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ON DIFFERENTIAL GEOMETRY OF THE LORENTZ SURFACES

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Abstract

In this paper we have defined the sign functions ε_1, ε_2, ε_3, ε_4, ε_5 and the vector fields X_u, X_v, n_u and n_v which have taken derivatives with (u,v) parameters of the tangent vector field X of any surface in Lorentz space and we get fundamental forms, Weingarten equations, Olin-Rodrigues and Gauss formulae. Beside these we calculate Gauss and mean curvatures.

***Keywords**:Lorenz Surface, Fundamental Forms, Curvatures, Weingarten Formulae*

Preliminaries

It is well known that in a Lorentzian Manifold we can find three types of submanifolds: Space-like (or Riemannian), time-like (Lorentzian) and light-like (degenerate or null), depending on the induced metric in the tangent vector space. Lorentz surfaces has been examined in numerous articles and books. In this article, however, we have examined some characteristics belonging to the surface by making some special choices on tangent space along the coordinate curves of the surface. Let IR^3 be endowed with the pseudoscalar product of X and Y is defined by

$$\langle X, Y\rangle = x_1y_1 + x_2y_2 - x_3y_3 \qquad X = (x_1, x_2, x_3),\ Y = (y_1, y_2, y_3)$$

$(IR^3, \langle,\rangle)$ is called 3-dimensional Lorentzian space denoted by L^3 [1]. The Lorentzian vector product is defined by

$$X \times Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

A vector fields X in L^3 is called a space-like, light-like, time-like vector field if $\langle X,X\rangle > 0$, $\langle X,X\rangle = 0$ or $\langle X,X\rangle < 0$ accordingly. For $X \in L^3$, the norm of X defined by

$$\|X\| = \sqrt{|\langle X, X\rangle|}$$

and X is called a unit vector if $\|X\| = 1$ [2].

1. INTRODUCTİON

Definition 1.1. A symmetric bilinear form b on vector space V is

i) positive [negative] definite provided $v \neq 0$ implies $b(v,v) > 0$ [< 0]

ii) positive [negative] semi-definite provided $v \geq 0$ [$v \leq 0$] for all $v \in V$

iii) non-degenerate provided $b(v,w) = 0$ for all $w \in V$ implies $v = 0$ [1].

Definition 1.2. A scalar product g on a vector space V is a non-degenerate symmetric bilinear form on V [1].

Definition 1.3. The index v of the symmetric bilinear form b on V is the largest integer that is the dimension of a subspace $W \subset V$ on which $g|_W$ is negative definite[1].

Lemma 1.4. A scalar product space $V \neq 0$ has an orthonormal basis for V , $\varepsilon_i = \langle e_i, e_i \rangle$. Then each $\alpha \in V$ has a unique expression [1],

$$\alpha = \sum_{i=1}^{n} \varepsilon_i \langle e_i, e_i \rangle e_i$$

Lemma 1.5. For any orthonormal basis $\{e_1, ..., e_n\}$ for V , the number of negative signs in the signature $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ is the index *v* of V [1].

Definition 1.6. A metric tensor *g* on a smooth manifold M is a symmetric nondegenerate (0, 2) tensor field on M of constant index [1].

Definition 1.7. A semi-Riemannian manifold is a smooth manifold furnished with a metric tensor g.

Definition 1.8. A semi-Riemannian submanifold M with (n−1)–dimensional of a semi-Riemannian manifold M with n-dimensional is called semi-Riemannian hypersurface of M [1].

2. FUNDAMENTAL FORMS

Let us denote space-like or time-like surface in L^3 as M and let the equation of M be $\vec{X}(u(t), v(t)) = \vec{X}$ with the parameter t ($t \in IR$). X_u and X_v are the tangent vector fields along coordinate curves on M and at any point these vector fields can be describe with the parameter t of the coordinate curve X respectively. The velocity vector of this curve at any point p(u, v) is,

$$X'(t) = \frac{dX}{dt} = \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt} = X_u \frac{du}{dt} + X_v \frac{dv}{dt}$$

and it is perpendicular to $X_u \times X_v$. Let ε_1, ε_2, ε_3, ε_4, ε_5 be sign functions and X_u, X_v, n tangent and normal vector fields on M and so we define the following equalities.

$$\langle n_u, n_u \rangle = \varepsilon_1 \|n_u\|^2, \quad \langle n_v, n_v \rangle = \varepsilon_2 \|n_v\|^2, \quad \langle n, n \rangle = \varepsilon_5 \|n\|^2$$

$$\langle X_u, X_u \rangle = \varepsilon_3 \|X_u\|^2, \quad \langle X_v, X_v \rangle = \varepsilon_4 \|X_v\|^2$$

$$\langle n_v, X_u \rangle = \frac{M}{\sqrt{\varepsilon_2 \varepsilon_3}}, \quad \langle n_v, X_v \rangle = \frac{N}{\sqrt{\varepsilon_2 \varepsilon_4}}, \quad \langle X_u, X_v \rangle = \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}, \quad \langle n_u, X_v \rangle = \frac{M}{\sqrt{\varepsilon_1 \varepsilon_4}}$$

$$\langle X_u, X_v \rangle = \varepsilon_3 E, \quad \langle X_v, X_v \rangle = \varepsilon_4 G, \quad \langle n_u, X_u \rangle = \frac{L}{\sqrt{\varepsilon_1 \varepsilon_3}}, \quad H = \varepsilon_3 \varepsilon_4 EG - \frac{F^2}{\varepsilon_3 \varepsilon_4}$$

Hence arc length of the curve X(t) is defined by,

[2.1] $$s = \int \|X'(t)\| dt = \varepsilon_3 E du^2 + \frac{2F}{\sqrt{\varepsilon_3 \varepsilon_4}} du dv + \varepsilon_4 G dv^2$$

Equation [2.1] is called F.F.F. quadratic form of M and we can define it as

[2.2] $$I = \varepsilon_3 E du^2 + \frac{2F}{\sqrt{\varepsilon_3 \varepsilon_4}} du dv + \varepsilon_4 G dv^2$$

Let the point $Q(u+\Delta u, v+\Delta v)$ be any near-by point at neigbourhood of $p(u,v)$ on the surface which belongs to the set C^2. Distance between tangent plane at $p(u,v)$ and at any point Q is $d=\langle \vec{n}, \overrightarrow{PQ}\rangle$ and using Taylor formula, then we can write,

$$\overrightarrow{PQ}=\Delta X=\Delta u X_u+\Delta v X_v+\frac{1}{2}\left(\Delta_u{}^2 X_{uu}+2\Delta u\Delta v X_{uv}+\Delta_v{}^2 X_{vv}+\varepsilon\right)$$

and we obtain,

$$d=\langle \vec{n}, \overrightarrow{\Delta X}\rangle=\frac{1}{2}\left(\Delta_u{}^2\langle n, X_{uu}\rangle+2\Delta u\Delta v\langle n, X_{uv}\rangle+\Delta_v{}^2\langle n, X_{vv}\rangle+\langle n,\varepsilon\rangle\right)$$

For $(\Delta u, \Delta v)\longrightarrow(0,0)$ we obtain,

$$\lim_{(\Delta_u,\Delta_v)\to(0,0)}\langle n,\varepsilon\rangle=0$$

so $\langle n,\varepsilon\rangle$ has no role to define the sign of d. For Q points which are sufficiently close to P, we can write,

$$d=\frac{1}{2}\left(\langle n, X_{uu}\rangle du^2+2\langle n, X_{uv}\rangle dudv+\langle n, X_{vv}\rangle dv^2\right)$$

Furthermore, X_u and X_v vectors are normal to n so $\langle n, X_u\rangle$ and $\langle n, X_v\rangle$ are equal to zero. Let us take derivative $\langle n, X_u\rangle$ and $\langle n, X_v\rangle$ with respect to *u* and *v*;

$$\langle n_u, X_u\rangle+\langle n, X_{uu}\rangle=0 \quad\Rightarrow\quad -\langle n_u, X_u\rangle=\langle n, X_{uu}\rangle=\frac{L}{\sqrt{\varepsilon_1\varepsilon_3}}$$

$$\langle n_v, X_v\rangle+\langle n, X_{vv}\rangle=0 \quad\Rightarrow\quad -\langle n_v, X_v\rangle=\langle n, X_{vv}\rangle=\frac{N}{\sqrt{\varepsilon_2\varepsilon_4}}$$

and use the formulae $II=\langle n, d^2X\rangle$ and $\langle n, d^2X\rangle=\langle -dn, dX\rangle$, then we obtain,

[2.3] $$II=\frac{L}{\sqrt{\varepsilon_1\varepsilon_3}}du^2+\left(\frac{1}{\sqrt{\varepsilon_1\varepsilon_4}}+\frac{1}{\sqrt{\varepsilon_2\varepsilon_3}}\right)Mdudv+\frac{N}{\sqrt{\varepsilon_2\varepsilon_4}}dv^2$$

Equation [2.3] is called S.F.F. quadratic form of M at P.

Corollary 2.1.

a) If X_u is time-like and X_v is space-like (resp. X_u space-like and X_v time-like) then surface is space-like.Thus F.F.F. for F=0,

$$I=-Edu^2+Gdv^2 \text{ (resp. } I=Edu^2-Gdv^2\text{)}$$

b) If surface is time-like then F.F.F.(resp. S.F.F) is

$$I=Edu^2+2Fdudv+Gdv^2, \text{ (resp. } II=du^2+Mdudv+dv^2\text{)}$$

3. WEİNGARTEN AND OLİN-RODRİGUES FORMULAS

Let M be a surface which has been defined with vectorial function $X=X(u,v)$ and $n(u,v)$ be unit normal vector at P(u, v) on M. Even though n_u and n_v perpendicular to $n(u,v)$, these vectors are parallel to the tangent plane at P so we can write X_u and X_v vectors in linear combination of n_u and n_v as,

$$n_u=a_{11}X_u+a_{12}X_v$$
$$n_v=a_{21}X_u+a_{22}X_v$$

We can find coefficients multiplying these equations by X_u and X_v both side and simplifying we get,

$$a_{11}=\frac{1}{EG-F^2}\left(\frac{MF}{\sqrt{\varepsilon_2\varepsilon_3}}-\frac{LG}{\sqrt{\varepsilon_1\varepsilon_3}}\right),\quad a_{12}=\frac{1}{EG-F^2}\left(\frac{LF}{\sqrt{\varepsilon_1\varepsilon_4}}-\frac{ME}{\sqrt{\varepsilon_2\varepsilon_4}}\right)$$

$$a_{21}=\frac{1}{EG-F^2}\left(\frac{NF}{\sqrt{\varepsilon_2\varepsilon_4}}-\frac{MG}{\sqrt{\varepsilon_2\varepsilon_3}}\right),\quad a_{22}=\frac{1}{EG-F^2}\left(\frac{MF}{\sqrt{\varepsilon_2\varepsilon_4}}-\frac{NE}{\sqrt{\varepsilon_2\varepsilon_3}}\right)$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ coefficients are called Weingarten coefficients and we show in matrix as $S=\lfloor a_{ij}\rfloor$. So the Gauss curvature of the surface will be follow.

$$K=\frac{\det II}{\det I}=\frac{\dfrac{4LN}{\varepsilon_2\sqrt{\varepsilon_1\varepsilon_4}}-\left(\dfrac{1}{\sqrt{\varepsilon_1\varepsilon_2}}-\dfrac{1}{\sqrt{\varepsilon_2\varepsilon_3}}\right)^2 M^2}{4EG-4F^2}$$

If $F=0$ and $M=0$ than,

$$K=\frac{\det II}{\det I}=\frac{LN}{\varepsilon_2\sqrt{\varepsilon_1\varepsilon_4}EG}$$

and mean curvature of M is

$$H=trc(S)=\frac{1}{EG-F^2}\left(\frac{MF}{\sqrt{\varepsilon_2\varepsilon_3}}-\frac{LG}{\sqrt{\varepsilon_1\varepsilon_3}}+\frac{MF}{\sqrt{\varepsilon_2\varepsilon_4}}-\frac{NE}{\sqrt{\varepsilon_2\varepsilon_3}}\right)$$

and since $F=0$ and $M=0$ than,

$$H=trc(S)=\frac{-1}{EG}\left(\frac{LG}{\sqrt{\varepsilon_1\varepsilon_3}}+\frac{NE}{\sqrt{\varepsilon_2\varepsilon_3}}\right)$$

Corollary 3.1.

a) The matrix S of the time-like surfaces for both $F\neq 0$ and $F=0$ will be as follows respectively.

$$\frac{1}{EG-F^2}\begin{bmatrix}MF-LG & LG-ME\\ NF-MG & MF-NE\end{bmatrix},\ (\text{resp. } -\frac{1}{EG}\begin{bmatrix}LG & ME\\ MG & NE\end{bmatrix})$$

b) Space-like surface's shape operator matrix for $F\neq 0$ and $F=0$ has complex coefficients.

If coordinate lines are perpendicular at every point on the surface then $\langle X_u, X_v\rangle=0$ and $F=0$. So the new equations are;

$$n_u=\frac{-L}{\sqrt{\varepsilon_1\varepsilon_3}E}X_u+\frac{-M}{\sqrt{\varepsilon_2\varepsilon_4}G}X_v,$$

$$n_v=\frac{-M}{\sqrt{\varepsilon_2\varepsilon_3}E}X_u+\frac{-N}{\sqrt{\varepsilon_2\varepsilon_3}G}X_v.$$

If $F=0$ and $M=0$ then we get

[3.1] $$n_u+\frac{r}{\sqrt{\varepsilon_1\varepsilon_3}}X_u=0\quad,\quad n_v+\frac{\bar{r}}{\sqrt{\varepsilon_2\varepsilon_3}}X_v=0$$

where r and $\bar{r}$ are

$$r=\frac{L}{E}\quad,\quad \bar{r}=\frac{N}{G}.$$

[3.1] equations which we have obtained are called Olin-Rodrigues formulae.

Corollary 3.2. Olin-Rodrigues equations of the time-like and space-like surface are

$$n_u+rX_u=0\quad,\quad n_v+\bar{r}X_v=0$$

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