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New Approaches for Evaluation Indeterminate Limits for Multivariable Functions in Undergraduate Mathematics Courses

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Zero divided by zero, L'Hôpital rule, Central finite differences, Forward finite differences, Backward finite differences, Richardson extrapolation method

Abstract – Zero divided by zero is one of the most important indeterminate forms obtained when evaluating limits for single variable functions and series in calculus education. Well-known method; L'Hôpital rule and its generalized form have been employed to simplify and resolve the indeterminate form such that zero divided by zero in terms of quotients of their derivatives for single variable functions as well as for multivariable functions. Nevertheless, L' Hôpital rule is impractical for the indeterminate limit forms of two variable functions in some cases such that isolated and nonisolated singularities, requirement of application of L'Hôpital rule more than once and complexity of taking derivative for some multivariable functions. So L'Hôpital rule cannot be preferred due to these reasons. By considering all these facts, new approaches including Finite Differences such as Central (CFD), Forward (FFD), Backward (BFD), High Accurate Central (HACFD), High Accurate Forward (HAFFD), High Accurate Backward (HABFD) methods, and Richardson Extrapolation method are presented that provide efficient ways to solve these limits instead of using L' Hôpital rule. Error analysis is also performed. All these methods are compared with each other in terms of accuracy and computational efficiency. It is observed that these approaches will be good alternatives instead of L'Hôpital rule for indeterminate form of two variable functions in calculus courses for both instructors and their students. Numerical examples are presented for this purpose.

1. Introduction

In calculus, the most famous and well-known method is L'Hôpital rule for evaluating the limits of indeterminate form: 0/0, for single variable functions. It has been employed comprehensively in the literature for this purpose and also for various applications (Aczel, 1990; Cooke, 1988; Corona-Corona, 2018; Duran, 1992; Estrada and Pavlovic, 2017; Gordon, 2017; Hartig, 1991; Huang, 1988; Muntean, 1993; Popa, 1999; Shishkina, 2007; Spigler and Vianello, 1993; Szabo, 1989; Takeuchi, 1995; Tian, 1993; Vianello, 1992; Vyborny and Nester, 1989; Zlobec, 2012). Besides, another significant and feature topic of interest in calculus is multivariable functions and their indeterminate form such that 0/0 (Fine and Kass, 1966; Ivlev, 2013; Young, 1910). L'Hôpital rule and its generalized form have been employed to overcome the complexity of indeterminate limit form: 0/0 by use of differentiation of both numerator and denominator (Ivlev and Shilin, 2014; Lawlor, 2020).

These methods often involve complex and lengthy processes that take derivative for some multivariable functions, especially for isolated and non isolated singularities. So applying L'Hôpital rule becomes inefficient and impractical method in these cases. It is the first time that Zlobec (2012) used L'Hôpital rule without derivative by employing Lagrange multiplier for only 0/0 indeterminate limit form for only single variable functions.

There is no other alternative numerical method instead of L'Hôpital rule taught in calculus courses for problem of indeterminate forms of multivariable functions in literature. This is why, new approaches including FFD, BFD, HAFFD, HABFD, CFD, HACFD techniques and Richardson Extrapolation methods (Chapra and Canale, 2010) are presented for the form: 0/0 of two variable functions to overcome the complexity and impossibility of

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L'Hôpital rule and its generalized form in this study. For the first time in literature Dinçkal (2024) introduced and employed finite difference methods for the solution of indeterminate problems of single variable functions.

There are three ways to get more accurate results when employing finite difference methods. One of them is to decrease step size. Other one is to employ a higher order formula which uses more points. Another one is to use the results of these finite difference methods to get more accurate approximations. For this reason, Richardson Extrapolation methods based on these finite difference methods are proposed, described and also compared with other finite difference method results.

Taking derivative and using Lagrange multiplier are not required in all approaches. These numerical techniques with proper step size here can be applied to all indeterminate limits for two variable functions conveniently.

After description of the methods proposed, numerical results from various complicated two variable functions are presented to prove the applicability of them.

2. Materials and Methods

2.1. L'Hôpital rule

Well-known rule; L'Hôpital rule for two variables functions has some steps and shown as follows (Dinçkal, 2024; Ivlev and Shilin, 2014; Lawlor, 2020)

$$\lim_{(x,y)\to(x_0,\,y_0)} \frac{f(x,y)}{g(x,y)} \tag{1}$$

where $\lim\limits_{(x,y)\to(x_0,\,y_0)} f(x,y) = 0$ or a number and $\lim\limits_{(x,y)\to(x_0,\,y_0)} g(x,y) = 0.$ So one should take partial derivative with respect to *x* and *y*, at points $(x_0,\ y_0)$ respectively (Ivlev and Shilin, 2014; Lawlor, 2020)

$$\frac{f_x(x_0,y_0)}{g_x(x_0,y_0)} \tag{2a}$$

$$\frac{f_y(x_0,y_0)}{g_y(x_0,y_0)} \tag{2b}$$

Provided that (2a) should be equal to (2b) such that

$$\frac{f_x(x_0,y_0)}{g_x(x_0,y_0)} = \frac{f_y(x_0,y_0)}{g_y(x_0,y_0)} = k_1 \tag{3}$$

Currently, if indeterminate form is found again as a result of (3), second order partial derivatives should be taken (Ivlev and Shilin, 2014):

$$\frac{f_{xx}(x_0,y_0)}{g_{xx}(x_0,y_0)} = \frac{f_{xy}(x_0,y_0)}{g_{xy}(x_0,y_0)} = \frac{f_{yx}(x_0,y_0)}{g_{yx}(x_0,y_0)} = \frac{f_{yy}(x_0,y_0)}{g_{yy}(x_0,y_0)} = k_2 \tag{4}$$

L'Hôpital rule for multivariable functions confirms the rules of partial derivation. So this method can be considered as being one more rule of partial derivation. However, this approach remains inadequate for isolated and non-isolated singularities.

2.2. Numerical methods

In some cases such as isolated and non isolated singularities, taking derivative is ineffective and impractical. Furthermore, sometimes, it is required to apply L'Hôpital rule more than once to overcome indeterminate limits. For this reason, new approaches including CFD, HACFD, FFD, HAFFD, BFD, HABFD and Richardson Extrapolation methods are proposed in this paper, correspondingly.

The assumptions have been made by changing x variable and not changing y variable in Taylor series expansions for all proposed methods. On the contrary, changing *y* variable and not changing *x* variable in Taylor series expansions do not change the results.

The originating idea of CFD, HACFD, FFD, HAFFD, BFD and HABFD techniques is based on well-known Taylor series.

2.2.1. CFD, HACFD, indeterminate limit formulations based on CFD and HACFD

Taylor series for forward form can be formulated as follows: by defining a step size $h = x_{i+1} - x_i$ and expressing as

$$(x_{i+1}, y) = f(x_i, y) + f'(x_i, y)h + \frac{f''(x_i,y)}{2!}h^2 + \frac{f^{(3)}(x_i,y)}{3!}h^3 + \cdots + \frac{f^{(n)}(x_i,y)}{n!}h^n + R_n \tag{5}$$

The remainder term for variables *x* and *y* is defined as

$$R_n = \frac{f^{(n+1)}(\varepsilon_1,\varepsilon_2)}{(n+1)!}h^{n+1} \tag{6}$$

The term in (6) corresponds to $O((x_{i+1} - x_i)^{n+1}, (y_{i+1} - y_i)^{n+1})$ which is $O(h^{n+1})$ called as error.

For backward form, Taylor series in (5) can be formulated as

$$f(x_{i-1}, y) = f(x_i, y) - f'(x_i, y)h + \frac{f''(x_i,y)}{2!}h^2 - \frac{f^{(3)}(x_i,y)}{3!}h^3 + \cdots \tag{7}$$

One of the ways to approximate the first derivative is to take differences of (7) and (5) to obtain:

$$f(x_{i+1}, y) = f(x_{i-1}, y) + 2f'(x_i, y)h + \frac{2f^{(3)}(x_i,y)}{3!}h^3 + \cdots \tag{8}$$

This can be solved for

$$f'(x_i, y) = \frac{f(x_{i+1},y) - f(x_{i-1},y)}{2h} - \frac{f^{(3)}(x_i,y)}{3!}h^2 - \cdots \tag{9}$$

Equation (9) can be also rewritten as

$$f'(x_i, y) = \frac{f(x_{i+1},y) - f(x_{i-1},y)}{2h} + O(h^2) \tag{10}$$

By substituting (10) into (3), one can get the combination of L'Hopital rule and CFD. As a result, one of the methods including CFD for solving indeterminate limit is formulated as

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{f(x^*{}_{i+1},y) - f(x^*{}_{i-1},y)}{2h} + O(h^2)}{\frac{g(x^*{}_{i+1},y) - g(x^*{}_{i-1},y)}{2h} + O(h^2)} \tag{11}$$

Both numerator and denominator in terms of CFD have errors which are $O(h^2)$. This demonstrates that errors are proportional to the square of the same step size for both $f(x, y)$ and $g(x, y)$. Error is of the order of h^2 in spite of the backward and forward approximations that are of the order of *h*. Therefore, Taylor series approximations yield the significant information that the centered formula is the most accurate demonstration of the derivative (Chapra and Canale, 2010; Dinçkal, 2024).

Level of accuracy depends on decreasing the step size and also increases the number of terms of the Taylor series. Hence, it is possible to reconstruct more accurate formulas called as HACFD by withholding more terms.

By substituting first order derivative in (10) into (5), CFD representation of the second order derivative based on error $O(h^2)$ can be found as

$$f''(x_i, y) = \frac{f(x_{i+1},y) - 2f(x_i,y) + f(x_{i-1},y)}{h^2} \tag{12}$$

Third order derivative based on error $O(h^2)$ by use of CFD:

$$f^{(3)}(x_i, y) = \frac{f(x_{i+2},y) - 2f(x_{i+1},y) + 2f(x_{i-1},y) - f(x_{i-2},y)}{2h^3} \tag{13}$$

To find the high-accurate form of first derivative based on error $O(h^4)$, one should use both (12) and (13), and substitute them into (5):

$$f(x_{i+1,}y) = f(x_i, y) + f'(x_i, y)h + \frac{\frac{f(x_{i+1}, y) - 2\mathrm{f}(x_i, y) + f(x_{i-1}, y)}{h^2}}{2!}h^2 + \frac{\frac{f(x_{i+2,}y) - 2\mathrm{f}(x_{i+1,}y) + 2f(x_{i-1}, y) - f(x_{i-2,}y)}{2h^3})}{3!}h^3 + \cdots \quad (14)$$

So $f'(x_i)$ based on error $O(h^4)$ can be obtained from (14) which is

$$f'(x_i, y) = \frac{-f(x_{i+2,}y)+8f(x_{i+1,}y)-8f(x_{i-1,}y)+\mathrm{f}(x_{i-2,}y)}{12h} + O(h^4) \quad (15)$$

By use of (15) and put into (3), one can get the final form of transition of L'Hopital rule and HACFD. Finally, one of the methods including HACFD for solving indeterminate limit becomes

$$\lim_{(x,y)\to(x^*,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{-f(x^*{}_{i+2,}y)+8f(x^*{}_{i+1,}y)-8f(x^*{}_{i-1,}y)+\mathrm{f}(x^*{}_{i-2,}y)}{12h}+O(h^4)}{\frac{-g(x^*{}_{i+2,}y)+8g(x^*{}_{i+1,}y)-8g(x^*{}_{i-1,}y)+g(x^*{}_{i-2,}y)}{12h}+O(h^4)} \quad (16)$$

2.2.2. FFD, HAFFD, indeterminate limit formulations based on FFD and HAFFD

By recalling the formulation given in (5), first derivative by FFD based on O(h) can be found as (Chapra and Canale, 2010)

$$f'(x_i, y) = \frac{f(x_{i+1,}y)-f(x_{i,}y)}{h} + O(h). \quad (17)$$

By use of (3) and (17), the transition from L'Hopital rule and FFD technique can be obtained. Solution for indeterminate limit is

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{f(x^*{}_{i+1,}y)-f(x^*{}_{i,}y)}{h}+O(h)}{\frac{g(x^*{}_{i+1,}y)-g(x^*{}_{i,}y)}{h}+O(h)} \quad (18)$$

By substituting first order derivative in (17) into (5), FFD representation of the second order derivative based on error $O(h^2)$ can be obtained as

$$f''(x_{i,}y) = \frac{f(x_{i+2,}y)-2\mathrm{f}(x_{i+1,}y)+f(x_{i,}y)}{h^2} \quad (19)$$

Third order derivative based on error $O(h^2)$ by use of FFD:

$$f^{(3)}(x_i, y) = \frac{f(x_{i+3,}y)-3\mathrm{f}(x_{i+2,}y)+3f(x_{i+1,}y)-f(x_{i,}y)}{h^3} \quad (20)$$

To find the high-accurate form of first derivative by FFD based on error $O(h^2)$, one should employ both (19) and (20), and substitute them into (5):

$$f'(x_i, y) = \frac{-f(x_{i+2,}y)+4f(x_{i+1,}y)-3f(x_{i,}y)}{2h} + O(h^2) \quad (21)$$

By substituting (21) into (3), the transition from L'Hopital rule to HAFFD method can be obtained. So indeterminate limit formulation based on HAFFD become

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{-f(x^*{}_{i+2,}y)+4f(x^*{}_{i+1,}y)-3f(x^*{}_{i,}y)}{2h}+O(h^2)}{\frac{-g(x^*{}_{i+2,}y)+4g(x^*{}_{i+1,}y)-3g(x^*{}_{i,}y)}{2h}+O(h^2)} \quad (22)$$

2.2.3. BFD, HABFD, indeterminate limit formulations based on BFD and HABFD

By use of (7), first derivative by BFD based on O(h) can be obtained as (Chapra and Canale, 2010)

$$f'(x_i, y) = \frac{f(x_{i,}y)-f(x_{i-1,}y)}{h} + O(h) \quad (23)$$

By use of both (3) and (23), final form of the proposed BFD technique with combination of L'Hopital rule can be found for solution of indeterminate limit. It is given as

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{f(x^*{}_i,y)-f(x^*{}_{i-1},y)}{h}+O(h)}{\frac{g(x^*{}_i,y)-g(x^*{}_{i-1},y)}{h}+O(h)} \tag{24}$$

By substituting first order derivative in (23) into (7), BFD representation of the second order derivative based on error $O(h^2)$ can be obtained as

$$f''(x_i,y) = \frac{f(x_i,y)-2f(x_{i-1},y)+f(x_{i-2},y)}{h^2} \tag{25}$$

Third order derivative based on error $O(h^2)$ by BFD computation:

$$f^{(3)}(x_i,y) = \frac{f(x_i,y)-3f(x_{i-1},y)+3f(x_{i-2},y)-f(x_{i-3},y)}{h^3} \tag{26}$$

To find the high-accurate form of first derivative by BFD based on error $O(h^2)$, one should use both (25) and (26), and substitute them into (7). So accurate form of BFD becomes

$$f'(x_i,y) = \frac{3f(x_i,y)-4f(x_{i-1},y)+f(x_{i-2},y)}{2h} + O(h^2) \tag{27}$$

To get the final form of L'Hopital rule transition to HABFD (27) should be substituted into (3). The following formulation can be obtained:

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{3f(x^*{}_i,y)-4f(x^*{}_{i-1},y)+f(x^*{}_{i-2},y)}{2h}+O(h^2)}{\frac{3g(x^*{}_i,y)-4g(x^*{}_{i-1},y)+g(x^*{}_{i-2},y)}{2h}+O(h^2)} \tag{28}$$

2.2.4. Richardson extrapolation based on CFD and HACFD, indeterminate limit formulations

To improve CFD and HACFD results, Richardson Extrapolation can be employed, conveniently. Target is to get more accurate results. For this reason, following steps are performed (Chapra and Canale, 2010):

$$C \cong C(h_2) + \frac{1}{\left(\left(\frac{h_1}{h_2}\right)^2-1\right)}\big(C(h_2)-C(h_1)\big) \tag{29}$$

where $h_2 = \frac{h_1}{2}$, by rearranging (29), final form of Richardson extrapolation can be found. It is

$$C \cong \frac{4}{3}C(h_2) - \frac{1}{3}C(h_1) \tag{30}$$

$C(h_2)$ and $C(h_1)$ can be obtained from the (10) for CFD based computation. If one can prefer computations based on HACFD, then $C(h_2)$ and $C(h_1)$ can be obtained from the (15) for HACFD based computation. Final form of transition from L'Hopital rule to new proposed method based on CFD with use of (3) and (10), one can obtain

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{4}{3}C1(h_2)-\frac{1}{3}C1(h_1)+O(h^2)}{\frac{4}{3}C2(h_2)-\frac{1}{3}C2(h_1)+O(h^2)} \tag{31}$$

where C1 is CFD based results for function $f(x,y)$ and C2 is CFD based results for function $g(x,y)$.

For CFD approximations, $O(h^2)$ yield a new estimate of new accuracy $O(h^4)$.

Final form of transition from L'Hopital rule to new proposed method based on HACFD with employ of (3) and (15) ,one can found the following formulation:

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{4}{3}C11(h_2)-\frac{1}{3}C11(h_1)+O(h^4)}{\frac{4}{3}C22(h_2)-\frac{1}{3}C22(h_1)+O(h^4)} \tag{32}$$

where C11 is HACFD based results for function $f(x,y)$ and C22 is HACFD based results for function $g(x,y)$. For HACFD approximations, $O(h^4)$ yield a new estimate of new accuracy $O(h^6)$.

2.2.5. Richardson extrapolation based on FFD and HAFFD, indeterminate limit formulations

To enhance FFD and HAFFD results and get more accurate results, Richardson Extrapolation can be employed. The steps are similar to those for CFD and HACFD based computations.

$$F \cong F(h_2) + \frac{1}{\left(\left(\frac{h_1}{h_2}\right)^2 - 1\right)}\big(F(h_2) - F(h_1)\big) \tag{33}$$

where $h_2 = \frac{h_1}{2}$, by rewriting (33), final form of Richardson extrapolation can be obtained which is

$$F \cong \frac{4}{3}F(h_2) - \frac{1}{3}F(h_1) \tag{34}$$

$F(h_2)$ and $F(h_1)$ can be computed from the (17) for FFD based computation. If one can choose computations based on HAFFD, then $F(h_2)$ and $F(h_1)$ can be obtained from the (21) for HAFFD based computation. Final form of transition from L'Hopital rule to new proposed method based on FFD with use of (3) and (17), one can obtain

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{4}{3}F1(h_2) - \frac{1}{3}F1(h_1) + O(h)}{\frac{4}{3}F2(h_2) - \frac{1}{3}F2(h_1) + O(h)} \tag{35}$$

Where F1 is FFD based results for function $f(x,y)$ and F2 is FFD based results for function $g(x,y)$.

For FFD approximations, O(h) yield a new estimate of new accuracy $O(h^2)$.

Final form of transition from L'Hopital rule to new proposed method based on HACFD with employ of (3) and (21) ,one can obtain the following formulation:

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{4}{3}F11(h_2) - \frac{1}{3}F11(h_1) + O(h^2)}{\frac{4}{3}F22(h_2) - \frac{1}{3}F22(h_1) + O(h^2)} \tag{36}$$

Where F11 is HAFFD based results for function $f(x,y)$ and F22 is HAFFD based results for function $g(x,y)$. For HAFFD approximations, $O(h^2)$ yield a new estimate of new accuracy $O(h^4)$.

2.2.6. Richardson extrapolation based on BFD and HABFD, indeterminate limit formulations

To upgrade BFD and HABFD results, Richardson extrapolation can be used, conveniently. Main aim is to get more accurate results. For this reason, following steps are executed (Chapra and Canale, 2010):

$$B \cong B(h_2) + \frac{1}{\left(\left(\frac{h_1}{h_2}\right)^2 - 1\right)}\big(B(h_2) - B(h_1)\big) \tag{37}$$

where $h_2 = \frac{h_1}{2}$, by rearranging (37), final form of Richardson extrapolation can be obtained as

$$B \cong \frac{4}{3}B(h_2) - \frac{1}{3}B(h_1) \tag{38}$$

$B(h_2)$ and $B(h_1)$ can be obtained from the (23) for BFD based computation. If one can prefer computations based on HABFD, then $B(h_2)$ and $B(h_1)$ can be obtained from the (27) for HABFD based computation. Final form of transition from L'Hopital rule to new proposed method based on BFD with use of (3) and (23), one can get

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{4}{3}B1(h_2) - \frac{1}{3}B1(h_1) + O(h)}{\frac{4}{3}B2(h_2) - \frac{1}{3}B2(h_1) + O(h)} \tag{39}$$

where B1 is BFD based results for function $f(x,y)$ and B2 is BFD based results for function $g(x,y)$.

For BFD approximations, O(h) yield a new estimate of new accuracy $O(h^2)$.

Final form of transition from L'Hopital rule to new proposed method based on HABFD with employ of (3) and (27), one can found the following formulation:

$$\lim_{(x,y)\to(x^*,\,y^*)} \frac{f(x,y)}{g(x,y)} = \frac{\frac{4}{3}B11(h_2) - \frac{1}{3}B11(h_1) + O(h^2)}{\frac{4}{3}B22(h_2) - \frac{1}{3}B22(h_1) + O(h^2)} \quad (40)$$

where B11 is HABFD based results for function $f(x,y)$ and B22 is HABFD based results for function $g(x,y)$. For HABFD approximations, $O(h^2)$ yield a new estimate of new accuracy $O(h^4)$.

3. Error Analysis

The condition and stability analysis of a problem relates to its sensitivity to changes in its input values. It is clear that a computation is numerically unstable if the uncertainty of the input values is over exaggerated by the numerical method (Chapra and Canale, 2010). Taylor series for two variables without second order and higher order terms can be written as:

$$f(x_{i+1}, y_{i+1}) = f(x_i, y_i) + f'(x_i, y)(x_{i+1} - x_i) + f'(x, y_i)(y_{i+1} - y_i) + ... \quad (41)$$

Where estimates of the errors in x is $(x_{i+1} - x_i)$ and y is $(y_{i+1} - y_i)$, respectively. These are assumed to be equal and denoted as h.

Equation (41) can be rearranged as

$$f(x_{i+1}, y_{i+1}) - f(x_i, y_i) = \left(f'(x_i, y) + f'(x, y_i)\right)\text{h} + ... \quad (42)$$

In (42),

$$f(x_{i+1}, y_{i+1}) - f(x_i, y_i) = \Delta f(\tilde{x}, \tilde{y})$$

For all examples, the function values are equal to 0 to become indeterminate limit condition.

An algorithm is stable if the following equality holds:

$$h = \frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{\left(f'(x_i, y) + f'(x, y_i)\right)} \quad (43)$$

If the numerator in (43) converges to 0, the problem becomes stable and well conditioned. This also leads to indeterminate limit form. For this reason, h is selected as very small values closer to 0.

This situation also fits with smaller step size selection in Taylor series.

Dropping the second and higher-order terms and rearranging (42) yields:

$$\Delta f(\tilde{x}, \tilde{y}) = \left(f'(x_i, y) + f'(x, y_i)\right)\text{h}. \quad (44)$$

It is possible to determine bounds for exact solutions of finite difference approaches by use of (44)

$$f(x_{i+1}, y_{i+1}) = f(x_{i+1}, y_{i+1}) \pm \left(f'(x_i, y) + f'(x, y_i)\right)\text{h} \quad (45)$$

Since, $f(x_{i+1}, y_{i+1}) = 0$ to make the limits indeterminate, upper and lower bounds should involve 0. So h should be selected as minimum as possible to involve exact solution for higher accuracy and sensitivity. As a result h should converge to 0. For this reason, $h=10^{-4}$ is selected as tolerance amount for step size choice such that h less than this amount, it is accepted that exact results can be always obtained.

4. Numerical Results and Discussion

Examples for indeterminate limit form: $\frac{0}{0}$ is presented in Table 1. A general code is generated for application of the proposed methods in Matlab R2022a. Numerical results are analyzed in terms of accuracy, error and computational time. These are presented in following subsections, respectively.

Table 1. Examples for indeterminate form: $\frac{0}{0}$

Example	Limit
1	$\lim_{(x,y)\to(1,-1)} \frac{(x^3+y^3)}{x+y}$
2	$\lim_{(x,y)\to(1,1)} \frac{(x^2-y^2)}{x-y}$
3	$\lim_{(x,y)\to(1,1)} \frac{(x^4-y^4)}{(\ln(x)-\ln(y))}$
4	$\lim_{(x,y)\to(0,0)} \frac{(x-y)}{(\sin(x)-\sin(y))}$
5	$\lim_{(x,y)\to(0,0)} \frac{(x-\sin(y))}{(\sin(x)-y)}$
6	$\lim_{(x,y)\to(1,-1)} \frac{(x+y)}{(x^2-y^2)}$

4.1. Accuracy

For the examples presented in Table 1, indeterminate limits are computed by each method and displayed in Figures 1-6.

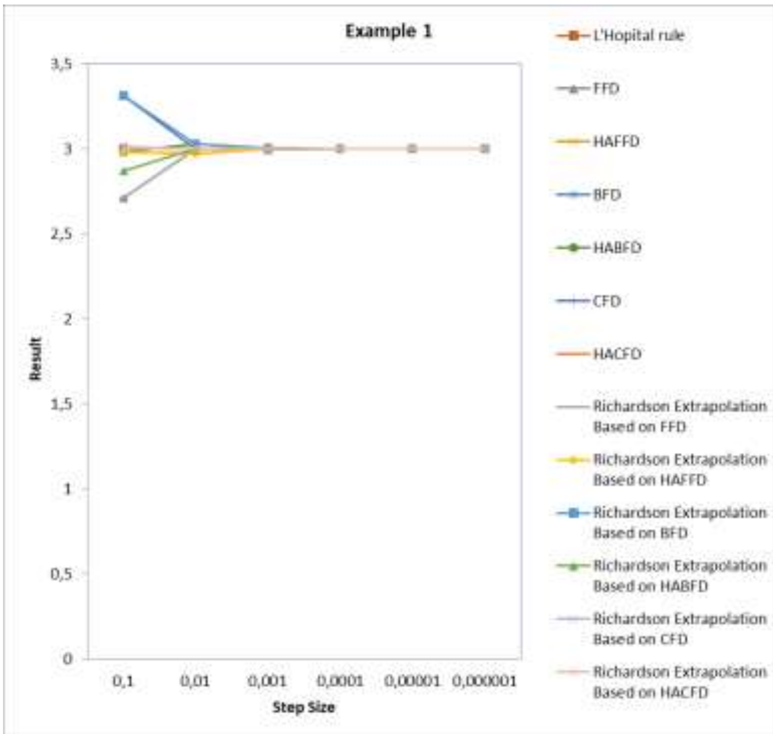

Figure 1. Computation results by all methods for Example 1

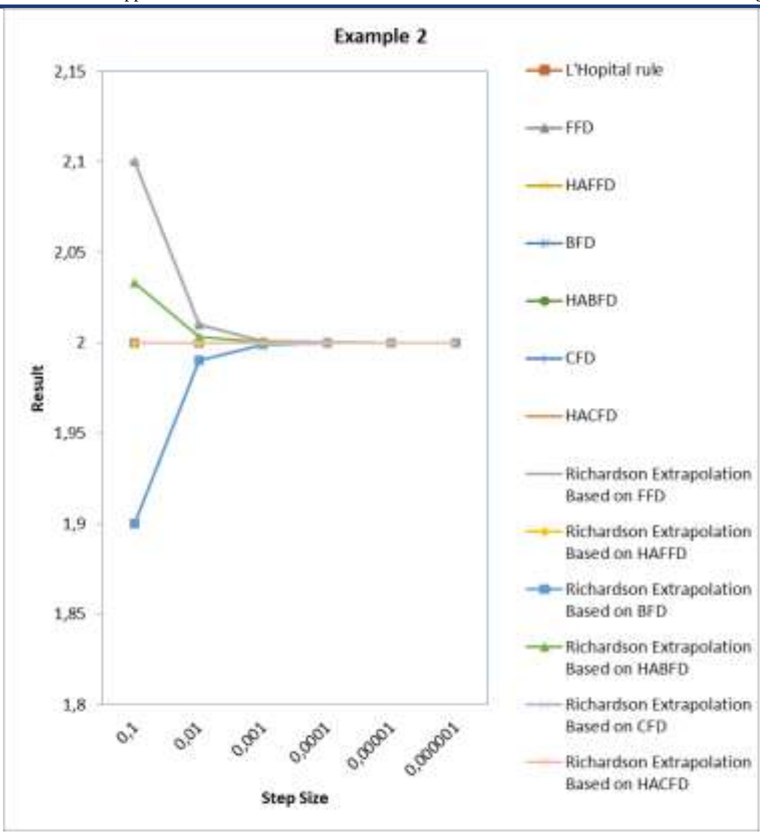

Figure 2. Computation results by all methods for Example 2

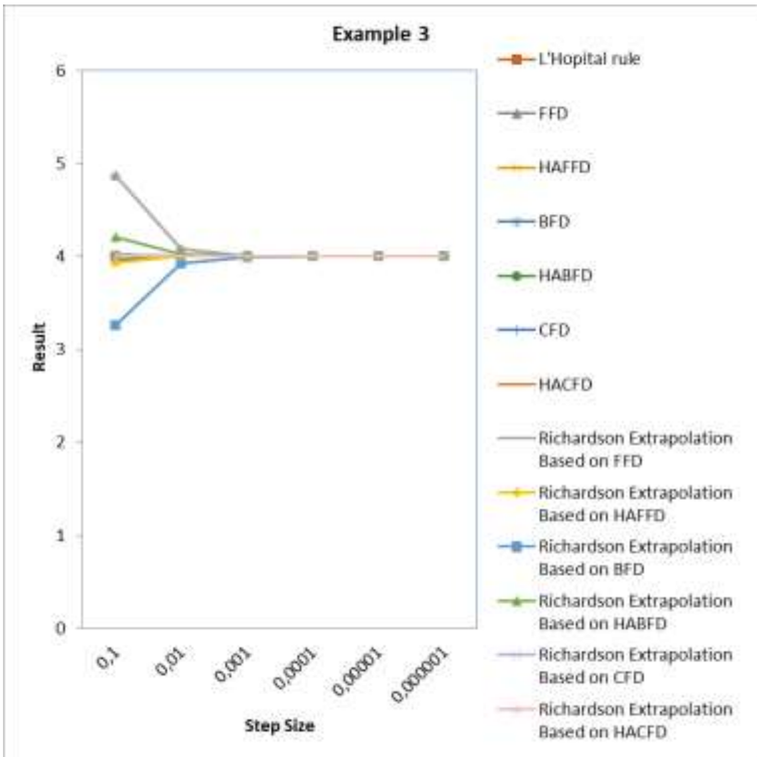

Figure 3. Computation results by all methods for Example 3

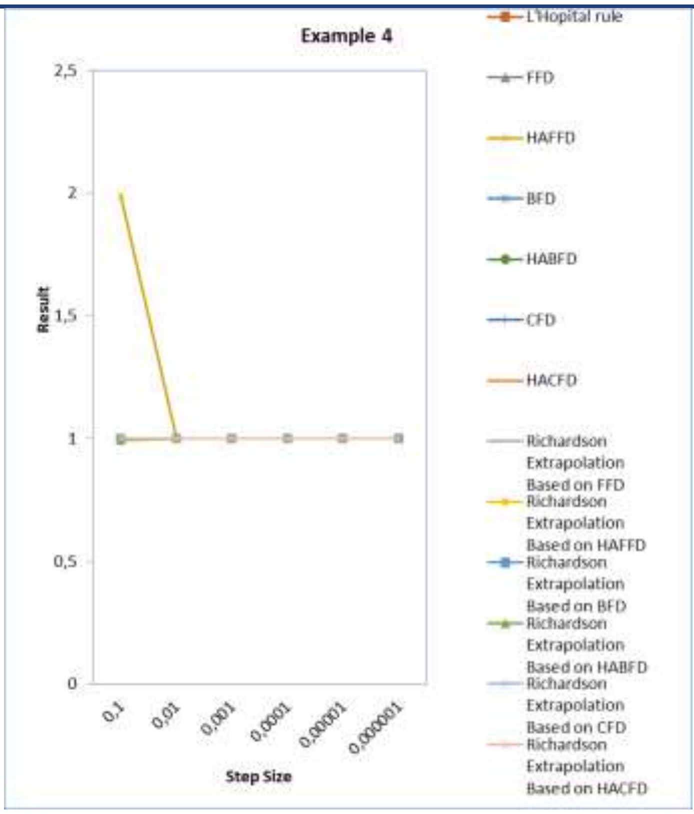

Figure 4. Computation results by all methods for Example 4

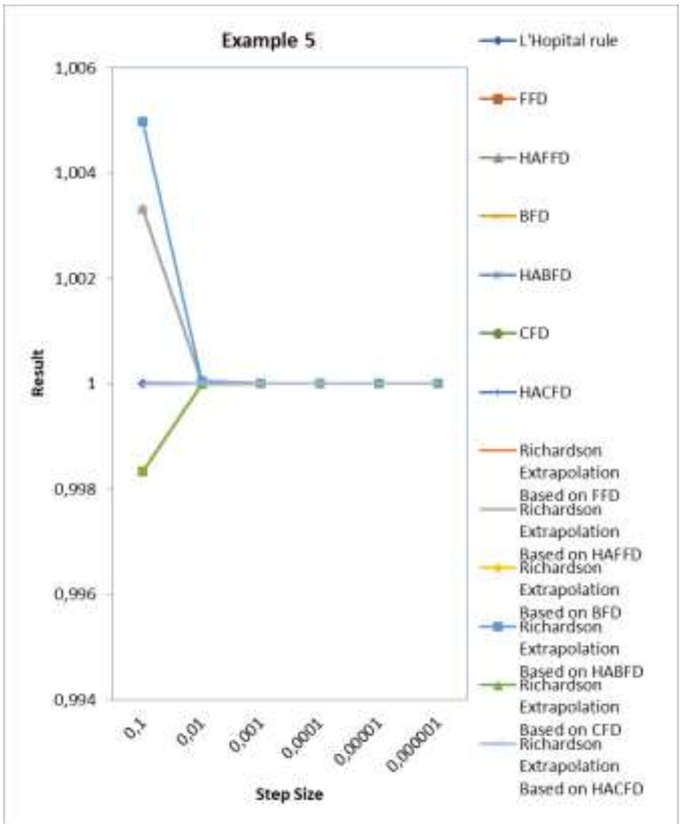

Figure 5. Computation results by all methods for Example 5

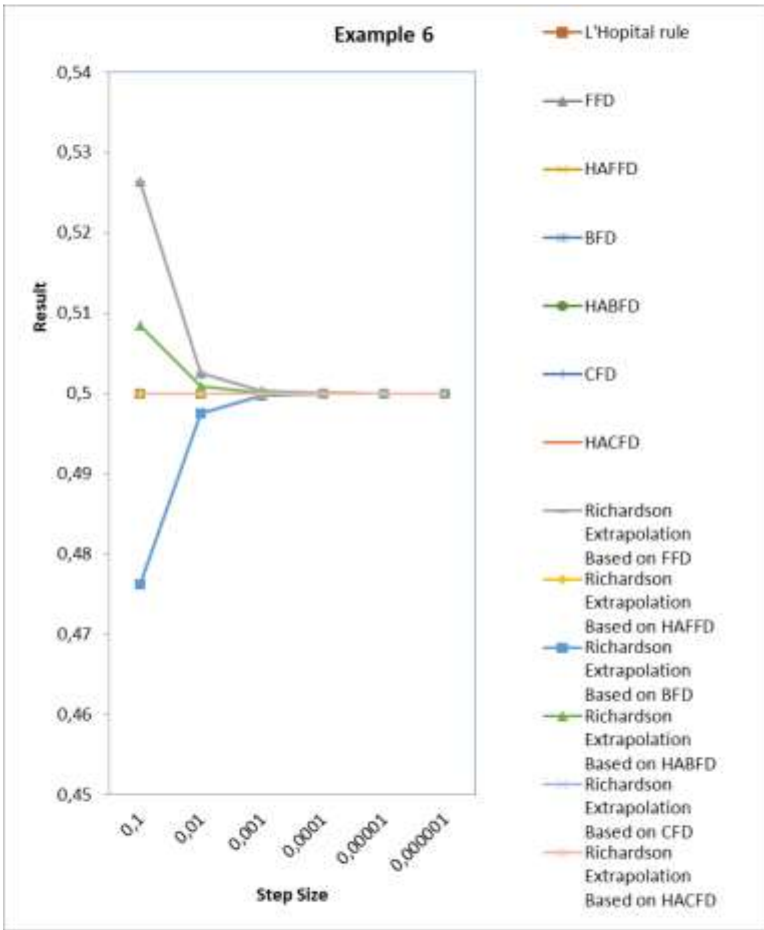

Figure 6. Computation results by all methods for Example 6

In common for Figures 1-6, when step size is 10^{-4}, exact results can be found for all examples in Table 1. For less amount of step size, FFD, BFD, Richardson Extrapolation based on BFD and FFD methods yield results diverge from true values. Notwithstanding, L'Hôpital rule HACFD, Richardson Extrapolation based on HABFD, HACFD and HAFFD converge to exact results fastly, even if step size is 10^{-1}.

4.2. Error

Error is computed in terms of true percent relative error. It is given as

$$\text{Error} = \left|\frac{\text{True Value} - \text{Approximation}}{\text{True Value}}\right| \times 100 \tag{46}$$

By using the formula in (46), amount of error for each example in Table 1 can be computed. The results for each example are displayed in Figures 7-12, respectively.

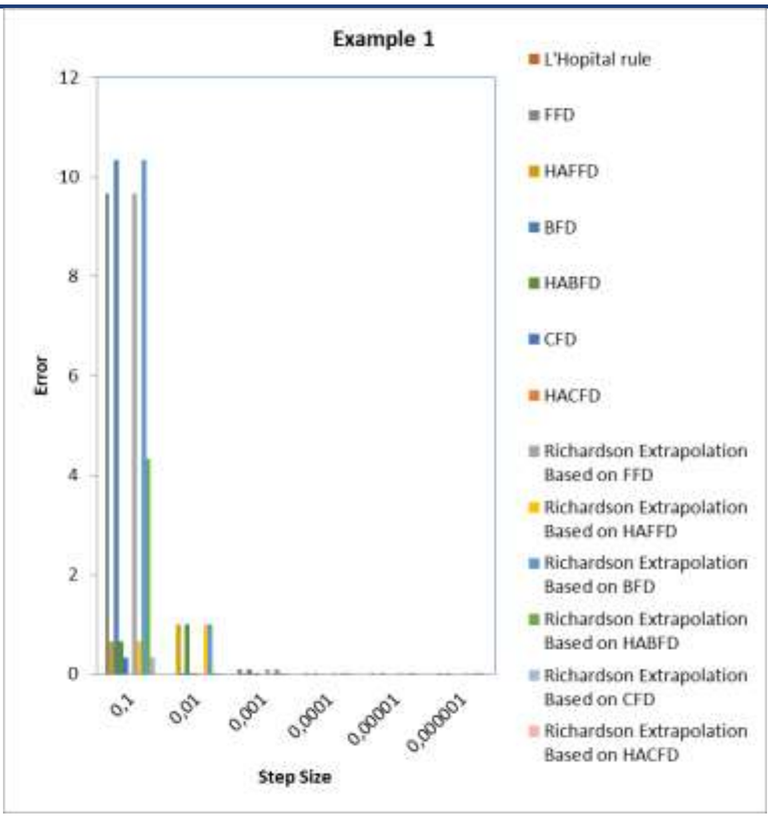

Figure 7. Error computation for Example 1

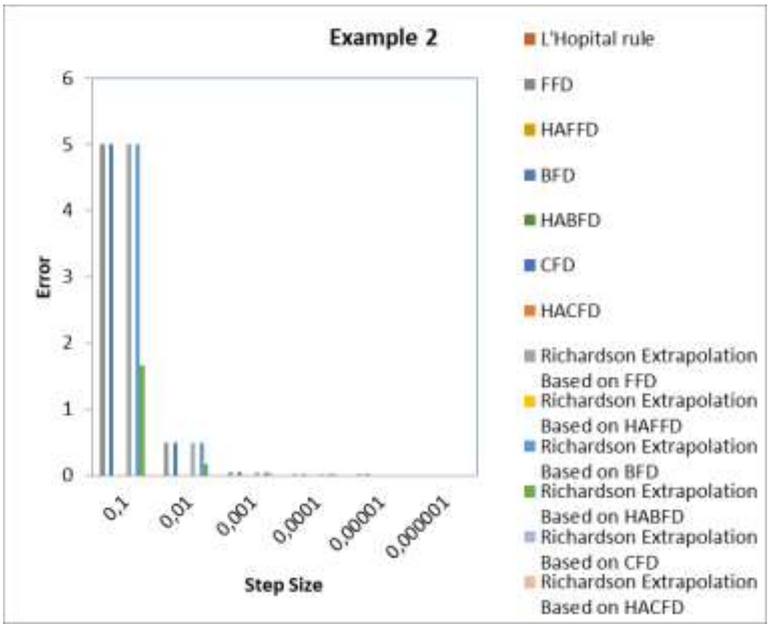

Figure 8. Error computation for Example 2

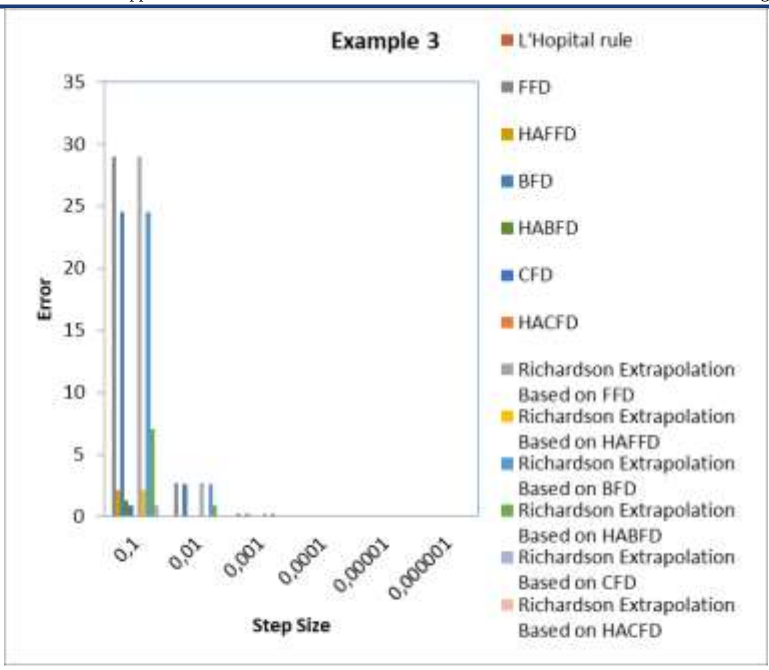

Figure 9. Error computation for Example 3

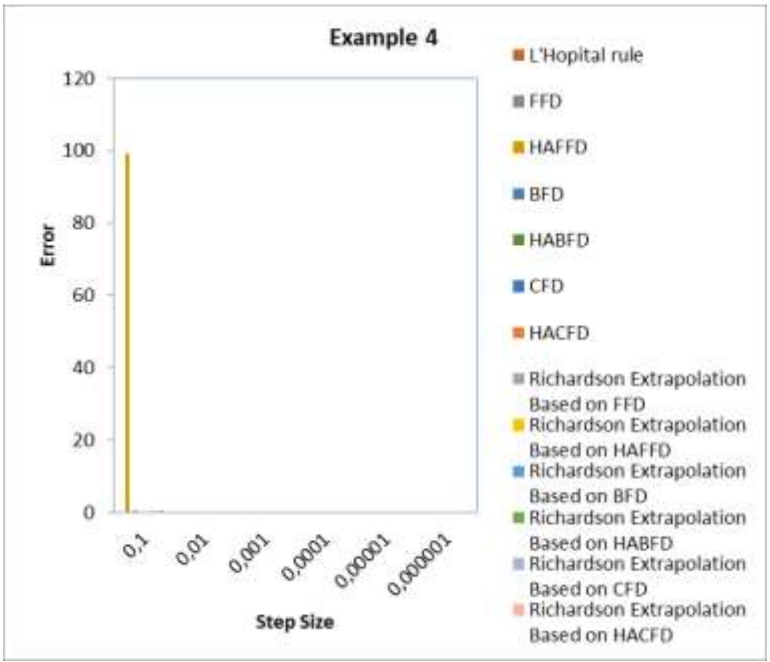

Figure 10. Error computation for Example 4

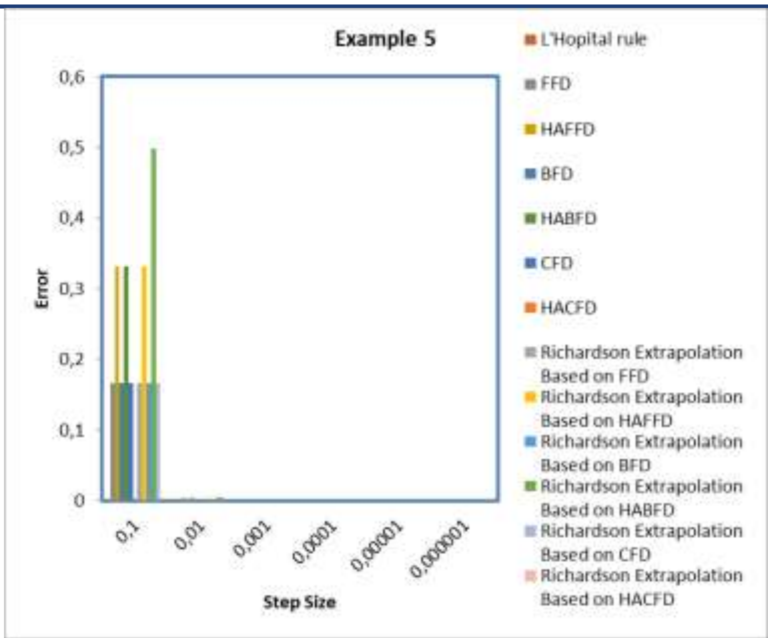

Figure 11. Error computation for Example 5

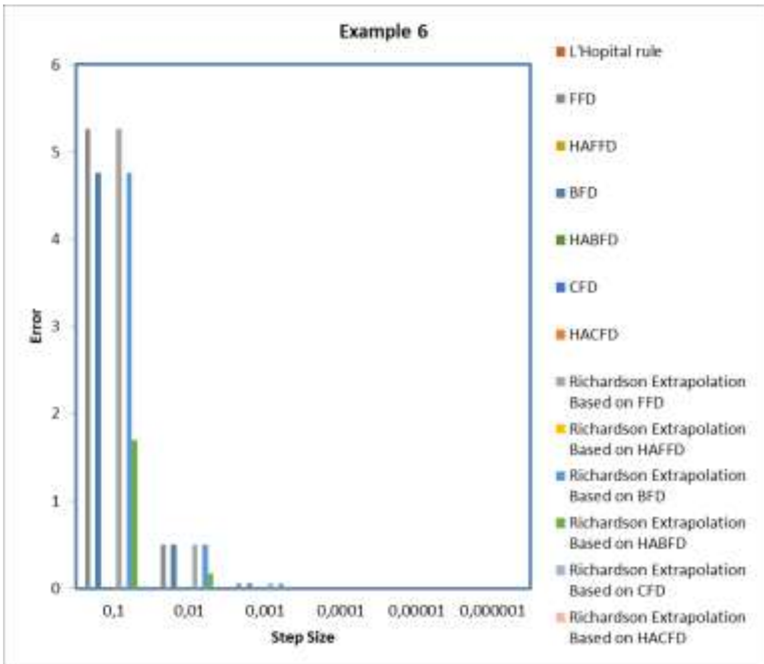

Figure 12. Error computation for Example 6

It is obvious that error is 0 when step size is 10^{-4} for all methods in Figures 7-12. Nevertheless, for h values less than 10^{-4}, there exists error amounts, especially for methods; BFD, FFD, Richardson extrapolation based on BFD and FFD.

4.3. Computational time

For the performance of the algorithm, computational time is measured with use of all numerical methods as well as L'Hôpital rule. The results are presented in Figures 13-18 for each example.

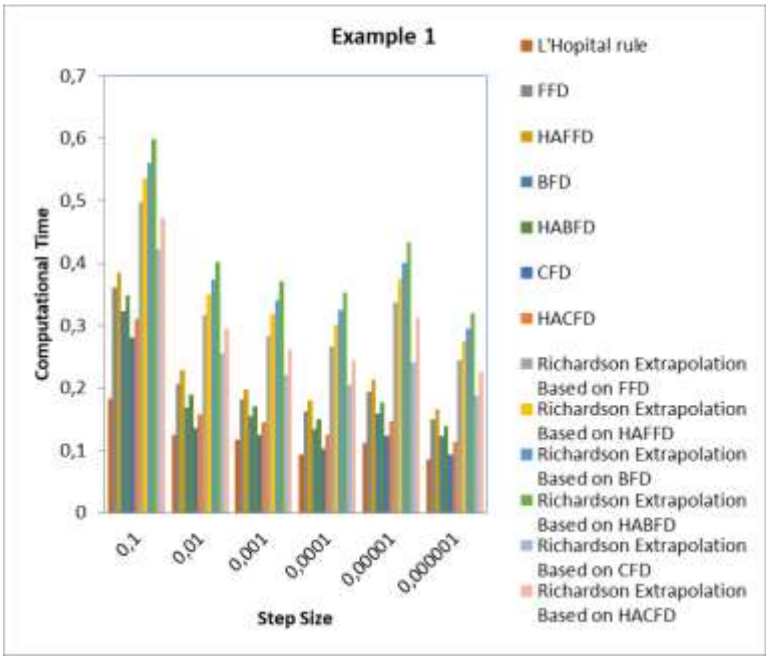

Figure 13. Computational time for Example 1

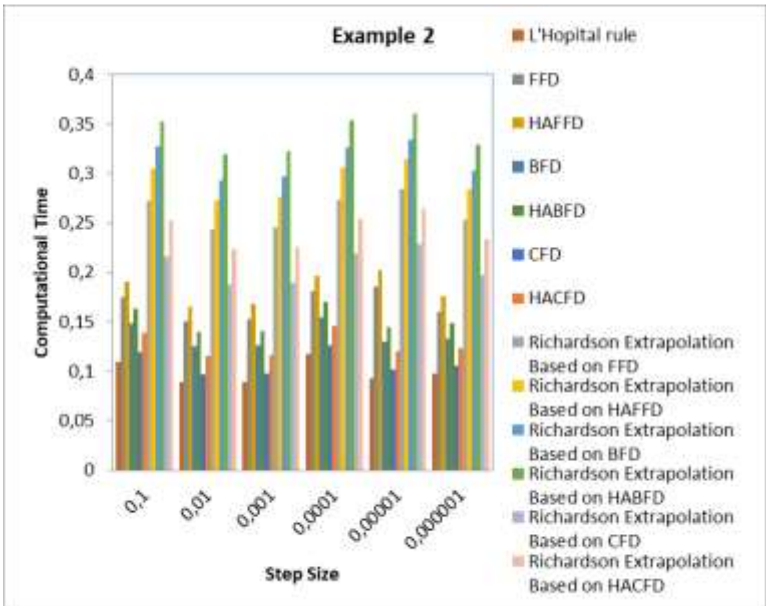

Figure 14. Computational time for Example 2

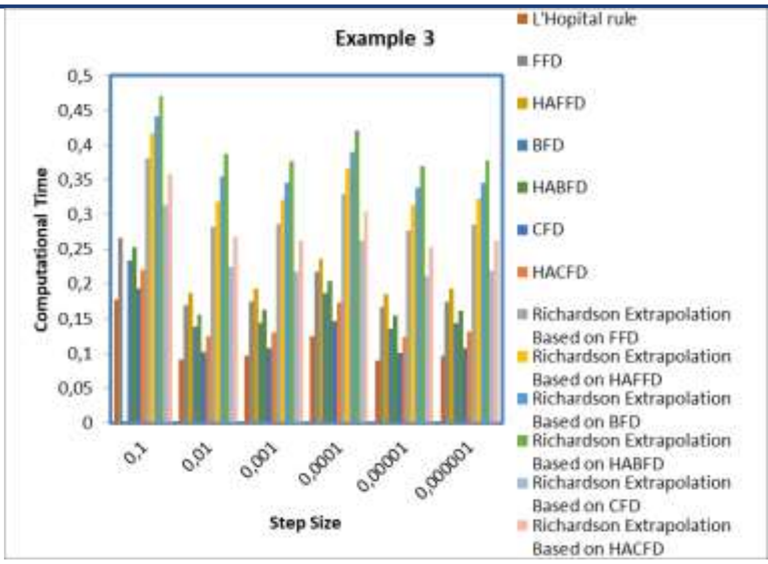

Figure 15. Computational time for Example 3

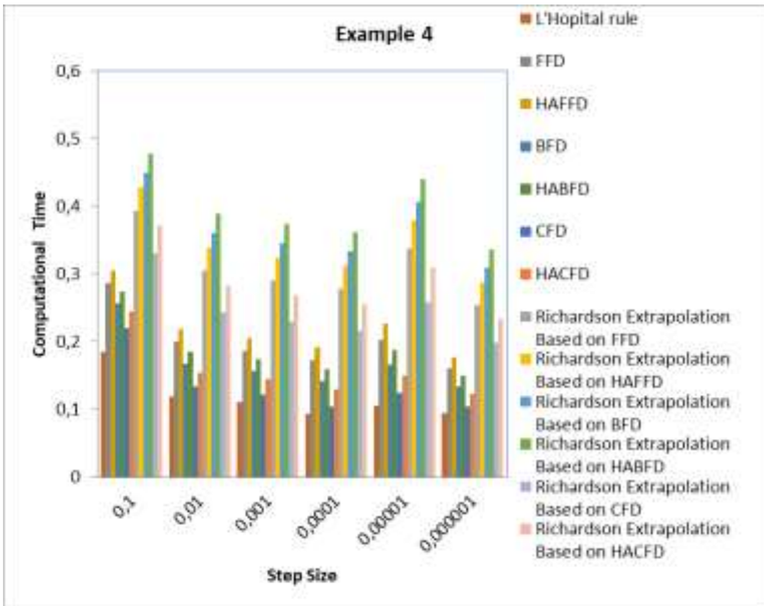

Figure 16. Computational time for Example 4

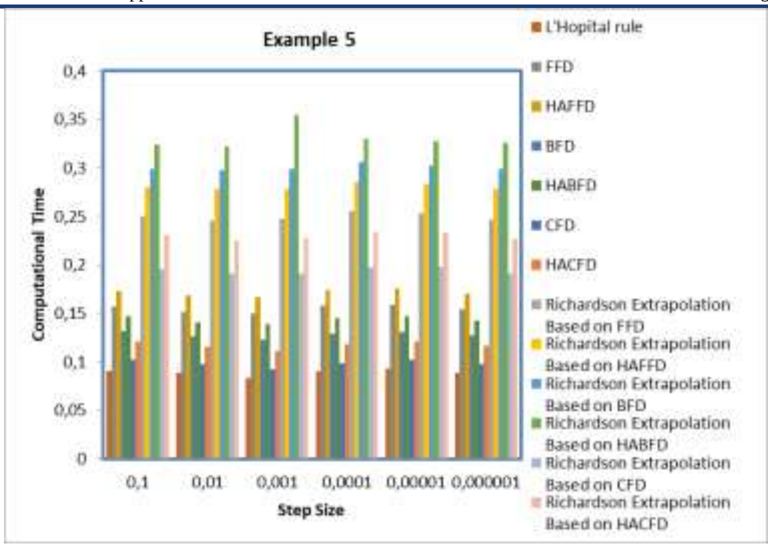

Figure 17. Computational time for Example 5

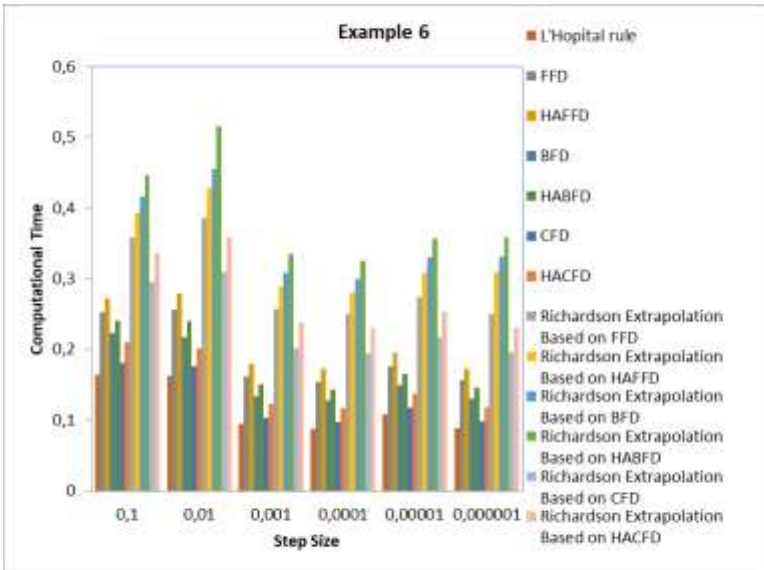

Figure 18. Computational time for Example 6

For each example, less amount of step size leads to a decrease in computational time for all methods. L' Hôpital rule spends the least amount of time while Richardson Extrapolation method based on HABFD spends the most computational time for each example. These situations are proved in Figures 13-18.

Furthermore, examples and numerical results for other indeterminate forms of small functions with isolated and non isolated singularities are also illustrated in Table 2.

Table 2. Numerical results for other indeterminate forms of small functions

Example	FFD	HAFFD	BFD	HABFD	Richardson Extrapolation Based on FFD	Richardson Extrapolation Based on HAFFD	Richardson Extrapolation Based on BFD	Richardson Extrapolation Based on HABFD
$\lim_{(x,y)\to(0,0)} \frac{x^{\alpha}y}{x^6+x^2y^2+y^6}$ $0 \le \alpha \le 7$	0	0	0	0	0	0	0	0
$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{x^4+2(\sin(x))^2(\sin(y))^2+y^4}$	1	1	1	1	1	1	1	1
$\lim_{(x,y)\to(0,0)} \frac{x^{\alpha}y}{x^4+y\sin(y)}$ $0 \le \alpha \le 4$	0	0	0	0	0	0	0	0

According to Table 2; L' Hôpital rule, CFD, HACFD, Richardson Extrapolation based on CFD and HACFD methods are inapplicable for indeterminate limit form: $\frac{\mathbf{0}}{\mathbf{0}}$. Even so, FFD, HAFFD, BFD, HABFD, Richardson Extrapolation based on FFD, HAFFD, BFD, HABFD techniques can be used conveniently for all step sizes ($h \le 0.1$). These methods give the same exact result. Computational time is in the interval such that at minimum: 0.0156 seconds and maximum: 0.1200 for the examples in Table 2.

Moreover, the effect of decreasing step size significantly causes accuracy for all methods proposed. This situation is proved by Figures 1-12.

5. Conclusion

This paper presents alternative methods with use of CFD, BFD, FFD, HACFD, HABFD, HAFFD, Richardson Extrapolation methods based on BFD, FFD, HACFD, HABFD and HAFFD for: 0/0 of two variable functions with isolated and also non isolated singularities. So, there is no need to use L' Hôpital rule with taking partial derivatives. Two variable functions are selected according to the difficulty and not applicability of L' Hôpital rule. Numerical results for some two variable functions are given for proof of the purpose of this study. It is the first time in the literature that numerical differentiation techniques are employed for indeterminate limits of two variable functions. L'Hôpital rule is impractical and complicated for evaluations of limits for some functions.

It is also shown that FFD, HAFFD, BFD, HABFD, Richardson Extrapolation methods based on BFD, FFD, HABFD and HAFFD can be applied to: 0/0 of both functions of two variables with isolated and non isolated singularities. According to error tolerance, step size selection should be at least 10^{-4}.

As one of the future works; functions with more than two variables can be employed for indeterminate limit computations. These methods can be applied to these cases. This leads to additional computational effort. For these reasons, new algorithms can be designed to simply computations. Other future work is to employ adaptive step size such that step size can be adjusted according to error computation for each iteration with use of same methods. This situation can also be adapted to functions with more than two variables.

Other indeterminate limit forms such as: $\frac{\infty}{\infty}$, $0.\infty$ can be considered. All these numerical differentiation methods can be applied to these forms.

As potential real-world applications, the proposed methods in this paper can be used conveniently by instructors and students in calculus courses for all multivariable functions with also both isolated and non isolated singularities. So these methods can be covered in lecture books and notes as a chapter in calculus lectures. New algorithms can be designed also simplify the computation overload and minimize the computation cost and time. These algorithms can also be an additional topic to be covered in calculus lectures and recitations. Beside well-known methods, these methods proposed in this study will be a good contribution to calculus lectures.

Ethics Permissions

There is no need for ethics committee permission in this study.

Conflict of Interest

There is no conflict of interest.

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