations are presented.

2. Hahn Sequence Space $h(C(p,q))$

This section focusses on constructing new Hahn sequence space $h(C(p,q))$, the relevant inclusion relations, some algebraic and topological properties of the aforementioned space and its basis.

The sequence $\nu = (\nu_r)$, which is the $C(p,q)$-transform of any sequence u, is expressed as

$$\nu_r = (C(p,q)u)_r = \sum_{s=0}^{r} \frac{p^{r-s}q^s}{[r+1]_{p,q}} u_s. \tag{2.1}$$

Now, we construct the new generalized Hahn sequence space $h(C(p,q))$ by using (p,q)-Cesàro matrix as follows

$$h(C(p,q)) = \left\{ u = (u_r) \in \omega : \sum_{r=1}^{\infty} r\,|\Delta(C(p,q)u)_r| < \infty \text{ and } \lim_{r\to\infty} (C(p,q)u)_r = 0 \right\}$$

where

$$\begin{aligned} \Delta(C(p,q)u)_r &= (C(p,q)u)_r - (C(p,q)u)_{r+1} \\ &= \sum_{s=0}^{r} \left(\frac{p^{r-s}q^s}{[r+1]_{p,q}} - \frac{p^{r+1-s}q^s}{[r+2]_{p,q}} \right) u_s - \frac{q^{r+1}}{[r+2]_{p,q}} u_{r+1} \quad (r \in \mathbb{N}). \end{aligned} \tag{2.2}$$

We see that $h(C(p,q)) = h_{C(p,q)}$. In other words, $h(C(p,q))$ is domain of $C(p,q)$ in h. It can be noted that, as $p \to 1$, the space $h(C(p,q))$ is reduced to the space $h(C^q)$ presented by Yaying et al. [23].

On the other hand, it is possible to rewrite equation (2.1) as

$$u_r = \sum_{s=r-1}^{r} (-1)^{r-s} \frac{p^{r-s}[s+1]_{p,q}}{q^r} v_s \tag{2.3}$$

assuming that terms of sequences with negative indexes are 0.

Theorem 2.1. *$h(C(p,q))$ is a BK-space with*

$$\|u\|_{h(C(p,q))} = \sum_{r=1}^{\infty} r\,|\Delta(C(p,q)u)_r| < \infty. \tag{2.4}$$

Proof. It seems reasonable to suggest that since the matrix $C(p,q)$ is triangular and h is BK-space with $||\cdot||_h$, according to Theorem 4.3.2 of [2, p.61], $h(C(p,q))$ is BK-space with (2.4). □

Theorem 2.2. *$h(C(p,q)) \cong h$.*

Proof. For all u in $h(C(p,q))$, describe the mapping $\tau : h(C(p,q)) \to h$ as $\tau u = C(p,q)u = v$. In this case, τ is linear and one-to-one. Assuming that $u = (u_s)$ is defined as in (2.3), then $v = (v_r)$ can be any sequence in h.

Given that $v \in h$, by taking into consideration (2.2) and (2.3), it is reached that

$$\begin{aligned}
\|u\|_{h(C(p,q))} &= \sum_{r=1}^{\infty} r\,|\Delta(C(p,q)u)_r| \\
&= \sum_{r=1}^{\infty} r \left| \sum_{s=0}^{r} \left(\frac{p^{r-s}q^s}{[r+1]_{p,q}} - \frac{p^{r+1-s}q^s}{[r+2]_{p,q}} \right) u_s - \frac{q^{r+1}}{[r+2]_{p,q}} u_{r+1} \right| \\
&= \sum_{r=1}^{\infty} r \left| \sum_{s=0}^{r} \left(\frac{p^{r-s}q^s}{[r+1]_{p,q}} - \frac{p^{r+1-s}q^s}{[r+2]_{p,q}} \right) \left(\sum_{j=s-1}^{s} (-1)^{s-j} \frac{p^{s-j}[j+1]_{p,q}}{q^s} v_j \right) \right. \\
&\quad \left. - \frac{q^{r+1}}{[r+2]_{p,q}} \left(\sum_{s=r}^{r+1} (-1)^{r+1-s} \frac{p^{r+1-s}[s+2]_{p,q}}{q^{r+1}} v_s \right) \right| \\
&= \sum_{r=1}^{\infty} r\,|\Delta v_r| = \|v\|_h < \infty.
\end{aligned}$$

Consequently, we understand that $u \in h(C(p,q))$ and τ is onto and preserves the norm. □

Theorem 2.3. *The following inclusion relations hold:*

1. $h \subset h(C(p,q))$
2. $h(C(p,q)) \subset \ell_1(C(p,q))$

Proof. 1. Let $0 < q < p \leq 1$. It is obvious that the inclusion $h \subset h(C(p,q))$ holds. Besides, let us consider the sequence $f = (f_s)_{s \in \mathbb{N}} = \left(\frac{q[s+1]_{p,q} - p[s]_{p,q}}{qp^{s+1}} \right)$. In that case,

$$\begin{aligned}
\lim_{s\to\infty} f_s &= \lim_{s\to\infty} \left(\frac{q[s+1]_{p,q} - p[s]_{p,q}}{qp^{s+1}} \right) \\
&= \lim_{s\to\infty} \left[\frac{1}{p-q} \left(1 - \left(\frac{q}{p} \right)^{s+1} \right) - \frac{1}{q} \left(1 - \left(\frac{q}{p} \right)^{s} \right) \right] \\
&= \frac{1}{p-q} - \frac{1}{q} = \frac{2q-p}{q(p-q)} \neq 0.
\end{aligned}$$

Thus f is not a sequence in h. On the other hand, $C(p,q)f = b = (b_s) = \left[\left(\frac{q}{p}\right)^s\right] \in h$. This follows from the following illustrations: We ensure that $\sum_s s|b_s - b_{s+1}| < \infty$, because $\left(\frac{q}{p}\right)^s \to 0$ for $s \to \infty$. We have

$$\begin{aligned}
\sum_s s|b_s - b_{s+1}| &= \left|\frac{q}{p} - \frac{q^2}{p^2}\right| + 2\left|\frac{q^2}{p^2} - \frac{q^3}{p^3}\right| + 3\left|\frac{q^2}{p^2} - \frac{q^3}{p^3}\right| + \cdots \\
&= \frac{q}{p^2}|p-q| + 2\frac{q^2}{p^3}|p-q| + 3\frac{q^3}{p^4}|p-q| + \cdots \\
&= \frac{q}{p^2}(p-q)\left(1 + 2\frac{q}{p} + 3\frac{q^2}{p^2} + \cdots\right) \\
&\leq \frac{q}{p^2}(p-q)\frac{1}{\left(1-\frac{q}{p}\right)^2} \\
&= \frac{q}{p-q} < \infty.
\end{aligned}$$

2. Consider the sequences $b_k = 2^k$ ($k \in \mathbb{N}$) and $\nu = (\nu_s)$ with

$$\nu_s = \begin{cases} 0, & s \neq 2^k, \\ \frac{1}{s}, & s = 2^k. \end{cases}$$

In that case, it is seen that the inclusion $h \subset \ell_1$ is strict. Consider that

$$u_s = \sum_{j=s-1}^{s} (-1)^{s-j}\frac{p^{s-j}[j+1]_{p,q}}{q^s}\nu_j$$

for each $s \in \mathbb{N}$. Since,

$$(C(p,q)u)_r = \sum_{s=0}^{r}\frac{p^{r-s}q^s}{[r+1]_{p,q}}u_s = \sum_{s=0}^{r}\frac{p^{r-s}q^s}{[r+1]_{p,q}}\sum_{j=s-1}^{s}(-1)^{s-j}\frac{p^{s-j}[j+1]_{p,q}}{q^s}\nu_j = \nu_r,$$

we obtain $C(p,q)u = \nu \in \ell_1 \backslash h$ and thus $\nu \in \ell_1(C(p,q)) \backslash h(C(p,q))$.

□

Theorem 2.4. *$h(C(p,q))$ has AK.*

Proof. Consider that $u = (u_r) \in h(C(p,q))$ with

$$(C(p,q)u)_r = \sum_{s=r}^{\infty}\left[(C(p,q)u)_s - (C(p,q)u)_{s+1}\right].$$

Then, it is reached that

$$r|(C(p,q)u)_r| \leq \sum_{s=r}^{\infty} s\left|(C(p,q)u)_s - (C(p,q)u)_{s+1}\right|$$

and consequently

$$\lim_{r\to\infty} r|(C(p,q)u)r| = 0. \tag{2.5}$$

By the relation (2.5), we obtain that

$$\|u - u^{[r]}\|_{C(p,q)} = r\left|(C(p,q)u)_{r+1}\right| + \sum_{s=r+1}^{\infty} s\left|(C(p,q)u)_s - (C(p,q)u)_{s+1}\right|$$

which tends to zero, as $r \to \infty$. □

Since every space that has AK also has AD, it can be given the next result:

Corollary 2.5. *$h(C(p,q))$ has AD.*

Theorem 2.6. *$h(C(p,q))$ is not normal.*

Proof. Let us take sequences $u=(u_r)=(1,-1,0,0,0,...)$ and $\nu=(\nu_r)=(1,1,0,0,0,...)$ such that $|u_r|\leq|\nu_r|$ for each positive integer r. Then, one can see that

$$\sum_{r=1}^{\infty} r|\Delta(C(p,q)u)_r| = \frac{q(p-q)^3}{p}\sum_{r=1}^{\infty}\frac{rp^rq^r}{(p^{r+1}-q^{r+1})(p^{r+2}-q^{r+2})}<\infty$$

that is, $u\in h(C(p,q))$ by D'Alembert's Ratio Test and

$$\sum_{r=1}^{\infty} r|\Delta(C(p,q)\nu)_r| = \sum_{r=1}^{\infty}\frac{rp^{r-1}2p^{r+2}-q^{r+2}-q^{r+1}p}{(p^{r+1}-q^{r+1})(p^{r+2}-q^{r+2})}=\infty.$$

Thus, it is obtained that $\nu\notin h(C(p,q))$. □

Theorem 2.7. *$h(C(p,q))$ is a wedge space.*

Proof. For $0<q<p\leq 1$, from the equation

$$\begin{aligned}
\|e^m-0\|_{C(p,q)} &= \sum_{r=1}^{\infty} r|\Delta(C(p,q)e^m)_r| \\
&= \frac{q^m(m-1)}{[m+1]_{p,q}}+\sum_{r=m}^{\infty} r|\Delta(C(p,q)e^m)_r| \\
&= \frac{q^m(m-1)}{[m+1]_{p,q}}+\sum_{r=0}^{\infty}(r+m)\left|\frac{p^rq^m}{[r+m+1]_{p,q}}-\frac{p^{r+1}q^m}{[r+m+2]_{p,q}}\right| \\
&= \frac{q^m(m-1)(p-q)}{p^{m+1}-q^{m+1}}+\sum_{r=0}^{\infty}(r+m)p^rq^m\left|\frac{[r+m+2]_{p,q}-p[r+m+1]_{p,q}}{[r+m+1]_{p,q}[r+m+2]_{p,q}}\right| \\
&= \frac{(m-1)(p-q)}{p\left(\frac{p}{q}\right)^m-q}+\sum_{r=0}^{\infty}(r+m)p^rq^m\left|\frac{q^{r+m+1}}{[r+m+1]_{p,q}[r+m+2]_{p,q}}\right| \\
&= \frac{(m-1)(p-q)}{p\left(\frac{p}{q}\right)^m-q}+\sum_{r=0}^{\infty}\frac{(r+m)p^rq^{r+2m+1}(p-q)^2}{(p^{r+m+1}-q^{r+m+1})(p^{r+m+2}-q^{r+m+2})},
\end{aligned}$$

we obtain that $e^m\to 0$ as $m\to\infty$ in $h(C(p,q))$, as desired. □

Theorem 2.8. *$h(C(p,q))$ isn't a conservative space.*

Proof. By choosing $u=e\in c$, we have

$$\lim_{r\to\infty}(C(p,q)u)_r=\lim_{r\to\infty}\frac{p^{r-s}q^s}{[r+1]_{p,q}}=\left(\frac{q}{p}\right)^s\frac{p-q}{p}\neq 0.$$

Consequently, $u\notin h(C(p,q))$. □

Theorem 2.9. *$h(C(p,q))$ isn't a semi-conservative space.*

Proof. Take the sequences $u=(u_r)=(1,1,0,0,0,...)\in c_0$ with the limit point 0. Then, one can see that

$$\sum_{r=1}^{\infty} r|\Delta(C(p,q)u)_r| = \sum_{r=1}^{\infty}\frac{rp^{r-1}2p^{r+2}-q^{r+2}-q^{r+1}p}{(p^{r+1}-q^{r+1})(p^{r+2}-q^{r+2})}=\infty$$

Thus, $u\notin h(C(p,q))$. □

A matrix domain $\Theta_{\mathscr{D}}$ has a basis iff Θ has a basis for a triangle $\mathscr{D}$ ([26]). It can be inferred that the Schauder basis of $h(C(p,q))$ is formed by the inverse image of the basis of h. This fact leads to the following outcomes:

Theorem 2.10. *Consider a sequence $b^{(s)} = \{b^{(s)}\}_{s\in\mathbb{N}}$ of the elements of the space $h(C(p,q))$ as*

$$b_r^{(s)} = \begin{cases} (-1)^{r-s}\dfrac{p^{r-s}[s+1]_{p,q}}{q^r} & , \quad s \le r \le s+1, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In this case, $\{b^{(s)}\}_{s\in\mathbb{N}}$ is a basis for $h(C(p,q))$, and any $u \in h(C(p,q))$ has a unique representation of the form

$$u = \sum_s \lambda_s b^{(s)}, \tag{2.6}$$

where $\lambda_s = (C(p,q)u)_s$ ($s \in \mathbb{N}$).

Proof. From the relation

$$C(p,q)b^{(s)} = e^s \in h, \tag{2.7}$$

we reach that $\{b^{(s)}\} \subset h(C(p,q))$. For $u \in h(C(p,q))$ and $n \in \mathbb{N}$, consider

$$u^{[n]} = \sum_s^n \lambda_s b^{(s)}. \tag{2.8}$$

In that case, it is obtained by applying $C(p,q)$ to (2.8) by the aid of (2.7) that

$$C(p,q)u^{[n]} = \sum_s^n \lambda_s C(p,q)b^{(s)} = \sum_s^n (C(p,q)u)_s e^s,$$

and

$$\left\{C(p,q)\left(u - u^{[n]}\right)\right\}_k = \begin{cases} 0, & 0 \le k \le n, \\ (C(p,q)u)_k, & k > n, \end{cases}$$

for all $k,n \in \mathbb{N}$. For an $\varepsilon > 0$, $\exists n_0 \in \mathbb{N} \ni$

$$|(C(p,q)u)_s| < \frac{\varepsilon}{2} \qquad (\forall n \ge n_0).$$

In that case,

$$\left\|u - u^{[n]}\right\|_{h(C(p,q))} = \sup_{r\ge n} |(C(p,q))_r| \le \sup_{r\ge n_0} |(C(p,q))_r| \le \frac{\varepsilon}{2} < \varepsilon$$

for all $n \ge n_0$, which proves that $u \in h(C(p,q))$ given by (2.6).

Consider another representation of u as $u = \sum_s \mu_s b^{(s)}$. From the continuity of the linear bijection τ described in the proof of the Theorem 2.2, it is obtained

$$(C(p,q)u)_r = \sum_s \mu\left[C(p,q)b^{(s)}\right]_r = \sum_k \mu e_r^{(s)} = \mu_r, (r \in \mathbb{N})$$

and this contradicts the situation $(C(p,q)u)_r = \lambda_r$. Hence, (2.6) is unique.

□

3. Dual Spaces

The aim of the current part is to ascertain duals of our novel sequence space. Hereafter, we refer to the set of all limited subsets of $\mathbb{N}$ as $\mathscr{N}$. Initially, let us provide a lemma which will be employed in the next results.

Lemma 3.1. *[17] The following claims are true:*

(i) $\mathscr{D}=(d_{rs})\in(h:\ell_1)$ *iff*

$$\sum_{r=1}^{\infty}|d_{rs}|<\infty,\ \ (s=1,2,...) \tag{3.1}$$

$$\sup_{s}\frac{1}{s}\sum_{r=1}^{\infty}\left|\sum_{j=1}^{s}d_{rj}\right|<\infty. \tag{3.2}$$

(ii) $\mathscr{D}=(d_{rs})\in(h:c)$ *iff*

$$\sup_{r,s}\frac{1}{s}\left|\sum_{j=1}^{s}d_{rj}\right|<\infty, \tag{3.3}$$

$$\lim_{r\to\infty}d_{rs}\ \textit{exists}\ (s=0,1,2,...). \tag{3.4}$$

(iii) $\mathscr{D}=(d_{rs})\in(h:c_0)$ *iff*

$$\lim_{r\to\infty}d_{rs}=0, \tag{3.5}$$

and (3.3) holds.

(iv) $\mathscr{D}=(d_{rs})\in(h:\ell_\infty)$ *iff (3.3) holds.*

(v) $\mathscr{D}=(d_{rs})\in(h:h)$ *iff* (3.5) *holds and*

$$\sum_{r=1}^{\infty}r\,|d_{rs}-d_{r+1,s}|<\infty,\ \ (s=1,2,...)$$

$$\sup_{s}\frac{1}{s}\sum_{r=1}^{\infty}r\left|\sum_{j=1}^{s}(d_{rj}-d_{r+1,j})\right|<\infty.$$

Theorem 3.2. *Define the sets Υ_1, Υ_2, Υ_3 and Υ_4, as follows:*

$$\Upsilon_1=\left\{u=(u_s)\in w:\sum_{r=1}^{\infty}\left|(-1)^{r-s}\frac{p^{r-s}[s+1]_{p,q}}{q^r}t_r\right|<\infty\right\},$$

$$\Upsilon_2=\left\{u=(u_s)\in w:\sup_{s}\frac{1}{s}\sum_{r=1}^{\infty}\left|\sum_{j=1}^{s}(-1)^{r-s}\frac{p^{r-s}[s+1]_{p,q}}{q^r}t_r\right|<\infty\right\},$$

$$\Upsilon_3=\left\{u=(u_s)\in w:\sup_{r,s}\frac{1}{s}\left|\sum_{i=1}^{s}\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right|<\infty\right\},$$

$$\Upsilon_4=\left\{u=(u_s)\in w:\exists(\eta_s)\in\omega\ni\lim_{s\to\infty}\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i=\eta_i\right\}$$

for all $i=1,2,\ldots$. Then the following statements holds:

1. $\{h(C(p,q)u))\}^{\alpha}=\Upsilon_1\cap\Upsilon_2,$
2. $\{h(C(p,q)u))\}^{\beta}=\Upsilon_3\cap\Upsilon_4,$
3. $\{h(C(p,q)u))\}^{\gamma}=\Upsilon_3.$

Proof. 1. Let us describe the matrix $G=(g_{rs})$ by aid of $t=(t_r)\in\omega$ by

$$g_{rs}=\begin{cases}(-1)^{r-s}\dfrac{p^{r-s}[s+1]_{p,q}}{q^r}t_r & , \quad (r-1\leqslant s\leqslant r)\\ 0 & , \quad (\text{otherwise})\end{cases}$$

for all $s,r\in\mathbb{N}$. By (2.3), it is reached that

$$t_ru_r=\sum_{s=r-1}^{r}(-1)^{r-s}\frac{p^{r-s}[s+1]_{p,q}}{q^r}t_rv_s=(Gv)_r,\quad (r\in\mathbb{N}). \tag{3.6}$$

It follows from (3.6), $tu=(t_ru_r)\in\ell_1$ whenever $u\in h(C(p,q))$ iff $Gv\in\ell_1$ whenever $v\in h$. Hence, by (3.1) and (3.2), it is concluded that $\{h(C(p,q))\}^{\alpha}=\Upsilon_1\cap\Upsilon_2$.

2. Let us define the matrix $T=(t_{si})$ using the sequence $t=(t_s)$ by

$$t_{si}=\begin{cases}\displaystyle\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i & , \quad (s\leqslant i\leqslant s+1),\\ 0 & , \quad (\text{otherwise}),\end{cases}$$

for all $r,s\in\mathbb{N}$. Assuming that $t=(t_s)\in\{h(C(p,q))\}^{\beta}$, the resulting sequence $tu=(t_su_s)\in cs$ converges for all $u=(u_s)\in\{h(C(p,q))\}$. To arrive at this conclusion, we examine the equality obtained by the r^{th} partial sum of the series $\sum_{s=0}^{r}t_su_s$ with (2.3)

$$\begin{aligned}\sum_{s=0}^{r}t_su_s&=\sum_{s=0}^{r}\left(\sum_{i=s-1}^{s}(-1)^{s-i}\frac{p^{s-i}[i+1]_{p,q}}{q^s}v_i\right)t_s\\ &=\sum_{s=0}^{r}\left(\frac{[s+1]_{p,q}}{q^s}v_s-\frac{p[s]_{p,q}}{q^s}v_{s-1}\right)t_s\\ &=\sum_{s=0}^{r-1}\left(\frac{[s+1]_{p,q}}{q^s}t_s-\frac{p[s+1]_{p,q}}{q^{s+1}}t_{s+1}\right)v_s+\frac{[r+1]_{p,q}}{q^r}t_rv_r\\ &=\sum_{s=0}^{r-1}\left(\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right)v_s+\frac{[r+1]_{p,q}}{q^r}t_rv_r\end{aligned} \tag{3.7}$$

for any $r\in\mathbb{N}$. Recognizing that $h(C(p,q))\cong h$, we consider the limit that r approaches infinity in (3.7). Given that the series $\sum_{s=0}^{r}t_su_s$ is convergent, the series

$$\sum_{s=0}^{r-1}\left(\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right)v_s$$

is also convergent and the term $\frac{[r+1]_{p,q}}{q^r}t_rv_r$ in the right side of (3.7) must tend to zero, as $r\to\infty$. Since $h\subset c_0$ this is achieved with $\frac{[r+1]_{p,q}}{q^r}t_rv_r\in\ell_\infty$, we therefore have

$$\sum_{s=0}^{\infty}t_su_s=\sum_{s=0}^{\infty}\left(\sum_{i=s}^{s+1}\frac{(-1)^{i-s}p^{i-s}[s+1]_{p,q}}{q^i}t_i\right)v_s=(Tv)_s \tag{3.8}$$

for any $s\in\mathbb{N}$. Hence, $T=(t_{si})\in(h:c)$. Thus, the conditions in (3.3) and (3.4) conditions are satisfied by the matrix T. Hence, $t=(t_s)\in\Upsilon_3\cap\Upsilon_4$.

Conversely, suppose that $t=(t_s)\in\Upsilon_3\cap\Upsilon_4$. Then, we again obtain the relation (3.8) by using (3.7). Therefore, since we have $T=(t_{si})\in(h:c)$ the series $\sum_{s=0}^{\infty}t_su_s$ is convergent for all $u=(u_s)\in h(C(p,q))$. Hence, $t=(t_s)\in\{h(C(p,q))\}^{\beta}$, that is, the conditions are sufficient.

3. We see from (3.3) that tu is an element of bs whenever u in $h(C(p,q))$ iff Tv is an element of ℓ_∞ for v in h. As a consequence, by (3.3), it is deduced that $\{h(C(p,q))\}^{\gamma}=\Upsilon_3$.

□

4. Matrix Mappings

Here, we provide some matrix mapping classes from $h(C(p,q))$ to $\mu \in \{c_0, c, \ell_\infty, \ell_1, h\}$. Define the infinite matrix $\mathscr{A}$ whose $(r,s)^{th}$ term a_{rs} is given by

$$a_{rs} = \sum_{i=s}^{s+1} \frac{(-1)^{i-s} p^{i-s} [s+1]_{p,q}}{q^i} d_{ri}$$

for all $r, s \in \mathbb{N}$.

Theorem 4.1. *$\mathscr{D} = (d_{rs}) \in (h(C(p,q)) : \mu)$ iff*

$$\mathscr{A} \in (h : \mu) \tag{4.1}$$

$$\left(\frac{[m+1]_{p,q}}{q^m} d_{rm} \right)_{m\in\mathbb{N}} \in \mu \tag{4.2}$$

for all $r, m \in \mathbb{N}$

Proof. Let $\mathscr{D} \in (h(C(p,q)) : \mu)$. Then, $\mathscr{D}u$ exists for all $u = (u_s) \in h(C(p,q))$, and belongs to the space μ. Thus, $\mathscr{D}_m \in \{h(C(p,q))\}^\beta$ which confirms the necessity of the conditions in (4.1) and (4.2).

Conversely, assume that the conditions in (4.1) and (4.2) hold. Let $u = (u_s) \in h(C(p,q))$. Then, $\mathscr{D}_m \in \{h(C(p,q))\}^\beta$ for each $m \in \mathbb{N}$, and $\mathscr{D}u$ exists. Therefore, we obtain the equality shown below:

$$\begin{aligned}\sum_{s=1}^{m} d_{rs} u_s &= \sum_{s=1}^{m} d_{rs} \left(\sum_{i=s-1}^{s} (-1)^{s-i} \frac{p^{s-i}[i+1]_{p,q}}{q^s} v_i \right) \\ &= \sum_{s=1}^{m-1} \left(\sum_{i=s}^{s+1} \frac{(-1)^{i-s} p^{i-s} [s+1]_{p,q}}{q^i} d_{ri} \right) v_s + \frac{[m+1]_{p,q}}{q^m} d_{rm} v_m \end{aligned} \tag{4.3}$$

for every $r, m \in \mathbb{N}$. In the light of the condition in (4.2) and passing to limits as $m \to \infty$ in (4.3), we deduce the following equality

$$\sum_{s=1}^{\infty} d_{rs} u_s = \sum_{s=1}^{\infty} a_{rs} v_s$$

for all $r, s \in \mathbb{N}$, where the matrix $A = (a_{rs})$ is defined as in (4.1). Thus A maps h into μ. This implies that $Av = \mathscr{D}u \in \mu$ as required. □

Now, combining Lemma 3.1 and Theorem 4.1, the following result is obtained:

Corollary 4.2. *The following claims are true:*

(i) *$\mathscr{D} \in (h(C(p,q)) : c_0)$ iff*

$$\sup_{r,s} \frac{1}{s} \left| \sum_{j=1}^{s} a_{rj} \right| < \infty, \tag{4.4}$$

$$\lim_{r\to\infty} a_{rs} \text{ exists } (s \in \mathbb{N}). \tag{4.5}$$

hold, and

$$\lim_{r\to\infty} a_{rs} = 0 \text{ for all } s \in \mathbb{N} \tag{4.6}$$

also holds.

(ii) *$\mathscr{D} \in (h(C(p,q)) : c)$ iff (4.4) and (4.5) hold, and*

$$\sup_{r,s} \frac{1}{s} \left| \sum_{j=1}^{s} a_{rj} \right| < \infty, \tag{4.7}$$

$$\lim_{r\to\infty} d_{rs} \text{ exists } (s \in \mathbb{N}).$$

also hold.

(iii) *$\mathscr{D} \in (h(C(p,q)) : \ell_\infty)$ iff (4.4), (4.5) and (4.7) hold.*

(iv) *$\mathscr{D} \in (h(C(p,q)) : \ell_1)$ iff (4.4) and (4.5) hold, and*

$$\sum_{r=1}^{\infty} |a_{rs}| < \infty, \quad (s = 1,2,...)$$
$$\sup_s \frac{1}{s} \sum_{r=1}^{\infty} \left| \sum_{j=1}^{s} a_{rj} \right| < \infty.$$

(v) *$\mathscr{D} \in (h(C(p,q)) : h)$ iff (4.4), (4.5) and (4.6) hold, and*

$$\sum_{r=1}^{\infty} r\,|a_{rs} - a_{r+1,s}| < \infty, \quad (s = 1,2,...)$$
$$\sup_s \frac{1}{s} \sum_{r=1}^{\infty} r \left| \sum_{j=1}^{s} (a_{rj} - a_{r+1,j}) \right| < \infty.$$

5. Conclusion

As an application of matrix summability methods to Banach sequence spaces, in this research we presented a BK sequence space $h(C(p,q))$, which is the domain of the conservative (p,q)-Cesàro matrix $C(p,q)$ (the (p,q)-analogue of the first order Cesàro mean) on the Hahn sequence space. This work is an example of the broader application of (p,q)-calculus in the construction of Banach spaces.

As a future scope, we will study the normed and paranormed domains of the (p,q)-Cesàro matrix in some well-known spaces.

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