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Research article

Spacetimes and Almost Ricci-Yamabe Solitons

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Abstract

This article investigates almost Ricci-Yamabe solitons and gradient almost Ricci-Yamabe solitons in spacetimes. Initially, we demonstrate that if a spacetime allows an almost Ricci-Yamabe soliton with a conformal vector field as potential vector field, then the spacetime turns into an Einstein spacetime. Next, we examine that if a spacetime admits an almost Ricci-Yamabe soliton with a recurrent vector field as potential vector field, then the spacetime becomes perfect fluid spacetime. Then, it is shown that if a generalized Robertson-Walker spacetime admits an almost Ricci-Yamabe soliton or a gradient almost Ricci-Yamabe soliton, then it represents a perfect fluid spacetime. Consequently, we derive a number of interesting corollaries. We conclude providing an example of an almost Ricci-Yamabe solitons.

Keywords: *Perfect fluid spacetimes; generalized Robertson-Walker spacetimes; almost Ricci-Yamabe solitons.*
AMS Subject Classification (2020): *53C50, 53E20, 53Z05, 83C05.*

1. Introduction

The geometric theory of gravity is the common name for Einstein's "Theory of General Relativity". Perhaps the finest broadly accepted theories of physics in the last century, general relativity (GR), has shown the fundamental relationship between physics and spacetime geometry. Over the past century, it has been regarded as among the most vibrant fields of study in both physics and mathematics. Apart from its pivotal role in theoretical study, GR has also found notable achievements in engineering when implemented in real-world situations. Finding alternative solution to Einstein's field equations has emerged as the greatest significant issues of our time and the most apparent answer is the Minkowski spacetime.The Schwartzchild solution, de-Sitter, Kerr, and other non-trivial solutions are also included. In GR, warped product Lorentzian manifolds were modified in order to provide a general solution to Einstein's field equations. Standard static spacetime and generalized Robertson-Walker spacetime (GRW) [3] are two well-known instances.

A spacetime is described as a 4-dimensional Lorentzian manifold M that is time-oriented. In [1] Alias et al. presented the idea of GRW spacetimes. When M^n is formed as a warped product $M = -I \times_{\varphi^2} M^*$, with $I \subset \mathbb{R}$, M^* representing a $(n-1)$-dimensional Riemannian manifold, and $\varphi > 0$ representing a warping function, the spacetime is referred to as a GRW spacetime. If M^* has dimension-3 and is of constant curvature, the spacetime becomes Robertson-Walker (RW) spacetime. There have been several studies of the geometrical and physical characteristics of GRW spacetimes (see, [3], [5], [12], [15]).

The Ricci tensor S in perfect fluid spacetime (PFS) has the shape:

$$S = ag + b\omega \otimes \omega, \tag{1.1}$$

in which a and b stand for scalars, ω is a non zero 1-form defined by $\omega(X_1)$=$g(X_1, \rho_1)$ for all X_1 and ρ_1 is a unit time-like vector field (Here after we will denote vector field by VF), that is, $g(\rho_1, \rho_1) = -1$, named the velocity VF or flow VF. In particular, if $b = 0$, then the PFS becomes an Einstein spacetime.

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The conformal curvature tensor C, also known as the Weyl tensor plays a vital role in GR, cosmology and in the differential geometry. The conformal curvature tensor C is demonstrated as

$$\begin{aligned} C(X_1,Y_1)Z_1 &= R(X_1,Y_1)Z_1 \\ &- \frac{1}{n-2}[g(QY_1,Z_1)X_1 - g(QX_1,Z_1)Y_1 + g(Y_1,Z_1)QX_1 - g(X_1,Z_1)QY_1] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1] \end{aligned} \tag{1.2}$$

for all X_1, Y_1, $Z_1 \in \mathfrak{X}(M)$ and r represents the scalar curvature, R is the Riemann curvature tensor, Q stands for the Ricci operator stated by $g(QX_1,Y_1) = S(X_1,Y_1)$.

Further, we are aware of that

$$\begin{aligned} (div\, C)(X_1,Y_1)Z_1 &= \frac{n-3}{n-2}[\{(\nabla_{X_1}S)(Y_1,Z_1) - (\nabla_{Y_1}S)(X_1,Z_1)\} \\ &- \frac{1}{2(n-1)}\{(X_1 r)g(Y_1,Z_1) - (Y_1 r)g(X_1,Z_1)\}], \end{aligned} \tag{1.3}$$

'div' denotes the divergence.

The term conformal VF refers to a VF V_1 on a semi-Riemannian manifold M if

$$\mathfrak{L}_{V_1} g = 2hg, \tag{1.4}$$

for some smooth function h. If h is constant then the VF V_1 is said to be homothetic VF.

A VF V_1 on M is named recurrent VF if

$$\nabla_{X_1} V_1 = A(X_1)V_1, \tag{1.5}$$

where A is a 1-form demonstrated by $A(X_1) = g(X_1,V_1)$ for all $X_1 \in \mathfrak{X}(M)$.

Sharma derived the following Theorems

Theorem A. ([20]) With a non-homothetic conformal VF and a divergence-free Weyl tensor C, a spacetime (M, g) is locally of type O or N.

Theorem B. ([21]) A PFS's Weyl tensor is divergence-free iff M is irrotational, shear-free, and has a constant energy density across the space like hypersurface orthogonal to the 4-velocity vector. In the situation that M allows a proper conformal VF, it is locally of type O or N and conformally flat.

Yano [23] presented the concept of a torse-forming VF V_1 and V_1 on M is said to be a torse-forming VF if for every VF X_1

$$\nabla_{X_1} V_1 = \phi X_1 + A(X_1)V_1 \tag{1.6}$$

in which ϕ indicates a scalar and A is the 1-form. If $A = 0$, then the VF V_1 is named as concircular[11].

The following relation is satisfied by a unit torse-forming and time-like VF V_1:

$$\nabla_{X_1} V_1 = \phi[X_1 + A(X_1)V_1]. \tag{1.7}$$

Additionally, in [15] the subsequent theorem has been derived:

Theorem C.([15]) An n ($n \geq 3$) dimensional Lorentzian manifold represents a GRW space-time iff it allows a time-like and unit torse-forming VF: $\nabla_{X_1}\rho_1 = \phi[X_1 + \omega(X_1)\rho_1]$, ϕ is a smooth function and ω stands for a one-form defined by $g(X_1,\rho_1) = \omega(X_1)$ for all $X_1 \in \mathfrak{X}(M)$, which is an eigenvector of the Ricci tensor.

Theorem D. [17]A PFS is a GRW spacetime iff $div\, C = 0$.

Theorem E. ([16]) In all GRW spacetime with a velocity vector ρ_1, (div $C)(X_1,Y_1)Z_1 = 0$ iff $C(X_1,Y_1)\rho_1 = 0$ for every X_1, Y_1, $Z_1 \in \mathfrak{X}_1(M)$.

In the recent years many researchers have shown interest in the theory of geometric flows, including Yamabe and Ricci flows, and associated solitons. In 2019, a new geometric flow that is a scalar combination of the

Yamabe and Ricci flows was introduced by Guler and Crasmareanu [13]. Another term for this is (α, β) type Ricci-Yamabe flow. The Ricci-Yamabe flow, stated as:[13]

$$\frac{\partial}{\partial t} g(t) = -2\alpha \mathcal{S}(t) + \beta r(t) g(t), \quad g_0 = g(0), \tag{1.8}$$

in which $\alpha, \beta \in \mathbb{R}$, and $\mathcal{S}$ indicates the Ricci tensor.

An almost Ricci-Yamabe soliton(ARYS) on (M, g) is stated by

$$\pounds_{W_1} g + 2\alpha S + (2\lambda - \beta r) g = 0, \tag{1.9}$$

where $\pounds$ being the Lie-derivative, λ is the smooth function on M, called soliton function. Here f is a smooth function on M and W_1 is the potential VF on M.

If the potential VF W_1 is the gradient of f (that is, $W_1 = Df$), then the above concept is called as gradient ARYS and then equation (1.9) becomes

$$\nabla^2 f + \alpha \mathcal{S} + (\lambda - \frac{1}{2}\beta r) g = 0, \tag{1.10}$$

$\nabla^2 f$ stands for the Hessian.

For $\lambda > 0$, the ARYS (or gradient ARYS) is called expanding; for $\lambda = 0$, it is said to be steady; and for $\lambda < 0$, it is said to be shrinking.

It should be observed that the equation (1.9) becomes

- almost Ricci soliton if $\alpha = 1, \ \beta = 0$;
- almost Yamabe soliton if $\alpha = 0, \ \beta = 1$;
- almost Einstein soliton if $\alpha = 1 \ , \ \beta = -1$;
- Ricci-Yamabe soliton if $\lambda = constant$;

Many researchers studied different type of geometric solitons on spacetime. For instance, De et al. ([9]) studied gradient Ricci solitons on PFS. Chen-Deshmukh [4] investigated Yamabe solitons on PFS. Blaga [2] studied η-Ricci solitons on PFS. Singh-Khatri [19] and Siddiqi [18] investigated Ricci-Yamabe solitons on PFS.

Recently, De et.al. ([7], [8]) studied gradient Ricci solitons, (m, τ)-quasi Einstein solitons in PFS and Ricci-Yamabe solitons on $f(R)$-gravity, respectively.

Motivated by the cited works in this article we study ARYS and AGRYS on spacetimes.

2. Preliminaries

Let us consider the potential VF $W_1 = \rho_1$ is a conformal VF and hence from (1.4), we obtain

$$g(\nabla_{X_1} \rho_1, Y_1) + g(X_1, \nabla_{Y_1} \rho_1) = 2h g(X_1, Y_1) \tag{2.1}$$

If the VF $W_1 = \rho_1$ is recurrent, then from (1.5) we have

$$g(\nabla_{X_1} \rho_1, Y_1) + g(X_1, \nabla_{Y_1} \rho_1) = 2\omega(X_1)\omega(Y_1) \tag{2.2}$$

If the potential VF $W_1 = \rho_1$ is unit torse-forming, then utilizing Theorem C, we provide

$$\nabla_{X_1} \rho_1 = \phi[X_1 + \omega(X_1)\rho_1] \tag{2.3}$$

and

$$S(X_1, \rho_1) = \xi \omega(X_1), \tag{2.4}$$

ϕ stands for a scalar and ξ indicates a eigenvector (non-zero).

Lemma 2.1. *([10]) For a GRW spacetime, we get*

$$R(X_1, Y_1)\rho_1 = (\rho_1 \phi + \phi^2)[\omega(Y_1)X_1 - \omega(X_1)Y_1] \tag{2.5}$$

and

$$S(X_1, \rho_1) = (n-1)(\rho_1 \phi + \phi^2)\omega(X_1). \tag{2.6}$$

Lemma 2.2. *([6]) For a GRW-spacetime, we provide*

$$g((\nabla_{\rho_1} Q)X_1, \rho_1) - g((\nabla_{X_1} Q)\rho_1, \rho_1) = 0. \tag{2.7}$$

3. Almost Ricci-Yamabe solitons

Let the spacetime (M, g) admit an ARYS with potential VF $W_1 = \rho_1$. Then from equation (1.9), we get

$$(\pounds_{W_1} g)(X_1, Y_1) + 2\alpha\mathcal{S}(X_1, Y_1) + (2\lambda - \beta r)g(X_1, Y_1) = 0, \tag{3.1}$$

which implies

$$g(\nabla_{X_1}\rho_1, Y_1) + g(X_1, \nabla_{Y_1}\rho_1) + 2\alpha\mathcal{S}(X_1, Y_1) + (2\lambda - \beta r)g(X_1, Y_1) = 0. \tag{3.2}$$

I. If the potential VF is conformal, then utilizing (2.1) in the foregoing equation, we obtain

$$\alpha\mathcal{S}(X_1, Y_1) = -(\lambda - \frac{1}{2}\beta r + h)g(X_1, Y_1), \tag{3.3}$$

which represents Einstein spacetime.

Theorem 3.1. *If a spacetime admits an ARYS whose potential VF is conformal, then the spacetime becomes an Einstein spacetime.*

Using (3.3) in the equation (1.3), we acquire $(div\, C)(X_1, Y_1)Z_1 = 0$. Hence, in view of the Theorem A, we write:

Corollary 3.1. *If a spacetime admits an ARYS with the potential VF as a conformal VF, then the spacetime is locally either of type O or N.*

II. If the potential VF is a recurrent, then making use of (2.2) in the equation (3.2), we get

$$\alpha\mathcal{S}(X_1, Y_1) = -(\lambda - \frac{1}{2}\beta r)g(X_1, Y_1) - \omega(X_1)\omega(Y_1), \tag{3.4}$$

which represents PFS.

Theorem 3.2. *If a spacetime admits an ARYS with the potential VF as a recurrent VF, then the spacetime becomes a PFS.*

If we assume $div\, C = 0$ in a 4-dimensional PFS , then in view of the theorem **B**, we state:

Corollary 3.2. *A spacetime admitting an ARYS whose potential VF is a recurrent VF and satisfying divergence-free conformal curvature tensor is irrotational, shear-free and its energy-density is constant over the spacelike hypersurface orthogonal to the 4-velocity vector.*

III. If the potential VF is a unit torse-forming VF, then using (2.3) in the equation (3.2), we provide

$$\alpha\mathcal{S}(X_1, Y_1) = -(\lambda - \frac{1}{2}\beta r + \phi)g(X_1, Y_1) - \phi\omega(X_1)\omega(Y_1), \tag{3.5}$$

which represents PFS.

Therefore, we can state the result as:

Theorem 3.3. *If a GRW spacetime allows an ARYS, then it becomes a PFS.*

In view of the Theorem D, we acquire $div\, C = 0$ and thus Theorem E entails that in a 4-dimensional GRW spacetime, $C(X_1, Y_1)\rho_1 = 0$ iff $(div\, C)(X_1, Y_1)Z_1 = 0$. The literal meaning of $C(X_1, Y_1)\rho_1 = 0$ is that the conformal or the Weyl tensor is purely electric [14] and from that we say the spacetime is of Petrov type I, D or O [22].
Therefore, we have:

Corollary 3.3. *If a 4-dimensional GRW spacetime admits an ARYS, then the Weyl tensor is purely electric and the spacetime is of Petrov type I, D or O.*

If we take $\alpha = 1$, $\beta = 0$, then equation (3.5) provides

$$\mathcal{S}(X_1, Y_1) = -(\lambda + \phi)g(X_1, Y_1) - \phi\omega(X_1)\omega(Y_1), \tag{3.6}$$

which means it is a PFS.

Setting $X_1 = Y_1 = \rho_1$ in (3.6) and using equation (2.6) we have

$$\lambda = -(n-1)(\rho_1\phi + \phi^2).$$

Corollary 3.4. *If a GRW spacetime allows an almost Ricci soliton, then the spacetime becomes a PFS and the soliton is expanding, steady or shrinking if* $(\rho_1\phi + \phi^2) <, = \text{ or } > 0$*, respectively.*

For $\alpha = 0$, $\beta = 1$, the equation (3.5) entails

$$(\lambda - \frac{1}{2}r + \phi)g(X_1, Y_1) = \phi\omega(X_1)\omega(Y_1), \tag{3.7}$$

Setting $X_1 = Y_1 = \rho_1$ in the previous equation yields

$$\lambda = \frac{1}{2}r.$$

Hence, we state:

Corollary 3.5. *If a GRW spacetime allows an almost Yamabe soliton, then* $r > 0, = 0$ *or* < 0*, respectively, indicates that the soliton is expanding, steady or shrinking.*

4. Gradient Almost Ricci-Yamabe solitons

Let the GRW spacetime allows a gradient ARYS. Then equation (1.10) yields

$$\nabla_{X_1} Df = -\alpha\mathcal{Q}X_1 - (\lambda - \frac{\beta}{2}r)X_1. \tag{4.1}$$

Covariant differentiation of (4.1) provides

$$\nabla_{Y_1}\nabla_{X_1} Df = -\alpha\nabla_{Y_1}\mathcal{Q}X_1 - (\lambda - \frac{\beta}{2}r)\nabla_{Y_1}X_1 + \frac{\beta}{2}(Y_1 r)X_1 - (Y_1\lambda)X_1. \tag{4.2}$$

Interchanging X_1 and Y_1 in (4.2), we get

$$\nabla_{X_1}\nabla_{Y_1} Df = -\alpha\nabla_{X_1}\mathcal{Q}Y_1 - (\lambda - \frac{\beta}{2}r)\nabla_{X_1}Y_1 + \frac{\beta}{2}(X_1 r)Y_1 - (X_1\lambda)Y_1. \tag{4.3}$$

Using the equation (4.1), we have

$$\nabla_{[X_1,Y_1]} Df = -\alpha\mathcal{Q}(\nabla_{X_1}Y_1 - \nabla_{Y_1}X_1) - (\lambda - \frac{\beta}{2}r)(\nabla_{X_1}Y_1 - \nabla_{Y_1}X_1). \tag{4.4}$$

In light of (4.2)-(4.4), we infer

$$\begin{aligned} \mathcal{R}(X_1, Y_1)Df \;&=\; -\alpha[(\nabla_{X_1}\mathcal{Q})Y_1 - (\nabla_{Y_1}\mathcal{Q})X_1] + \frac{\beta}{2}[(X_1 r)Y_1 - (Y_1 r)X_1] \\ &\quad -(X_1\lambda)Y_1 + (Y_1\lambda)X_1. \end{aligned} \tag{4.5}$$

Taking inner product of (4.5) with ρ_1 and using lemma 2.2, we infer

$$g(\mathcal{R}(X_1, Y_1)Df, \rho_1) = \frac{\beta}{2}[(X_1 r)\omega(Y_1) - (Y_1 r)\omega(X_1)] - (X_1\lambda)\omega(Y_1) + (Y_1\lambda)\omega(X_1). \tag{4.6}$$

Again from equation (2.5) it follows that

$$g(\mathcal{R}(X_1, Y_1)\rho_1, Df) = (\rho_1\phi + \phi^2)[(X_1 f)\omega(Y_1) - (Y_1 f)\omega(X_1)]. \tag{4.7}$$

Jointly the equations (4.6) and (4.7) produce

$$\begin{aligned} (\rho_1\phi + \phi^2)[(X_1 f)\omega(Y_1) - (Y_1 f)\omega(X_1)] \;&=\; -\frac{\beta}{2}[(X_1 r)\omega(Y_1) - (Y_1 r)\omega(X_1)] \\ &\quad -(X_1\lambda)\omega(Y_1) + (Y_1\lambda)\omega(X_1). \end{aligned} \tag{4.8}$$

Setting $Y_1 = \rho_1$ in the above equation we get

$$(\rho_1\phi+\phi^2)[(X_1f)+(\rho_1f)\omega(X_1)] = -\frac{\beta}{2}[(X_1r)+(\rho_1r)\omega(X_1)] +(X_1\lambda)+(\rho_1\lambda)\omega(X_1). \tag{4.9}$$

If we take $r = constant$ and $\lambda = f$, then equation (4.9) provides

$$(\rho_1\phi+\phi^2-1)[(X_1f)+(\rho_1f)\omega(X_1)] = 0, \tag{4.10}$$

which implies

$$[(X_1f)+(\rho_1f)\omega(X_1)] = 0,\ since(\rho_1\phi+\phi^2-1)\neq 0. \tag{4.11}$$

The above equation reduces to

$$Df = -(\rho_1f)\rho_1. \tag{4.12}$$

The covariant derivative of equation (4.12) yields

$$\nabla_{X_1}Df = -\{X_1(\rho_1f)\}\rho_1 - \phi(\rho_1f)\{X_1+\omega(X_1)\rho_1\}, \tag{4.13}$$

Equations (4.1) and (4.13) together implies

$$\{X_1(\rho_1f)\}\omega(Y_1)+\phi(\rho_1f)[g(X_1,Y_1)+\omega(X_1)\omega(Y_1)] = \alpha\mathcal{S}(X_1,Y_1) +(\lambda-\frac{1}{2}\beta r)g(X_1,Y_1). \tag{4.14}$$

Setting $Y_1 = \rho_1$ in (4.14) entails that

$$\{X_1(\rho_1f)\} = -\{\alpha(n-1)(\rho_1\phi+\phi^2)+(\lambda-\frac{1}{2}\beta r)\}\omega(X_1). \tag{4.15}$$

Using (4.15) in the equation (4.14) we obtain

$$\alpha\mathcal{S}(X_1,Y_1) = \{\phi(\rho_1f)-(\lambda-\frac{1}{2}\beta r)\}g(X_1,Y_1)+\{\alpha(n-1)(\rho_1\phi+\phi^2)+\phi(\rho_1f)\}\omega(X_1)\omega(Y_1), \tag{4.16}$$

which represent a PFS.

Theorem 4.1. *A GRW spacetime having constant scalar curvature and admitting a gradient ARYS is a PFS, provided $\lambda = f$.*

Remark: Corollary 3.3 also holds for gradient almost Ricci-Yamabe soliton.

5. Example

Let $M^4 = \{(x,y,z,t)\in\mathbb{R}^4, t\neq 0\}$, in which (x,y,z,t) indicates the standard coordinate of $\mathbb{R}^4$. Let us choose

$$\upsilon_1 = e^t\frac{\partial}{\partial x},\quad \upsilon_2 = e^t\frac{\partial}{\partial y},\quad \upsilon_2 = e^t\frac{\partial}{\partial z},\quad \upsilon_3 = \frac{\partial}{\partial t}. \tag{5.1}$$

The VFs { υ_1, υ_2, υ_3 , υ_4 } are linearly independent and describe the semi-Riemannian metric g as

$$g(\upsilon_1,\upsilon_2) = g(\upsilon_1,\upsilon_3) = g(\upsilon_1,\upsilon_4) = g(\upsilon_2,\upsilon_3) = g(\upsilon_2,\upsilon_4) = g(\upsilon_3,\upsilon_4) = 0,$$

$$g(\upsilon_1,\upsilon_1) = g(\upsilon_2,\upsilon_2) = g(\upsilon_3,\upsilon_3) = 1, g(\upsilon_4,\upsilon_4) = -1.$$

Suppose ω is a 1-form described by $\omega(X_1) = g(X_1, \rho_1)$ for all $X_1 \in \mathfrak{X}(M^4)$.

The Lie brackets are calculated as

$$\begin{aligned}[][\upsilon_1, \upsilon_4] &= \upsilon_1\upsilon_4 - \upsilon_4\upsilon_1 \\ &= e^t \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\right)\left(e^t \frac{\partial}{\partial x}\right) \\ &= e^t \frac{\partial^2}{\partial x \partial t} - e^t \frac{\partial^2}{\partial t \partial x} - e^t \frac{\partial}{\partial x} \\ &= -\upsilon_1. \end{aligned} \tag{5.2}$$

Similarly,

$$[\upsilon_1, \upsilon_1] = [\upsilon_1, \upsilon_2] = [\upsilon_1, \upsilon_3] = 0, \quad [\upsilon_2, \upsilon_2] = [\upsilon_2, \upsilon_3] = 0, \quad [\upsilon_2, \upsilon_4] = -\upsilon_2,$$
$$[\upsilon_3, \upsilon_3] = 0, \quad [\upsilon_3, \upsilon_4] = -\upsilon_3, \quad [\upsilon_4, \upsilon_4] = 0.$$

Therefore, the semi-Riemannian connection ∇ is written by

$$\begin{gathered} \nabla_{\upsilon_1}\upsilon_1 = \upsilon_4, \quad \nabla_{\upsilon_1}\upsilon_2 = 0, \quad \nabla_{\upsilon_1}\upsilon_3 = 0, \quad \nabla_{\upsilon_1}\upsilon_4 = -\upsilon_1, \\ \nabla_{\upsilon_2}\upsilon_1 = 0, \quad \nabla_{\upsilon_2}\upsilon_2 = \upsilon_4, \quad \nabla_{\upsilon_2}\upsilon_3 = 0, \quad \nabla_{\upsilon_2}\upsilon_4 = -\upsilon_2, \\ \nabla_{\upsilon_3}\upsilon_1 = 0, \quad \nabla_{\upsilon_3}\upsilon_2 = 0, \quad \nabla_{\upsilon_3}\upsilon_3 = \upsilon_4, \quad \nabla_{\upsilon_3}\upsilon_4 = -\upsilon_3, \\ \nabla_{\upsilon_4}\upsilon_1 = 0, \quad \nabla_{\upsilon_4}\upsilon_2 = 0, \quad \nabla_{\upsilon_4}\upsilon_3 = \upsilon_4, \quad \nabla_{\upsilon_4}\upsilon_4 = 0. \end{gathered}$$

The non-zero Riemannian curvature tensor are

$$\begin{gathered} R(\upsilon_1, \upsilon_2)\upsilon_2 = -\upsilon_1, \quad R(\upsilon_1, \upsilon_3)\upsilon_3 = -\upsilon_1, \quad R(\upsilon_1, \upsilon_4)\upsilon_4 = -\upsilon_1, \quad R(\upsilon_1, \upsilon_2)\upsilon_1 = \upsilon_2, \\ R(\upsilon_1, \upsilon_3)\upsilon_1 = -\upsilon_3, \quad R(\upsilon_1, \upsilon_4)\upsilon_1 = \upsilon_4, \quad R(\upsilon_2, \upsilon_3)\upsilon_3 = -\upsilon_2, \quad R(\upsilon_2, \upsilon_3)\upsilon_2 = \upsilon_3, \\ R(\upsilon_2, \upsilon_4)\upsilon_2 = \upsilon_4, \quad R(\upsilon_2, \upsilon_4)\upsilon_4 = -\upsilon_2, \quad R(\upsilon_3, \upsilon_4)\upsilon_3 = \upsilon_4, \quad R(\upsilon_4, \upsilon_3)\upsilon_4 = \upsilon_3. \end{gathered}$$

Thus, we get

$$\mathcal{S}(\upsilon_1, \upsilon_1) = \mathcal{S}(\upsilon_2, \upsilon_2) = \mathcal{S}(\upsilon_3, \upsilon_3) = \mathcal{S}(\upsilon_4, \upsilon_4) = -3.$$

and

$$r = \mathcal{S}(\upsilon_1, \upsilon_1) + \mathcal{S}(\upsilon_2, \upsilon_2) + \mathcal{S}(\upsilon_3, \upsilon_3) + \mathcal{S}(\upsilon_4, \upsilon_4) = -12.$$

Also, we have

$$\begin{gathered} (\pounds_{\upsilon_4} g)(\upsilon_1, \upsilon_1) = (\pounds_{\upsilon_4} g)(\upsilon_2, \upsilon_2) = (\pounds_{\upsilon_4} g)(\upsilon_3, \upsilon_3) = -2, \\ (\pounds_{\upsilon_4} g)(\upsilon_4, \upsilon_4) = 0. \end{gathered}$$

and

$$(\omega \otimes \omega)(\upsilon_1, \upsilon_1) = (\omega \otimes \omega)(\upsilon_2, \upsilon_2) = (\omega \otimes \omega)(\upsilon_3, \upsilon_3) = 0, \quad (\omega \otimes \omega)(\upsilon_4, \upsilon_4) = 1.$$

If we assume that $W_1 = \upsilon_4$, $\alpha = -\frac{1}{6}$, $\beta = -\frac{1}{12}$ and $\lambda = 1$, then $(g, W_1, \alpha, \beta, \lambda)$ is an ARYS on the manifold M^4.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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