

## A NUMERICAL SOLUTION OF THE ADVECTION-DIFFUSION EQUATION BY USING EXTENDED CUBIC B-SPLINE FUNCTIONS

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### ABSTRACT

In this paper, numerical solution of the advection-diffusion equation is obtained by using extended cubic B-spline functions. For space discretization, the extended cubic B-spline Galerkin method is used to integrate the advection-diffusion equation and for time discretization, the Crank-Nicolson method is employed to obtain the fully integrated advection-diffusion equation. The maximum error norm has been used to show the accuracy of the method. Robustness of the suggested method is shown by studying some classical test problems and comparing the results with some earlier ones.

**Keywords:** Extended cubic B-spline, Galerkin finite element method, Advection-diffusion equation

### 1. INTRODUCTION

It is well-known that many real life problems in physics and engineering can be modelled by the advection-diffusion equation (ADE) which describes phenomena including weak nonlinearity and dispersion waves. Since the solutions of the ADE include the sharp behavior with some selection of parameters, the numerical methods are of interest due to modelling of steep solutions. Many studies have been existed for the numerical solutions of the ADE using spline functions so far [1-23]. By keeping the continuity, adding higher order terms with a free parameter to the B-spline which is a piecewise function, the extended B-spline function is obtained. The use of different free parameters changes the shape of extended B-spline. The effect of the additional term and free parameters for the extended B-splines are discussed in the studies [24, 25].

Our aim in this work is to investigate the results obtained by using extended B-spline functions with a combination of Galerkin and Crank-Nicolson methods to find numerical solutions of the ADE. Although the extended B-spline function has not yet been extensively used for the numerical solutions of partial differential equations, several studies are available in the literature [14, 26, 27]. In parallel with those, we use the extended cubic B-spline Galerkin method (ECBSGM) that is not performed before.

### 2. GOVERNING EQUATION

The one dimensional ADE which describes the transport and diffusion processes is

$$\frac{\partial u}{\partial t} + \xi \frac{\partial u}{\partial x} - \mu \frac{\partial^2 u}{\partial x^2} = 0, \quad (1)$$

where the function  $u(x,t)$  indicates the concentration at position  $x$  and time  $t$ ,  $\xi$  and  $\mu$  represent the uniform flow velocity parameter and the constant diffusion coefficient, respectively. The initial and boundary conditions of Eq. (1) are

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$$u(x, 0) = u_0(x), \quad 0 \leq x \leq L \tag{2}$$

$$u(0, t) = f_0(t), \quad u(L, t) = f_L(t) \quad \text{or} \quad -\mu \frac{\partial u}{\partial x} \Big|_L = \varphi_L(t) \tag{3}$$

where the parameters  $L$  and  $\varphi_L$  demonstrate the length of the channel and the flux at the boundary  $x = L$ , respectively and  $u_0, f_0, f_L$  are imposed functions.

### 3. EXTENDED CUBIC B-SPLINE GALERKIN METHOD

Let we take a uniform mesh with the knots  $x_l = x_0 + lh$  on  $[a, b]$  and  $Q_l(x)$  be the extended cubic B-splines at these points with knots  $x_l, l = -3, -2, -1, N+1, N+2, N+3$  outside the interval  $[a, b]$  and having a finite support on the four consecutive intervals  $[x_l + kh, x_l + (k+1)h]_{k=-3}^0, l = 0, \dots, N+2$ . The extended cubic B-spline is given as

$$Q_l(x) = \frac{1}{24h^2} \begin{cases} 4h(1-\lambda)(x-x_{l-2})^3 + 3\lambda(x-x_{l-2})^4 & , \quad \text{if } x \in [x_{l-2}, x_{l-1}]; \\ (4-\lambda)h^4 + 12h^3(x-x_{l-1}) + 6h^2(2+\lambda)(x-x_{l-1})^2 \\ -12h(x-x_{l-1})^3 - 3\lambda(x-x_{l-1})^4 & , \quad \text{if } x \in [x_{l-1}, x_l]; \\ (4-\lambda)h^4 + 12h^3(x_{l+1}-x) + 6h^2(2+\lambda)(x_{l+1}-x)^2 \\ -12h(x_{l+1}-x)^3 - 3\lambda(x_{l+1}-x)^4 & , \quad \text{if } x \in [x_l, x_{l+1}]; \\ 4h(1-\lambda)(x_{l+2}-x)^3 + 3\lambda(x_{l+2}-x)^4 & , \quad \text{if } x \in [x_{l+1}, x_{l+2}]; \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{4}$$

The values of  $Q_l(x), Q'_l(x)$  and  $Q''_l(x)$  at the knots  $x_l$ 's are tabulated in Table 1. The  $Q_l(x), l = -1, K, N+1$  form a basis for functions defined on the interval  $[a, b]$ .

**Table 1.** Extended cubic B-spline values at the knot points

	$x_{l-2}$	$x_{l-1}$	$x_l$	$x_{l+1}$	$x_{l+2}$
$Q_l(x)$	0	$\frac{4-\lambda}{24}$	$\frac{8+\lambda}{12}$	$\frac{4-\lambda}{24}$	0
$Q'_l(x)$	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
$Q''_l(x)$	0	$\frac{2+\lambda}{2h^2}$	$-\frac{2+\lambda}{h^2}$	$\frac{2+\lambda}{2h^2}$	0

We seek an approximation  $U$  to the analytical solution  $u$  in terms for the extended cubic B-splines

$$u(x, t) \approx U(x, t) = \sum_{l=-1}^{N+1} Q_l(x) \delta_l(t) \tag{5}$$

where  $\delta_l(t)$  are unknown parameters to be obtained by Galerkin approach to the ADE.  $U$  and its derivatives can be calculated at each knot by the following equations:

$$\begin{aligned}
 U_l &= U(x_l, t) = \frac{4-\lambda}{24} \delta_{l-1} + \frac{8+\lambda}{12} \delta_l + \frac{4-\lambda}{24} \delta_{l+1}, \\
 U'_l &= U'(x_l, t) = -\frac{1}{2h} \delta_{l-1} + \frac{1}{2h} \delta_{l+1}, \\
 U''_l &= U''(x_l, t) = \frac{2+\lambda}{2h^2} \delta_{l-1} - \frac{2+\lambda}{h^2} \delta_l + \frac{2+\lambda}{2h^2} \delta_{l+1}.
 \end{aligned}
 \tag{6}$$

When we choose the extended cubic B-splines as weight functions over the interval  $[0, L]$ , we obtain following integral for the ADE by Galerkin method:

$$\int_0^L Q_l(x) (u_t + \xi u_x - \mu u_{xx}) dx = 0.
 \tag{7}$$

Over the sub element  $[x_k, x_{k+1}]$ , Eqs. (5) and (7) can be rewritten as

$$U^e = Q_{k-1}(x) \delta_{k-1}(t) + Q_k(x) \delta_k(t) + Q_{k+1}(x) \delta_{k+1}(t) + Q_{k+2}(x) \delta_{k+2}(t)
 \tag{8}$$

and

$$\int_{x_k}^{x_{k+1}} Q_j(x) (u_t + \xi u_x - \mu u_{xx}) dx
 \tag{9}$$

respectively, where quantities  $\delta_i(t)$ ,  $i = k-1, \dots, k+2$  are element parameters and  $Q_j(x)$ ,  $j = k-1, \dots, k+2$  are known as the element shape functions.

According to the Galerkin discretization scheme, replacing  $U_l, U_x, U_{xx}$  which are derivatives of the approximate solution  $U^e$  in Eq. (8), into  $u_t, u_x, u_{xx}$  which are derivatives of the exact solution  $u$ , respectively, we have

$$\sum_{i=k-1}^{k+2} \left\{ \left( \int_{x_k}^{x_{k+1}} Q_j Q_i dx \right) \dot{\delta}_i + \xi \left( \int_{x_k}^{x_{k+1}} Q_j Q_i' dx \right) \delta_i - \mu \left( \int_{x_k}^{x_{k+1}} Q_j Q_i'' dx \right) \delta_i \right\}
 \tag{10}$$

where  $j = k-1, \dots, k+2$ ;  $k = 0, 1, K, N-1$  and  $\dot{\phantom{x}}$  symbolizes time derivative.

Let the integrals in Eq. (10) are denoted by

$$P_{ji}^e = \int_{x_k}^{x_{k+1}} Q_j Q_i dx, \quad R_{ji}^e = \int_{x_k}^{x_{k+1}} Q_j Q_i' dx, \quad S_{ji}^e = \int_{x_k}^{x_{k+1}} Q_j Q_i'' dx
 \tag{11}$$

and  $\mathbf{P}^e$ ,  $\mathbf{R}^e$  and  $\mathbf{S}^e$  are the element matrices of which dimensions are  $4 \times 4$ . With these matrices we can write the following instead of the (10):

$$\mathbf{P}^e \dot{\boldsymbol{\delta}}^e + (\xi \mathbf{R}^e - \mu \mathbf{S}^e) \boldsymbol{\delta}^e
 \tag{12}$$

where  $\delta^e = (\delta_{k-1}, \dots, \delta_{k+2})^T$ .

Collecting the systems (12) over all elements, the following global system can be written

$$\mathbf{P} \dot{\delta} + (\xi \mathbf{R} - \mu \mathbf{S}) \delta = 0 \tag{13}$$

where  $\mathbf{P}$ ,  $\mathbf{R}$  and  $\mathbf{S}$  are reproduced by the corresponding element matrices  $\mathbf{P}^e$ ,  $\mathbf{R}^e$  and  $\mathbf{S}^e$  and  $\delta = (\delta_{-1}, \dots, \delta_{N+1})^T$  includes all element parameters.

By using the Crank-Nicolson method for the unknown parameters  $\delta$

$$\delta = \frac{\delta^{n+1} + \delta^n}{2}, \quad \dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t},$$

we have the following iterative formula:

$$\left[ \mathbf{P} + \frac{\Delta t}{2} (\xi \mathbf{R} - \mu \mathbf{S}) \right] \delta^{n+1} = \left[ \mathbf{P} - \frac{\Delta t}{2} (\xi \mathbf{R} - \mu \mathbf{S}) \right] \delta^n. \tag{14}$$

To incorporate boundary conditions into the remaining system (14) before starting the iteration process, we eliminate the terms  $\delta_{-1}^{n+1}$  and  $\delta_{N+1}^{n+1}$  from the system with the following equations:

$$u(0, t) = \frac{4 - \lambda}{24} \delta_{-1}^n + \frac{8 + \lambda}{12} \delta_0^n + \frac{4 - \lambda}{24} \delta_1^n = f_0(t),$$

$$u(L, t) = \frac{4 - \lambda}{24} \delta_{N-1}^n + \frac{8 + \lambda}{12} \delta_N^n + \frac{4 - \lambda}{24} \delta_{N+1}^n = f_L(t).$$

Thus, a septa-diagonal matrix is obtained.

To carry on the iteration of the system (14), the initial parameters  $\delta^0$  must be obtained from the initial condition (2) and the derivatives of the boundary conditions (3) at both ends:

$$u'(x_0, 0) = -\frac{1}{2h} \delta_{-1} + \frac{1}{2h} \delta_1$$

$$u(x_k, 0) = \frac{4 - \lambda}{24} \delta_{k-1} + \frac{8 + \lambda}{12} \delta_k + \frac{4 - \lambda}{24} \delta_{k+1}, \quad k = 0, \dots, N, \tag{15}$$

$$u'(x_N, 0) = -\frac{1}{2h} \delta_{N-1} + \frac{1}{2h} \delta_{N+1}.$$

The matrix equation (15) is solved by the way of Thomas algorithm. Thus the approximate solution  $U$  (5) can be determined by using these  $\delta$  values.

### 3. TEST PROBLEMS

In this section, two test problems are addressed where the correctness of given algorithm is measured by error norm

$$L_\infty = \|u^{\text{exact}} - u^{\text{numeric}}\|_\infty = \max_{0 \leq j \leq N} |u_j^{\text{exact}} - u_j^{\text{numeric}}|. \quad (16)$$

In numerical calculations, the value of  $\lambda$  in the extended cubic B-spline is experimentally selected by scanning the predetermined interval with a small increment in a way that the best numerical solutions are obtained for the test problems. The Courant number is defined as

$$C_r = \xi \frac{\Delta t}{h}.$$

### 3.1. First Problem

In the first example, we consider the pure advection that is  $\mu = 0$ , in an infinitely long channel is of long constant cross-section, bottom slope and in which constant velocity is  $\xi = 0.5 \text{ m/s}$ . The analytical solution is

$$u(x,t) = 10 \exp\left(-\frac{1}{2\rho^2}(x - x_0 - \xi t)^2\right) \quad (17)$$

where the standard deviation is chosen as  $\rho = 264 \text{ m}$  and the distribution is  $x_0 = 2 \text{ km}$  away from the start at the beginning. The initial concentration can be obtained from (17) by taking  $t = 0$ . At the boundaries of interval  $[0,9000]$ , the values of Eq. (17) are chosen as zero. Figure 1 shows the transportation of initial distribution after  $t = 9600 \text{ s}$ .

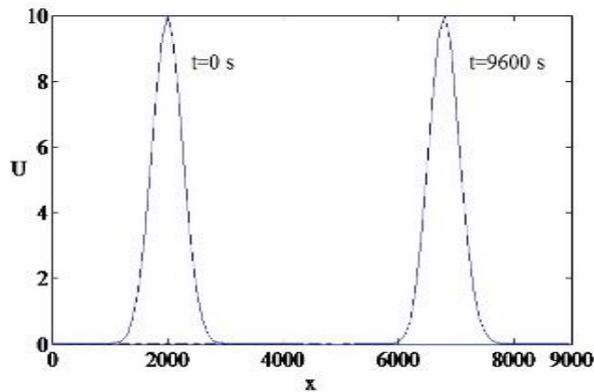


Figure 1. Transportation of the initial distribution with  $C_r = 0.25$  and  $\Delta t = 50$ .

To see the errors along the whole domain for various Courant numbers, Table 2 is documented. According to this table, the results of ECBSGM for various Courant numbers are obtained to be same accuracy with those produced by Dag et al. [5] and Irk et al. [14]. For various values of  $\lambda$ , Table 3 is shown the errors with  $C_r = 0.25$  at  $t = 9600 \text{ s}$ .

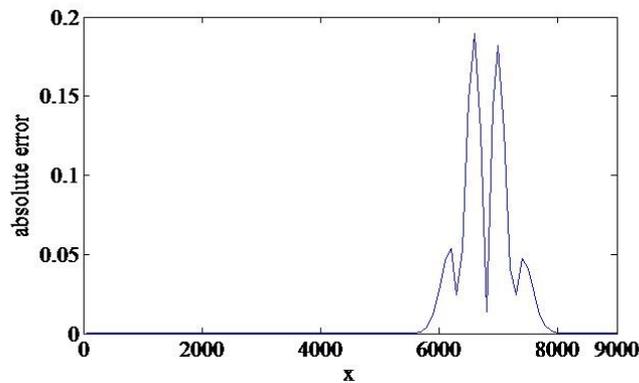
Table 2. Errors at  $t = 9600 \text{ s}$  with  $\xi = 0.5 \text{ m/s}$ .

$C_r$	$h$	$\Delta t$	$\lambda$	ECBSGM	[14]	[5]
0.125	200	50	-0.568169	2.18E-1	1.29	5.18E-1
0.25	100	50	-0.142055	1.90E-1	3.25E-1	3.76E-1
0.50	50	50	-0.032925	1.90E-1	1.98E-1	3.73E-1
0.50	10	10	-0.027064	7.50E-3	7.51E-3	
0.50	1	1	-0.027064	7.50E-5	7.50E-5	
0.50	0.5	0.5	-0.498015	1.88E-5	1.88E-5	

**Table 3.** Errors at  $t = 9600$  s with  $\xi = 0.5$  m/s and  $C_r = 0.25$ .

$\lambda$	ECBSGM
-10	0.9200226
-5	0.4351456
-1	0.1968777
-0.5	0.1908966
-0.142055	0.1896909
0	0.1898749
0.142055	0.1904208
0.5	0.1933445
1	0.2009420
5	0.3666941
10	0.6572077

The absolute error distribution of the ECBSGM at  $t = 9600$  is illustrated in Figure 2. Maximum error occurs around the peak concentration.



**Figure 2.** Absolute error distribution at  $t = 9600$  s with  $C_r = 0.25$  and  $\Delta t = 50$ .

### 3.2. Second Problem

As a second test problem, we deal with both advection and diffusion. The analytical solution to the one-dimensional ADE of a Gaussian pulse of unit height over the domain  $[0, 9]$  is given as

$$u(x, t) = \frac{1}{\sqrt{4t + 1}} \exp\left(-\frac{(x - x_0 - \xi t)^2}{\mu\sqrt{4t + 1}}\right) \tag{18}$$

where  $\xi$  is the velocity,  $\mu$  is diffusion coefficient and  $x_0$  is the center of the initial Gaussian pulse [28].

The initial condition is chosen as the analytical value of the Eq. (18) for  $t = 0$  and the boundary conditions are chosen as

$$u(0, t) = u(9, t) = 0.$$

The results presented here are computed for time step  $\Delta t = 0.0125$  s. Parameters in the equation are used as  $\mu = 0.005$  m<sup>2</sup>/s and  $\xi = 0.8$  m/s. Figure 3 shows the behavior of the numerical solutions for

various times until the simulation terminating time  $t = 5$ . Thus, the decay in time of the initial pulse is modeled. So that the effect of the diffusion term has been observed in this test problem. The absolute error distribution of the ECBSGM at  $t = 5$  is illustrated in Figure 4.

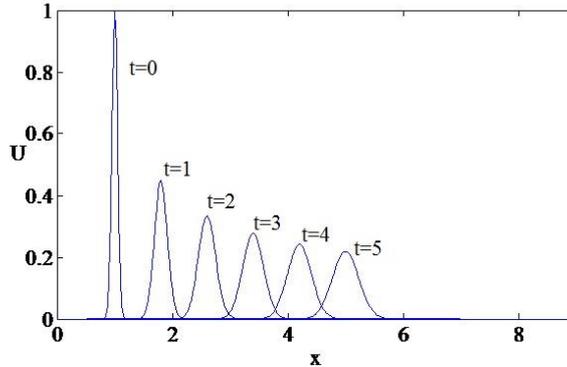


Figure 3. Distributions of an initial Gaussian pulse

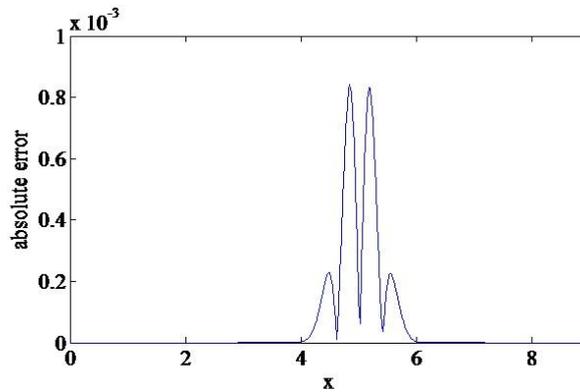


Figure 4. Absolute error distribution at  $t = 5$  with  $h = 0.025$ ,  $\Delta t = 0.0125$ .

For comparison, the ADE is solved for various Courant numbers and computed errors at  $t = 5$  s are presented in Table 4.

Table 4. Error norm at  $t = 5$ ,  $\xi = 0.8$  m/s,  $\mu = 0.005$  m<sup>2</sup>/s,  $\Delta t = 0.0125$ .

$C_r$	$h$	$\lambda$	ECBSGM	Method I [16]	Method II [16]
0.05	0.2	-0.238247	0.1326156	0.1253926	0.1361437
0.10	0.1	-0.238247	0.0042296	0.0069553	0.0145554
0.20	0.05	-0.200000	0.0008429	0.0012117	0.0002886
0.40	0.025	-0.106431	0.0008426	0.0003071	0.0000181

#### 4. CONCLUSION

In this paper, we have proposed a new algorithm for the numerical solution of the ADE. This algorithm is obtained by employing extended cubic B-spline functions to the well-known Galerkin finite element method. To see achievement of the method, two test problems is studied. The resulting numerical solutions for various Courant numbers are compared with the previous studies in Tables 2 and 4. Accordingly, we can say that the proposed method gives acceptable results.

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