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On the *k*- Pell Quaternions and the *k*- Pell-Lucas Quaternions

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ABSTRACT: The quaternions form a four-dimensional associative and non-commutative algebra over the set of real numbers. In this paper, firstly, we give some relations for k - Pell quaternions and k - Pell-Lucas quaternions. Then, by using Binet's formula, we obtain their sums formulas, their the identities such as Cassini's identity and generating function, also derive relationships between these quaternions.

Keywords: k - Pell numbers, k - Pell-Lucas numbers, quaternions

k- Pell Kuaterniyonlar ve *k*- Pell-Lucas Kuaterniyonlar Üzerine

ÖZET: Kuaterniyonlar, reel sayılar kümesinde dört boyutlu birleşmeli ve değişmeli olmayan bir cebir oluştururlar. Bu makalede ilk olarak, k - Pell kuaterniyonlar ve k - Pell-Lucas kuaterniyonların bazı bağıntılarını verdik. Daha sonra, Binet formülünü kullanarak toplam formüllerini, Cassini özdeşliği ve geren fonksiyonu gibi özdeşliklerini elde ettik, ayrıca bu kuaterniyonların arasındaki ilişkileri türettik.

Anahtar Kelimeler: k - Pell sayıları, k - Pell-Lucas sayıları, kuaterniyonlar

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INTRODUCTION

In recent years, number sequences such as Fibonacci, Pell, Lucas etc. play an important role in many fields of science (Koshy, 2001). Topics in these sequences has attracted the attention of several researchers (see (Everest, 2005; Cerin and Gianella, 2006; Cerin and Gianella, 2007; Falcon and Plaza, 2007; Bolat and Köse, 2010; Falcon, 2011; Catarino, 2013; Catarino and Vasco, 2013; Catarino and Vasco, 2013)).

For $n \in N$ and $n \geq 2$, the Pell numbers $\{P_n\}$ are defined by the recursive recurrence

$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$,

the Pell-Lucas numbers $\{p_n\}$

$p_0 = 2, p_1 = 2, p_n = 2p_{n-1} + p_{n-2}$.

It is well known that the relationship between $\{P_n\}$ and $\{p_n\}$

$p_n = P_n + P_{n-1}$(Horadam, 1971).

k-Pell and k-Lucas numbers which are the generalizations of Pell and the Pell-Lucas numbers, their some properties which are studied Binet formulas, sum formulas, several identities and generating functions and many applications have been studies by some authors (Cerin and Gianella, 2006; Cerin and Gianella, 2007; Catarino, 2013; Catarino and Vasco, 2013; Catarino and Vasco, 2013; Vasco et al., 2015).

For any positive real number k and $n \geq 2$, k-Pell numbers, k- Pell-Lucas numbers and Modified k-Pell numbers are defined by

$$P_{k,0} = 0, P_{k,1} = 1, P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \quad (1)$$

$$p_{k,0} = 2, p_{k,1} = 2, p_{k,n+1} = 2p_{k,n} + kp_{k,n-1}, \quad (2)$$

$$q_{k,0} = 1, q_{k,1} = 1, q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}, \quad (3)$$

respectively. The relationship between these numbers is presented by

$q_{k,n} = P_{k,n} + kP_{k,n-1}$ and $p_{k,n} = 2(P_{k,n} + kP_{k,n-1})$ since $2q_{k,n} = p_{k,n}$ (Vasco et al., 2015).

We now summarize some properties given for the k-Pell and the k-Pell Lucas numbers in literature. For more details about these sequences, see (Catarino, 2013; Catarino and Vasco, 2013; Catarino and Vasco, 2013; Vasco et al., 2015).

The characteristic equation of these sequences is $r^2 - 2r - k = 0$. The roots of this equation are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. Also, There are the following identities:

$$r_1 + r_2 = 2, r_1 - r_2 = 2\sqrt{1+k}, r_1 r_2 = -k.$$

Binet formula for n-th the k-Pell, the k-Pell-Lucas numbers and Modified k-Pell numbers is $P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$, $p_{k,n} = r_1^n + r_2^n$ and $q_{k,n} = \frac{r_1^n + r_2^n}{2}$, respectively.

Catalan's identities for the k-Pell numbers and the k-Pell-Lucas numbers are

$$P_{k,n-r} P_{k,n+r} - P_{k,n}^2 = (-1)^{n-r+1} k^{n-r} P_{k,r}^2,$$

$$p_{k,n-r} p_{k,n+r} - p_{k,n}^2 = (-k)^{n-r} \left(p_{k,r}^2 - 4(-k)^r\right).$$

The generating functions for these numbers are

$$\left(P_{k,n}; x\right) = \frac{x}{1-2x-kx^2} \text{ and } \left(p_{k,n}; x\right) = \frac{2-2x}{1-2x-kx^2}.$$

There exists closely relationship between the Modified k-Pell and the k-Pell-Lucas numbers where $2q_{k,n} = p_{k,n}$. Thus, we only deal with the k-Pell and the k-Pell-Lucas numbers.

MATERIALS AND METHODS

The *k*-Pell quaternions and *k*-Pell-Lucas quaternions

The quaternions of the sequences firstly are introduced by Horadam (Horadam, 1993). These quaternions have been investigated by several authors. For example, some relations of the Fibonacci and Lucas quaternions have been defined in (Horadam, 1963; Iyer, 1969; Iyer, 1969; Swamy, 1973; Iakin, 1981; Horadam, 1993; Halici, 2012; Ramirez, 2015), and Pell quaternions and Pell-Lucas quaternions have been defined and obtained some properties about these quaternions in (Çimen and İpek, 2016; Szynal-Liana and Wloch, 2016). Paulo have introduced the Modified Pell, the Modified k-Pell quaternions and their octions (Catarino P, 2016). The researches working on quaternions of the sequences deal with Binet formulas, the generating functions and summation formulas for these quaternions. In this section, we introduce the k-Pell quaternions and k-Pell-Lucas quaternions and in section 3, we give some properties for these quaternions. Furthermore, we obtain some summation formulas for these quaternions and relationships between these quaternions.

n -th Pell quaternion and Pell-Lucas quaternion numbers are defined as follow:

$$QP_n = P_n + iP_{n+1} + jP_{n+2} + \Bbbk P_{n+3},$$

$$QPL_n = p_n + ip_{n+1} + jp_{n+2} + \Bbbk p_{n+3},$$

where P_n and p_n are n-th Pell and Pell-Lucas numbers (Çimen and İpek, 2016).

In here, there exist the basis $i, j, \Bbbk$ which satisfies the following:

$i^2 = j^2 = \Bbbk^2 = ij\Bbbk = -1$ and $ij = \Bbbk = -ji$, $\Bbbk i = j = -i\Bbbk$, $j\Bbbk = i = -\Bbbk j$.

A quaternion is a hyper-complex number and is defined by

$q = q_0 + iq_1 + jq_2 + \Bbbk q_3$.

q^* the conjugate of the quaternion q equal $q^* = q_0 - iq_1 - jq_2 - \Bbbk q_3$.

The conjugates of QP_n and QPL_n are defined by, respectively,

$$QP_n^* = P_n - iP_{n+1} - jP_{n+2} - \Bbbk P_{n+3},$$

$$QPL_n^* = p_n - ip_{n+1} - jp_{n+2} - \Bbbk p_{n+3}.$$

The norm of QP_n and QPL_n are defined by, respectively,

$$QP_nQP_n^* = P_n^2 + P_{n+1}^2 + P_{n+2}^2 + P_{n+3}^2,$$

$$QPL_nQPL_n^* = p_n^2 + p_{n+1}^2 + p_{n+2}^2 + p_{n+3}^2.$$

The k- Pell quaternion $QP_{k,n}$ is defined by

$$QP_{k,n} = P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + \Bbbk P_{k,n+3},$$

where $P_{k,n}$ is the are n-th k-Pell number for $n \geq 0$. The k- Pell-Lucas quaternion

$QPL_{k,n}$ is defined by

$$QPL_{k,n} = p_{k,n} + ip_{k,n+1} + jp_{k,n+2} + \Bbbk p_{k,n+3},$$

where $p_{k,n}$ is the are n-th k-Pell-Lucas number for $n \geq 0$.

RESULTS AND DISCUSSION

Some identities of the k-Pell quaternions and k-Pell-Lucas quaternions and relationships between these quaternions

In this section, we mainly focus on the k-Pell and k-Pell-Lucas quaternions to get some important results. We give some relations about these quaternions as in the following.

Proposition 1.

i. $QP_{k,n}^2 = 2P_{k,n}QP_{k,n} - QP_{k,n}QP_{k,n}^*$

ii. $QPL_{k,n}^2 = 2p_{k,n}QPL_{k,n} + QPL_{k,n}QPL_{k,n}^*$

iii. $QP_{k,n} + QP_{k,n}^* = 2P_{k,n}$

iv. $QPL_{k,n} + QPL_{k,n}^* = 2p_{k,n}$

v. $2QP_{k,n+1} + kQP_{k,n} = QP_{k,n+2}$

vi. $2QPL_{k,n+1} + kQPL_{k,n} = QPL_{k,n+2}$.

Proof.

i. $QP_{k,n}^2 = P_{k,n}^2 - P_{k,n+1}^2 - P_{k,n+2}^2 - P_{k,n+3}^2 + 2i\left(P_{k,n}P_{k,n+1}\right) + 2jP_{k,n}P_{k,n+2} + 2\Bbbk P_{k,n}P_{k,n+3}$

$= 2P_{k,n}\left(P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + \Bbbk P_{k,n+3}\right) - P_{k,n}^2 - P_{k,n+1}^2 - P_{k,n+2}^2 - P_{k,n+3}^2$

$= 2P_{k,n}QP_{k,n} - QP_{k,n}QP_{k,n}^*$

iii. $QP_{k,n} + QP_{k,n}^* = \left(P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + \Bbbk P_{k,n+3}\right) + \left(P_{k,n} - iP_{k,n+1} - jP_{k,n+2} - \Bbbk P_{k,n+3}\right) = 2P_{k,n}$

v. $2QP_{k,n+1} + kQP_{k,n} = 2\left(P_{k,n+1} + iP_{k,n+2} + jP_{k,n+3} + \Bbbk P_{k,n+4}\right) + k\left(P_{k,n} + iP_{k,n+1} + jP_{k,n+2} + \Bbbk P_{k,n+3}\right) = P_{k,n+2} + iP_{k,n+3} + jP_{k,n+4} + \Bbbk P_{k,n+5} = QP_{k,n+2}$.

The proofs of the others similarly have been done.

Proposition 2.

i. $QP_{k,n+1} + kQP_{k,n} = \frac{1}{2}QPL_{k,n+1}$

ii. $QP_{k,n+1} - QP_{k,n} = \frac{1}{2}QPL_{k,n}$

iii. $(2k+2)QP_{k,n}+QPL_{k,n}= QPL_{k,n+1}$.

Proof

i. Since $p_{k,n}=2(P_{k,n}+kP_{k,n-1})$, we get

$$QP_{k,n+1}+kQP_{k,n}=P_{k,n+1}+iP_{k,n+2}+jP_{k,n+3}+\text{ḱ}P_{k,n+4}+k\big(P_{k,n}+ iP_{k,n+1}+jP_{k,n+2}+\text{ḱ}P_{k,n+3}\big).$$

$$QP_{k,n+1}+kQP_{k,n}=\tfrac{1}{2}\big(p_{k,n+1}+ip_{k,n+2}+jp_{k,n+3}+\text{ḱ}p_{k,n+4}\big)=\tfrac{1}{2}QPL_{k,n+1}.$$

ii. Since $q_{k,n}=P_{k,n+1}-P_{k,n}$ (Horadam, 1971), we get

$$QP_{k,n+1}-QP_{k,n}=P_{k,n+1}+iP_{k,n+2}+jP_{k,n+3}+\text{ḱ}P_{k,n+4} -\big(P_{k,n}+iP_{k,n+1}+jP_{k,n+2}+\text{ḱ}P_{k,n+3}\big)=\tfrac{1}{2}QPL_{k,n}.$$

iii. From $2(k+1)P_{k,n}=p_{k,n}+kp_{k,n-1}$ (Vasco et al., 2015) and $p_{k,n+1}=2p_{k,n}+kp_{k,n-1}$, we have

$$2(k+1)QP_{k,n}+QPL_{k,n}=2(k+1)\big(P_{k,n}+iP_{k,n+1}+jP_{k,n+2}+\text{ḱ}P_{k,n+3}\big)+p_{k,n}+ ip_{k,n+1}+jp_{k,n+2}+\text{ḱ}p_{k,n+3}=2(k+1)P_{k,n}+p_{k,n}+i\left(2(k+1)P_{k,n+1}+ p_{k,n+1}\right)+j\left(2(k+1)P_{k,n+2}+p_{k,n+2}\right)+\text{ḱ}\left(2(k+1)P_{k,n+3}+p_{k,n+3}\right)= QPL_{k,n+1}.$$

In (Catarino, 2016), the author obtained some properties for the modified k-Pell quaternion. Now, as a different approximation we will prove the following results. We give the following theorem (Catarino, 2016) by adapting our using symbols.

Theorem 1. (Binet's formula)

Binet's formula for $QP_{k,n}$ and $QPL_{k,n}$, respectively, are as the following equations $QP_{k,n}=\frac{\hat{r}_1r_1^n-\hat{r}_2r_2^n}{r_1-r_2}$ and $QPL_{k,n}=\hat{r}_1r_1^n+\hat{r}_2r_2^n$, where $\hat{r}_1=1+ir_1+jr_1^2+\text{ḱ}r_1^3$ and $\hat{r}_2=1+ir_2+jr_2^2+\text{ḱ}r_2^3$.

Proof

The characteristic equation in (Catarino, 2013) is $r^2-2r-k=0$. Morever, the initial values are $QP_{k,0}=(0,1,2,k+4)$ and $QP_{k,1}=(1,2,k+4,4k+8)$. Thus, $QP_{k,n}=Ar_1^n+Br_2^n$. Then, we have $QP_{k,0}=A+B$ and $QP_{k,1}=Ar_1+Br_2$. We obtain that

$$A=\frac{QP_{k,1}-r_2QP_{k,0}}{r_1-r_2}=\frac{1+ir_1+jr_1^2+\mathfrak{k}r_1^3}{2\sqrt{1+k}}$$

$$B=-\frac{QP_{k,1}-r_1QP_{k,0}}{r_1-r_2}=-\frac{1+ir_2+jr_2^2+\mathfrak{k}r_2^3}{2\sqrt{1+k}}$$

$$QP_{k,n}=\frac{1}{2\sqrt{1+k}}\left((1+ir_1+jr_1^2+\mathfrak{k}r_1^3)r_1^n-(1+ir_2+jr_2^2+\mathfrak{k}r_2^3)r_2^n\right)$$

$$=\frac{(1+ir_1+jr_1^2+\mathfrak{k}r_1^3)r_1^n-(1+ir_2+jr_2^2+\mathfrak{k}r_2^3)r_2^n}{r_1-r_2}$$

$$=\frac{\hat{r}_1r_1^n-\hat{r}_2r_2^n}{r_1-r_2}.$$

Similarly, the initial values are $QPL_{k,0}=(2,2,2k+4,6k+8)$ and $QPL_{k,1}=(2,2k+4,6k+8,2k^2+16k+16)$ for k-Pell-Lucas quaternion. Then, we have $QPL_{k,0}=A+B$ and $QPL_{k,1}=Ar_1+Br_2$. We obtain that

$$A=\frac{QPL_{k,1}-r_2QPL_{k,0}}{r_1-r_2}=\frac{(r_1-r_2)(1+ir_1+jr_1^2+\mathfrak{k}r_1^3)}{r_1-r_2}=1+ir_1+jr_1^2+\mathfrak{k}r_1^3,$$

$$B=-\frac{QPL_{k,1}-r_1QPL_{k,0}}{r_1-r_2}=-\frac{(r_2-r_1)(1+ir_2+jr_2^2+\mathfrak{k}r_2^3)}{r_1-r_2}=1+ir_2+jr_2^2+\mathfrak{k}r_2^3$$

and $QPL_{k,n}=\hat{r}_1r_1^n+\hat{r}_2r_2^n$.

Theorem 2. (Catalan's identity) We have the following equations:

$$QP_{k,n-r}QP_{k,n+r}-QP_{k,n}^2=(-1)^{n-r+1}k^{n-r}\hat{r}_1\hat{r}_2\frac{(r_1^r-r_2^r)^2}{(r_1-r_2)^2}=(-1)^{n-r+1}k^{n-r}\hat{r}_1\hat{r}_2P_{k,r}^2,$$

$$QPL_{k,n-r}QPL_{k,n+r}-QPL_{k,n}^2=(-1)^{n-r+1}(4+4k)k^{n-r}\hat{r}_1\hat{r}_2P_{k,r}^2.$$

Proof: By using the Binet formula and $r_1r_2=-k$, we obtain

$$QP_{k,n-r}QP_{k,n+r}-QP_{k,n}^2=\left(\frac{\hat{r}_1r_1^{n-r}-\hat{r}_2r_2^{n-r}}{r_1-r_2}\right)\left(\frac{\hat{r}_1r_1^{n+r}-\hat{r}_2r_2^{n+r}}{r_1-r_2}\right)-\left(\frac{\hat{r}_1r_1^{n}-\hat{r}_2r_2^{n}}{r_1-r_2}\right)^2$$

$$=(-1)^{n-r+1}k^{n-r}\hat{r}_1\hat{r}_2\frac{(r_1^r-r_2^r)^2}{(r_1-r_2)^2}=(-1)^{n-r+1}k^{n-r}\hat{r}_1\hat{r}_2P_{k,r}^2.$$

Similarly, we get

$QPL_{k,n-r}QPL_{k,n+r} - QPL_{k,n}^2 = (\hat{r}_1 r_1^{n-r} + \hat{r}_2 r_2^{n-r})(\hat{r}_1 r_1^{n+r} + \hat{r}_2 r_2^{n+r}) - (\hat{r}_1 r_1^n + \hat{r}_2 r_2^n)^2 = (-1)^{n-r}(4+4k)k^{n-r}\hat{r}_1\hat{r}_2 P_{k,r}^2.$

Theorem 3. (Cassini's identity) For $n \geq 1$,

$$QP_{k,n-1}QP_{k,n+1} - QP_{k,n}^2 = (-1)^n k^{n-1}\hat{r}_1\hat{r}_2,$$

$$QPL_{k,n-1}QPL_{k,n+1} - QPL_{k,n}^2 = (-1)^{n-1}(4+4k)k^{n-1}\hat{r}_1\hat{r}_2.$$

For $r = 1$ in the above theorem and using the initial conditions of these sequences, we hold the Cassini's identity for the k- Pell quaternions and k- Pell-Lucas quaternions.

Theorem 4. If $m > n$, then we get

$$QP_{k,m}QP_{k,n+1} - QP_{k,m+1}QP_{k,n} = \hat{r}_1\hat{r}_2(-k)^n P_{k,m-n},$$

$$QPL_{k,m}QPL_{k,n+1} - QPL_{k,m+1}QPL_{k,n} = (-1)^{n+1}4\hat{r}_1\hat{r}_2 k^n(1+k)P_{k,m-n}.$$

Proof

$$QP_{k,m}QP_{k,n+1} - QP_{k,m+1}QP_{k,n} = \left(\frac{\hat{r}_1 r_1^m - \hat{r}_2 r_2^m}{r_1 - r_2}\right)\left(\frac{\hat{r}_1 r_1^{n+1} - \hat{r}_2 r_2^{n+1}}{r_1 - r_2}\right) - \left(\frac{\hat{r}_1 r_1^{m+1} - \hat{r}_2 r_2^{m+1}}{r_1 - r_2}\right)\left(\frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}\right)$$

$$= \hat{r}_1\hat{r}_2(r_1 r_2)^n (r_1 - r_2)\left(\frac{r_1^{m-n} - r_2^{m-n}}{(r_1 - r_2)^2}\right)$$

$$= \hat{r}_1\hat{r}_2(-k)^n\left(\frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2}\right)$$

$$= \hat{r}_1\hat{r}_2(-k)^n P_{k,m-n}.$$

Similarly, the identities $QPL_{k,m}QPL_{k,n+1} - QPL_{k,m+1}QPL_{k,n} = (-1)^{n+1}4\hat{r}_1\hat{r}_2 k^n(1+k)P_{k,m-n}$ are proved by using Theorem 1 and Theorem 2.

Theorem 5.

$$QPL_{k,n}^2 - QP_{k,n}^2 = (3+4k)QP_{k,n}^2 + 4\hat{r}_1\hat{r}_2(-k)^n.$$

Proof

$$QPL_{k,n}^2 - QP_{k,n}^2 = (\hat{r}_1 r_1^n + \hat{r}_2 r_2^n)^2 - \left(\frac{\hat{r}_1 r_1^n - \hat{r}_2 r_2^n}{r_1 - r_2}\right)^2$$

$$= \frac{(r_1 - r_2)^2(\hat{r}_1^2 r_1^{2n} + 2\hat{r}_1\hat{r}_2 r_1^n r_2^n + \hat{r}_2^2 r_2^{2n}) - \hat{r}_1^2 r_1^{2n} + 2\hat{r}_1\hat{r}_2 r_1^n r_2^n - \hat{r}_2^2 r_2^{2n}}{(r_1 - r_2)^2}$$

$$= (3+4k)QP_{k,n}^2 + 4\hat{r}_1\hat{r}_2(-k)^n.$$

Theorem 6. Suppose that the generating function for the k-Pell quaternion is $G(QP_{k,n};x)=\sum_{n=0}^{\infty} QP_{k,n}x^n$. Then, $G(QP_{k,n};x)=\frac{QP_{k,0}+QP_{k,1}x-2QP_{k,0}x}{1-2x-kx^2}$ (Catarino, 2016).

Corollary 1. Let be $G(QP_{k,n};x)$, $H(QPL_{k,n};x)$ the generating functions for the k-Pell and k-Pell-Lucas quaternions, respectively.

$$G(QP_{k,n};x)=\frac{x+i+(2+xk)j+(4+k+2kx)\mathfrak{k}}{1-2x-kx^2},$$

$$H(QPL_{k,n};x)=2\frac{1-x+(1+xk)i+(k+2+xk)j+(xk(k+2)+3k+4)\mathfrak{k}}{1-2x-kx^2}.$$

Proof: The generating function for $\{QP_{k,n}\}_{n=0}^{\infty}$ is

$$G(QP_{k,n};x)=QP_{k,0}+QP_{k,1}x+QP_{k,2}x^2+\cdots+QP_{k,m}x^m+\cdots.$$

Multiplying both side of equation with $2x$ and k, we obtain

$$2xG(QP_{k,n};x)=2QP_{k,0}x+2QP_{k,1}x^2+2QP_{k,2}x^3+\cdots+2QP_{k,m}x^{m+1}+\cdots,$$

$$kx^2G(QP_{k,n};x)=kQP_{k,0}x^2+kQP_{k,1}x^3+kQP_{k,2}x^4+\cdots+kQP_{k,m}x^{m+2}+\cdots.$$

And then, adding these equations, we get

$$(1-2x-kx^2)G(QP_{k,n};x)=QP_{k,0}+QP_{k,1}x-2QP_{k,0}x$$

$$G(QP_{k,n};x)=\frac{QP_{k,0}+QP_{k,1}x-2QP_{k,0}x}{1-2x-kx^2}$$

$$=\frac{x+i+(2+xk)j+(4+k+2kx)\mathfrak{k}}{1-2x-kx^2}.$$

Similarly, we obtain $H(QPL_{k,n};x)$ as follows;

$$H(QPL_{k,n};x)=QPL_{k,0}+QPL_{k,1}x+QPL_{k,2}x^2+\cdots+QPL_{k,m}x^m+\cdots$$

$$2xH(QPL_{k,n};x)=2QPL_{k,0}x+2QPL_{k,1}x^2+2QPL_{k,2}x^3+\cdots+2QPL_{k,m}x^{m+1}+\cdots$$

$$kx^2H(QPL_{k,n};x)=kQPL_{k,0}x^2+kQPL_{k,1}x^3+kQPL_{k,2}x^4+\cdots+kQPL_{k,m}x^{m+2}+\ldots$$

$$(1-2x-kx^2)H(QPL_{k,n};x)=QPL_{k,0}+QPL_{k,1}x-2QPL_{k,0}x$$

$$H(QPL_{k,n};x)=\frac{QPL_{k,0}+(QPL_{k,1}-2QPL_{k,0})x}{1-2x-kx^2}.$$

$$H(QPL_{k,n};x)=2\frac{1-x+(1+xk)i+(k+2+xk)j+(xk(k+2)+3k+4)\mathfrak{k}}{1-2x-kx^2}.$$

Theorem 7. For the k-Pell quaternions $QP_{k,n}$ and the k-Pell-Lucas quaternions $QPL_{k,n}$,

$$\sum_{i=0}^{n} QP_{k,mi+j}=\begin{cases}\frac{(-k)^m QP_{k,mn+j}-QP_{k,mn+m+j}+QP_{k,j}+(-k)^j\,QP_{k,m-j}}{(-k)^m-p_{k,m}+1}, & if\ j<m;\\ \frac{(-k)^m QP_{k,mn+j}-QP_{k,mn+m+j}+QP_{k,j}-(-k)^m\,QP_{k,j-m}}{(-k)^m-p_{k,m}+1}, & otherwise\end{cases}$$

$$\sum_{i=0}^{n} QPL_{k,mi+j}=\begin{cases}\frac{(-k)^m QPL_{k,mn+j}-QPL_{k,mn+m+j}+QPL_{k,j}-(-k)^j\,QPL_{k,m-j}}{(-k)^m-p_{k,m}+1}, & if\ j<m;\\ \frac{(-k)^m QPL_{k,mn+j}-QPL_{k,mn+m+j}+QPL_{k,j}-(-k)^m\,QPL_{k,j-m}}{(-k)^m-p_{k,m}+1}, & otherwise\end{cases}.$$

Proof.

$$\sum_{i=0}^{n} QP_{k,mi+j}=\sum_{i=0}^{n}\frac{\hat{r}_1 r_1^{mi+j}-\hat{r}_2 r_2^{mi+j}}{r_1-r_2}=\frac{1}{2\sqrt{1+k}}\left(\hat{r}_1 r_1^{j}\sum_{i=0}^{n} r_1^{mi}-\hat{r}_2 r_2^{j}\sum_{i=0}^{n} r_2^{mi}\right)$$

$$=\frac{1}{2\sqrt{1+k}}\left(\hat{r}_1 r_1^{j}\frac{r_1^{mn+m}-1}{r_1^{m}-1}-\hat{r}_2 r_2^{j}\frac{r_2^{mn+m}-1}{r_2^{m}-1}\right)$$

$$=\frac{1}{2\sqrt{1+k}}\frac{1}{(r_1r_2)^m-(r_1^m+r_2^m)+1}\Big(\hat{r}_1 r_1^{mn+m+j} r_2^{m}-\hat{r}_1 r_1^{mn+m+j}-\hat{r}_1 r_1^{j} r_2^{m}+$$

$$\hat{r}_1 r_1^{j}-\hat{r}_2 r_2^{mn+m+j} r_1^{m}+\hat{r}_2 r_2^{mn+m+j}+\hat{r}_2 r_2^{j} r_1^{m}-\hat{r}_2 r_2^{j}\Big)$$

$$=\frac{1}{2\sqrt{1+k}}\frac{1}{(-k)^m-p_{k,m}+1}\Big(\big(\hat{r}_1 r_1^{mn+m+j} r_2^{m}-\hat{r}_2 r_2^{mn+m+j} r_1^{m}\big)-$$

$$\big(\hat{r}_1 r_1^{mn+m+j}-\hat{r}_2 r_2^{mn+m+j}\big)-\big(\hat{r}_1 r_1^{j} r_2^{m}-\hat{r}_2 r_2^{j} r_1^{m}\big)+\big(\hat{r}_1 r_1^{j}-\hat{r}_2 r_2^{j}\big)\Big)$$

$$=\frac{(-k)^m QP_{k,mn+j}-QP_{k,mn+m+j}+QP_{k,j}}{(-k)^m-p_{k,m}+1}-\frac{\frac{\hat{r}_1 r_1^{j} r_2^{m}-\hat{r}_2 r_2^{j} r_1^{m}}{2\sqrt{1+k}}}{(-k)^m-p_{k,m}+1}.$$

In addition to $\frac{\hat{r}_1 r_1^{j} r_2^{m}-\hat{r}_2 r_2^{j} r_1^{m}}{2\sqrt{1+k}}=\begin{cases}-(-k)^j\,QP_{k,m-j}, & if\ j<m;\\ (-k)^m\,QP_{k,j-m}, & otherwise\end{cases}$,

Then,

$$\sum_{i=0}^{n} QP_{k,mi+j}=\begin{cases}\frac{(-k)^m QP_{k,mn+j}-QP_{k,mn+m+j}+QP_{k,j}+(-k)^j\,QP_{k,m-j}}{(-k)^m-p_{k,m}+1}, & if\ j<m;\\ \frac{(-k)^m QP_{k,mn+j}-QP_{k,mn+m+j}+QP_{k,j}-(-k)^m\,QP_{k,j-m}}{(-k)^m-p_{k,m}+1}, & otherwise.\end{cases}$$

The second part of the proof is found in a similar way.

Corollary 2. For the k-Pell quaternions $QP_{k,n}$,

$$\sum_{i=0}^{n} QP_{k,mi} = \frac{(-k)^m QP_{k,mn} - QP_{k,mn+m} + QP_{k,0} + QP_{k,m}}{(-k)^m - p_{k,m} + 1},$$

$$\sum_{i=0}^{n} QP_{k,i} = \frac{kQP_{k,n} + QP_{k,n+1} - QP_{k,0} - QP_{k,1}}{k+1}.$$

Moreover, for the k-Pell-Lucas quaternions $QPL_{k,n}$,

$$\sum_{i=0}^{n} QPL_{k,mi} = \frac{(-k)^m QPL_{k,mn} - QPL_{k,mn+m} + QPL_{k,0} - QPL_{k,m}}{(-k)^m - p_{k,m} + 1},$$

$$\sum_{i=0}^{n} QPL_{k,i} = \frac{kQPL_{k,n} + QPL_{k,n+1} - QPL_{k,0} + QPL_{k,1}}{k+1}.$$

Theorem 8. For $n \geq 0$, we obtain the following summation formula:

$$\sum_{i=0}^{n} QP_{k,i} = \frac{QPL_{k,n+1} - 2\left(QP_{k,0} + QP_{k,1}\right)}{p_{k,2} - p_{k,1}}.$$

Proof. From Corollary 2, we know that $\sum_{i=0}^{n} QP_{k,i} = \frac{kQP_{k,n}+QP_{k,n+1}-QP_{k,0}-QP_{k,1}}{k+1}$.

$$\sum_{i=0}^{n} QP_{k,i} = \frac{kQP_{k,n}+QP_{k,n+1}-QP_{k,0}-QP_{k,1}}{k+1} = \frac{\frac{1}{2}QPL_{k,n+1}-QP_{k,0}-QP_{k,1}}{\frac{1}{2}\left(p_{k,2}-p_{k,1}\right)}$$

$$= \frac{QPL_{k,n+1}-2\left(QP_{k,0}+QP_{k,1}\right)}{p_{k,2}-p_{k,1}}.$$

Theorem 9.

$$\sum_{r=0}^{\infty} QP_{k,r+s} = \frac{-1}{2(k+1)} QPL_{k,s},$$

$$\sum_{r=0}^{\infty} QPL_{k,r+s} = \frac{-1}{k+1}\left(QPL_{k,s} + kQPL_{k,s-1}\right).$$

Proof By using Binet's formula, we obtain the following equations:

$$\sum_{r=0}^{\infty} QP_{k,r+s} = \sum_{r=0}^{\infty} \frac{\hat{r}_1 r_1^{r+s} - \hat{r}_2 r_2^{r+s}}{r_1 - r_2}$$

$$= \frac{\hat{r}_1 r_1^s}{r_1 - r_2} \sum_{r=0}^{\infty} r_1^r - \frac{\hat{r}_2 r_2^s}{r_1 - r_2} \sum_{r=0}^{\infty} r_2^r$$

$$= \frac{\hat{r}_1 r_1^s}{r_1 - r_2} \frac{1}{1 - r_1} - \frac{\hat{r}_2 r_2^s}{r_1 - r_2} \frac{1}{1 - r_2}$$

$$= \frac{1}{2\sqrt{1+k}} \left(\frac{\hat{r}_1 r_1^s}{1 - r_1} - \frac{\hat{r}_2 r_2^s}{1 - r_2} \right)$$

$$= \frac{-1}{k+1} \left(QP_{k,s} + kQP_{k,s-1} \right)$$

$$= \frac{-1}{2(k+1)} QPL_{k,s}.$$

The second sum is found in a similar manner.

Theorem 10. Let be *G(x)* the generating function for the *k*-Pell quaternions and *H(x)* the generating function for the *k*-Pell-Lucas quaternions. Then, there exists in the following equation:

following equation:

$$2(k+1)G(x) + H(x) = \frac{2\left[1+kx+(kx+k+2)i+\left(k^2x+2kx+3k+4\right)j+\left(3k^2x+4kx+k^2+8k+8\right)k\right]}{1-2x-kx^2}.$$

Proof.

$$G\left(QP_{k,n}; x\right) = \frac{QP_{k,0} + QP_{k,1}x - 2QP_{k,0}x}{1 - 2x - kx^2}$$

$$H\left(QPL_{k,n}; x\right) = \frac{QPL_{k,0} + QPL_{k,1}x - 2QPL_{k,0}x}{1 - 2x - kx^2}$$

From Proposition 2 (iii) and Proposition 1 (vi), we get

$$2(k+1)G(x) + H(x) = 2(k+1)\frac{QP_{k,0}+QP_{k,1}x-2QP_{k,0}x}{1-2x-kx^2} + \frac{QPL_{k,0}+QPL_{k,1}x-2QPL_{k,0}x}{1-2x-kx^2}$$

$$= \frac{QPL_{k,1} + QPL_{k,2}x - 2QPL_{k,1}x}{1 - 2x - kx^2} = \frac{QPL_{k,1} + kxQPL_{k,0}}{1 - 2x - kx^2}.$$

$$2(k+1)G(x) + H(x) = \frac{2\left[1+kx+(kx+k+2)i+\left(k^2x+2kx+3k+4\right)j+\left(3k^2x+4kx+k^2+8k+8\right)k\right]}{1-2x-kx^2}.$$

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