Communications in Advanced Mathematical Sciences
ISSN (Online) 2651-4001

Vol. 8, No. 1, 49-56, 2025
https://doi.org/10.33434/cams.1605646
Research Article

On the Solution of a Class of Discontinuous Sturm-Liouville Problems

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Abstract
This study examines boundary value problems consisting of a second-order differential equation with discontinuous coefficients and boundary conditions. Asymptotic formulas for the eigenvalues and eigenfunctions of the problem are derived, and an expansion formula is obtained based on the eigenfunctions.

Keywords: Asymptotic formulas, Discontinuous coefficient, Expansion formula, Sturm-Liouville problem
2012 AMS: 34B24, 34K10, 47E05

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Received: 22 December 2024, **Accepted:** 14 March 2025, **Available online:** 24 March 2025
How to cite this article: Kh. R. Mamedov, D. K. Dinsever, U. Çelik *On the solution of a class of discontinuous Sturm-Liouville problems*, Commun. Adv. Math. Sci., 8(1) (2025), 49-56.

1. Introduction

Boundary value problems involving discontinuities are prevalent in various fields, including mathematics, mechanics, physics, and other natural sciences. Applications of such boundary value problems in geophysics can be found in [1], [2]. Certain aspects of direct and inverse problems for differential operators with discontinuity conditions are discussed in [3]- [5]. The direct and inverse problems for the Sturm-Liouville operator are studied in [6] and [7]. In [8], an integral representation of the solution for the Sturm-Liouville operator is provided.

For a boundary value problem with a discontinuous coefficient, the direct and inverse problems concerning the Weyl function are examined in [9, 10]. For a similar problem, the fundamental equation, which plays a crucial role in solving the inverse problem, is formulated in [11]. The necessary and sufficient conditions for the solution of the inverse problem are analyzed in [12]. For a boundary value problem defined on a finite interval, consisting of an equation with discontinuous coefficients and boundary conditions with spectral parameters, both direct and inverse problems for the Sturm-Liouville and Dirac operators are considered in [13, 14].

Consider the following boundary value problem defined on the interval $[0,\pi]$:

$$-y'' + q(x)y = \lambda^2 \upsilon(x)y, \tag{1.1}$$

$$U_1(y) := y'(0) - hy(0) = 0, \tag{1.2}$$

$$U_2(y) := y(\pi) = 0, \tag{1.3}$$

where $q(x) \in L_2[0,\pi]$ is a real-valued function, λ is a spectral parameter, $h \neq 0$ is an arbitrary real number and

$$\upsilon(x) = \begin{cases} 1, & 0 \leq x \leq a, \\ \alpha^2, & a \leq x \leq \pi. \end{cases}$$

2. Preliminaries

Let us show the special solutions of equation (1.1) with $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ satisfying the conditions

$$\phi(0,\lambda) = 0, \quad \phi'(0,\lambda) = h, \tag{2.1}$$

$$\vartheta(\pi,\lambda) = 0, \quad \vartheta'(\pi,\lambda) = 1. \tag{2.2}$$

For the solution of the (1.1) equation, the following integral representation is obtained:

$$e(x,\lambda) = e_0(x,\lambda) + \int_{-\eta^+(x)}^{\eta^+(x)} K(x,t) e^{i\lambda t} dt,$$

where $\eta^{\pm}(x) = \pm x\sqrt{\upsilon(x)} + a(1 \mp \sqrt{\upsilon(x)})$, $K(x,.) \in L_1\left(-\eta^+(x), \eta^+(x)\right)$ and

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & 0 \leq x \leq a, \\ \frac{1}{2}\left(1 + \frac{1}{\sqrt{\upsilon(x)}}\right) e^{i\lambda\eta^+(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\upsilon(x)}}\right) e^{i\lambda\eta^-(x)}, & a \leq x \leq \pi. \end{cases}$$

In addition, the K_x derivative exists and provides the following properties

$$\frac{d}{dx} K\left(x, \eta^+(x)\right) = \frac{1}{4\sqrt{\upsilon(x)}}\left(1 + \frac{1}{\sqrt{\upsilon(x)}}\right) q(x), \tag{2.3}$$

$$\frac{d}{dx}\left\{K\left(x, \eta^-(x) + 0\right) - K\left(x, \eta^+(x) - 0\right)\right\} = \frac{1}{4\sqrt{\upsilon(x)}}\left(1 - \frac{1}{\sqrt{\upsilon(x)}}\right) q(x), \tag{2.4}$$

$$K\left(x, -\eta^+(x)\right) = 0. \tag{2.5}$$

Besides these properties, the following also apply if $q(x)$ is a differentiable function:

$$\upsilon(x) K''_{tt} - K''_{xx} + q(x) K = 0, \quad |t| < \eta^+(x), \tag{2.6}$$

$$\int_{-\eta^+(x)}^{\eta^+(x)} |K(x,t)| dt \leq C\left(\exp\left\{\int_0^x |q(t)| dt\right\} - 1\right), \quad 0 < C \tag{2.7}$$

The special solution of equation (1.1) satisfying condition (2.1) is of the form

$$\phi(x,\lambda) = \phi_0(x,\lambda) + \int_0^{\eta^+(x)} A(x,t) \cos\lambda t\, dt + h \int_0^{\eta^+(x)} \tilde{A}(x,t) \frac{\sin\lambda t}{\lambda} dt, \tag{2.8}$$

and the kernel $A(x,t) = K(x,t) + K(x,-t)$ satisfies the conditions (2.3)-(2.7).

We define

$$\Gamma(\lambda) = \langle \phi(x,\lambda), \vartheta(x,\lambda) \rangle = \phi(x,\lambda)\vartheta'(x,\lambda) - \phi'(x,\lambda)\vartheta(x,\lambda). \tag{2.9}$$

The characteristic function $\Gamma(\lambda)$ is the Wronskian of the functions ϕ and ϑ. From Liouville's theorem, it can be seen that $\Gamma(\lambda)$ is independent of $x \in [0,\pi]$. From equation (2.9), if $x = 0$ and $x = \pi$ respectively, we obtain

$$\Gamma(\lambda) = U_2(\phi) = U_1(\vartheta).$$

Lemma 2.1. *The square of the zeros $\{\lambda_n\}_{n=0}^{\infty}$ of the characteristic function coincide with the eigenvalues of the boundary value problem (1.1)-(1.3). There is also a sequence $\{k_n\}_{n=0}^{\infty}$ such that $\vartheta(x,\lambda_n) = k_n\phi(x,\lambda_n)$ for each eigenvalue λ_n, where $\phi(x,\lambda_n)$ and $\vartheta(x,\lambda_n)$ are the eigenfunctions corresponding to the eigenvalue λ_n.*

Let us define the normalized numbers of the boundary value problem (1.1)-(1.3) as

$$\alpha_n := \int_0^\pi \phi^2(x,\lambda_n)\upsilon(x)dx.$$

Lemma 2.2. *The following relation holds*

$$-\dot{\Gamma}(\lambda_n) = 2\lambda_n k_n \alpha_n, \tag{2.10}$$

where $\dot{\Gamma}(\lambda) = \dfrac{d}{d\lambda}\Gamma(\lambda)$.

Proof. Since

$$-\phi''(x,\lambda_n) + q(x)\phi(x,\lambda_n) = \lambda_n^2\upsilon(x)\phi(x,\lambda_n),$$
$$-\vartheta''(x,\lambda) + q(x)\vartheta(x,\lambda) = \lambda^2\upsilon(x)\vartheta(x,\lambda),$$

we have

$$\frac{d}{dx}\langle\phi(x,\lambda_n),\vartheta(x,\lambda)\rangle = (\lambda_n^2-\lambda^2)\upsilon(x)\phi(x,\lambda_n)\vartheta(x,\lambda).$$

Integrating the last equation in the interval $[0,\pi]$ and considering the conditions (2.1), (2.2), we obtain

$$\Gamma(\lambda_n) - \Gamma(\lambda) = (\lambda_n^2-\lambda^2)\int_0^\pi \phi(x,\lambda_n)\vartheta(x,\lambda)\upsilon(x)dx.$$

The desired result is obtained by taking the limit for $\lambda \to \lambda_n$. □

3. Asymptotic Formulas of the Eigenvalues

Theorem 3.1. *The eigenvalues $\{\lambda_n\}$ and the eigenfuynctions $\phi(x,\lambda_n)$, $\vartheta(x,\lambda_n)$ are real. All zeros of $\Gamma(\lambda)$ are simple. Eigenfunctions related to different eigenvalues are orthogonal in $L_2(0,\pi)$.*

Lemma 3.2. *When $q(x) \equiv 0$, the eigenvalues of the boundary value problem (1.1)-(1.3) have the following asymptotic form:*

$$\left(\lambda_n^0\right)^2 = n + \vartheta(n), \quad \sup_n|\vartheta(n)| < +\infty.$$

Lemma 3.3. *The λ_n^0 roots of the function $\Gamma_0(\lambda)$ are discrete, i.e.*

$$\inf_{n\neq k}|\lambda_n^0 - \lambda_k^0| = \tau > 0.$$

Proof. Assume the opposite, that there are sequences $\{\lambda_k^{0'}\}$ and $\{\lambda_k^{0''}\}$ such that $\lambda_k^{0'} \neq \lambda_k^{0''}$, $\lambda_k^{0'} \to +\infty$, $\lambda_k^{0''} \to +\infty$ and

$$\lim_{k\to+\infty}\left[\lambda_k^{0'} - \lambda_k^{0''}\right]$$

for the zeros of the function $\Gamma_0(\lambda)$. Since the eigenfunctions of the boundary value problem (1.1)-(1.3) are orthogonal, we obtain

$$\begin{aligned}
0 =& \lambda_k^{0'}\lambda_k^{0''}\int_0^\pi \phi_0\left(x,\lambda_k^{0'}\right)\phi_0\left(x,\lambda_k^{0''}\right)\upsilon(x)dx \\
&= I_k + \int_0^\pi \left(\lambda_k^{0'}\right)^2\phi_0^2\left(x,\lambda_k^{0'}\right)\upsilon(x)dx \\
&\geq I_k + \int_0^a \left(\lambda_k^{0'}\right)^2\phi_0^2\left(x,\lambda_k^{0'}\right)\upsilon(x)dx \\
&= I_k + \frac{a}{2} - \frac{\sin 2\lambda_k^{0'}a}{4\lambda_k^{0'}},
\end{aligned} \tag{3.1}$$

where $I_k = \int_0^{\pi} \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)\left[\lambda_k^{0''} \phi_0\left(x, \lambda_k^{0''}\right) - \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)\right] \upsilon(x) dx$.

Now let us show that $I_k \to 0$ when $k \to +\infty$.

Since $|\lambda_k^{0''} \phi_0\left(x, \lambda_k^{0''}\right) - \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)| \leq C|\lambda_k^{0''} - \lambda_k^{0'}|$, for $\forall x \in [0, \pi]$,

$$\lim_{k \to +\infty} |\lambda_k^{0''} \phi_0\left(x, \lambda_k^{0''}\right) - \lambda_k^{0'} \phi_0\left(x, \lambda_k^{0'}\right)| = 0.$$

In inequality (3.1), if the limit is taken when $k \to +\infty$, it can be shown that $0 \geq \frac{a}{2}$. This is in contradiction to the definition of the coefficient $\upsilon(x)$ in equation (1.1). It can, therefore, be concluded that the proof is complete. □

Lemma 3.4. *The set of eigenvalues of the boundary value problem (1.1)-(1.3) are countable and of the form*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{\eta_n}{n}$$

where λ_n^0 is the zeros of the characteristic function $\Gamma_0(\lambda)$, d_n is a finite sequence and $\{\eta_n\} \in l_2$.

Proof. From the condition (1.3), $\phi(\pi, \lambda) = \Gamma(\lambda)\mid_{x=\pi}$ can be written. From the (2.8) representations of the function $\phi(\pi, \lambda)$, we get

$$\Gamma(\lambda) = \Gamma_0(\lambda) + \int_0^{\eta^+(\pi)} A(\pi, t) \cos \lambda t dt + h \int_0^{\eta^+(\pi)} \tilde{A}(\pi, t) \frac{\sin \lambda t}{\lambda} dt. \tag{3.2}$$

Let $\sigma < \frac{\tau}{2}$ be a sufficiently small positive number and let $G_\sigma = \{\lambda : |\lambda - \lambda_n^0| \geq \sigma\}$. From [15],

$$\mid \Gamma_0(\lambda) \mid \geq C_\sigma \frac{e^{|Im\lambda|\eta^+(\pi)}}{\lambda}, \quad \lambda \in G_\sigma.$$

On the other hand, if $f(x) \in L_1(0, \pi)$, from the expression

$$\lim_{|\lambda| \to \infty} e^{-|Im\lambda\pi|} \int_0^{\pi} f(x) \cos \lambda dx = \lim_{|\lambda| \to \infty} e^{-|Im\lambda\pi|} \int_o^{\pi} f(x) \sin \lambda dx$$

then

$$\Gamma(\lambda) - \Gamma_0(\lambda) = O\left(\frac{e^{|Im\lambda|\eta^+(\pi)}}{|\lambda|}\right), \quad |\lambda| \to \infty.$$

Therefore, for a sufficiently large n, the inequality
$\mid \Gamma(\lambda) - \Gamma_0(\lambda) \mid \leq \mid \Gamma_0(\lambda) \mid$ is satisfied on the $\Omega_n = \{\lambda : |\lambda| = |\lambda_n^0| + \frac{\tau}{2}\}$ curves. Appliyng now Rouche's theorem to the curve $\omega_n(\sigma) = \{\lambda : |\lambda - \lambda_n^0| \leq \sigma\}$, we conclude that for sufficiently large n, in $\omega_n(\sigma)$ there is exactly one zero of $\Gamma(\lambda)$, namely λ_n.

Now let's find the eigenvalues λ_n. For an arbitrary number $\sigma > 0$,

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty \tag{3.3}$$

is obtained. Substituting (3.3) into (3.2) we get,

$$\Gamma(\lambda_n^0 + \varepsilon_n) = \Gamma_0(\lambda_n^0 + \varepsilon_n) + \int_0^{\eta^+(\pi)} A(\pi, t) \cos\left(\lambda_n^0 + \varepsilon_n\right) t dt + h \int_0^{\eta^+(\pi)} \tilde{A}(\pi, t) \frac{\sin\left(\lambda_n^0 + \varepsilon_n\right) t}{\lambda_n^0 + \varepsilon_n} dt = 0. \tag{3.4}$$

Considering the equations $\Gamma_0(\lambda_n^0) = 0$ and (3.4) together in relation $\Gamma_0(\lambda_n^0 + \varepsilon_n) = \dot{\Gamma}_0(\lambda_n^0)\varepsilon_n + o(\varepsilon_n^2)$, we find

$$\dot{\Gamma}_0(\lambda_n^0)\varepsilon_n + \int_0^{\eta^+(\pi)} A(\pi, t) \cos\left(\lambda_n^0 + \varepsilon_n\right) t dt + h \int_0^{\eta^+(\pi)} \tilde{A}(\pi, t) \frac{\sin\left(\lambda_n^0 + \varepsilon_n\right) t}{\lambda_n^0 + \varepsilon_n} dt \approx 0.$$

Using properties (2.4) and (2.5), since $\varepsilon_n = o(1)$ while $n \to \infty$ we find

$$\begin{aligned}
\varepsilon_n \approx & \frac{1}{\dot{\Gamma}_0(\lambda_n^0)\lambda_n^0}\left\{\frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\upsilon(t)}}\left(1-\frac{1}{\sqrt{\upsilon(t)}}\right)\sin\left(\lambda_n^0\eta^-(\pi)\right)q(t)dt\right.\\
&\left. -\frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\upsilon(t)}}\left(1+\frac{1}{\sqrt{\upsilon(t)}}\right)\sin\left(\lambda_n^0\eta^-(\pi)\right)q(t)dt - \int_0^{\eta^+(\pi)} A_t'(\pi,t)\cos(\lambda_n^0 t)dt\right\}\\
& + \frac{h}{\dot{\Gamma}_0(\lambda_n^0)\lambda_n^0}\left\{\frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\upsilon(t)}}\left(1-\frac{1}{\sqrt{\upsilon(t)}}\right)\frac{\cos\left(\lambda_n^0\eta^-(\pi)\right)}{\lambda_n^0}q(t)dt\right.\\
&\left. +\frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\upsilon(t)}}\left(1+\frac{1}{\sqrt{\upsilon(t)}}\right)\frac{\cos\left(\lambda_n^0\eta^-(\pi)\right)}{\lambda_n^0}q(t)dt - \frac{1}{\lambda_n^0}\int_0^{\eta^+(\pi)} \tilde{A}_t'(\pi,t)\cos(\lambda_n^0 t)dt\right\}\\
& = \frac{1}{\dot{\Gamma}_0(\lambda_n^0)\lambda_n^0}\left\{d_n + \eta_n + \frac{\tilde{\eta}_n}{\lambda_n^0}\right\}
\end{aligned}$$

where $\eta_n := \int_0^{\eta^+(\pi)} A_t'(\pi,t)\sin(\lambda_n^0 t)dt$, $\tilde{\eta}_n := \int_0^{\eta^+(\pi)} \tilde{A}_t'(\pi,t)\cos(\lambda_n^0 t)dt$ and $\eta_n, \tilde{\eta}_n \in l_2$. □

4. Expansion Formula with respect to Eigenfunctions

In this section, the completeness of the eigenfunctions of the boundary value problem (1.1)-(1.3) is shown, and the expansion formula for the eigenfunctions is obtained. Let

$$G(x,t;\lambda) := -\frac{1}{\Gamma(\lambda)}\begin{cases}\phi(t,\lambda)\vartheta(x,\lambda), & t \le x,\\ \vartheta(t,\lambda)\phi(x,\lambda), & t > x.\end{cases}$$

Consider the function

$$y(x,\lambda) = \int_0^\pi G(x,t;\lambda)f(t)\upsilon(t)dt. \tag{4.1}$$

Theorem 4.1. *The system of eigenfunctions $\{\phi(x,\lambda_n)\}_{n\ge 0}$ of the boundary value problem (1.1)-(1.3) is complete in $L_{2,\upsilon}[0,\pi]$.*

Proof. Using (2.10) and Lemma 2.1, we get

$$\vartheta(x,\lambda_n) = -\frac{\dot{\Gamma}(\lambda_n}{2\lambda_n\alpha_n}\phi(x,\lambda_n). \tag{4.2}$$

From (4.1) and (4.2), we have

$$Res_{\lambda=\lambda_n} y(x,\lambda) = \frac{1}{2\lambda_n\alpha_n}\phi(x,\lambda_n)\int_0^\pi \phi(t,\lambda_n)f(t)\upsilon(t)dt. \tag{4.3}$$

Let us assume $f(x) \in L_{2,\upsilon}[0,\pi]$ and $\int_0^\pi \phi(t,\lambda_n)\overline{f(t)}\upsilon(t)dt$. Then $Res_{\lambda=\lambda_n} y(x,\lambda) = 0$ is obtained. Thus, for each fixed $x \in [0,\pi]$, $y(x,\lambda)$ is entire with respect to λ. If $f(x) \in L_1(0,\pi)$, the equations

$$\lim_{|\lambda|\to\infty}\max_{x\in[0,\pi]}\left\{e^{-|Im\lambda|x}\left|\int_0^x f(t)\cos\lambda t dt\right|\right\} = 0$$

$$\lim_{|\lambda|\to\infty}\max_{x\in[0,\pi]}\left\{e^{-|Im\lambda|x}\left|\int_0^x f(t)\sin\lambda t dt\right|\right\} = 0$$

are satisfied. Also, for $|\lambda| \to \infty$, we get

$$\phi(x,\lambda) = O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\eta^+(x)}\right),$$

$$\phi'(x,\lambda) = \phi_0'(x,\lambda) + O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\eta^+(x)}\right) = O\left(e^{|Im\lambda|\eta^+(x)}\right),$$

$$\vartheta(x,\lambda)=O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\left(\eta^{+}(\pi)-\eta^{+}(x)\right)}\right),$$

$$\vartheta'(x,\lambda)=\vartheta_0'(x,\lambda)+O\left(\frac{1}{|\lambda|}e^{|Im\lambda|\left(\eta^{+}(\pi)-\eta^{+}(x)\right)}\right)=O\left(e^{|Im\lambda|\left(\eta^{+}(\pi)-\eta^{+}(x)\right)}\right).$$

Then the following inequality is satisfied:

$$|\Gamma(\lambda)|\geq C_\sigma\frac{1}{|\lambda|}e^{|Im\lambda|\eta^{+}(\pi)},\quad \lambda\in G_\sigma.$$

From equation (4.1) it follows that $|y(x,\lambda)|\leq\dfrac{C_\sigma}{|\lambda|},\quad \lambda\in G_\sigma,\quad |\lambda|\geq\lambda^*$ for $\sigma>0$ and sufficiently large $\lambda^*>0$. Thus $f(x)\equiv 0$ is obtained almost everywhere in the interval $[0,\pi]$. Therefore, the proof is complete.

□

Theorem 4.2. *Let $f(x)$ be an absolutely continuous function on $[0,\pi]$. Then*

$$f(x)=\sum_{n=1}^{\infty}a_{,n}\phi(x,\lambda_n),\quad a_n=\frac{1}{\alpha_n}\int_0^{\pi}\phi(t,\lambda_n)f(t)\upsilon(t)dt, \tag{4.4}$$

and the series converges uniformly on $[0,\pi]$. For $f(x)\in L_{2,\upsilon}[0,\pi]$, the series (4.4) converces in $L_{2,\upsilon}[0,\pi]$ and the following Parseval's equality is satisfied:

$$\int_0^{\pi}|f(x)|^2\upsilon(x)dx=\sum_{n=1}^{\infty}\alpha_n|a_n|^2.$$

Proof. Let $f(x)\in AC[0,\pi]$. Since $\phi(x,\lambda)$ and $\vartheta(x,\lambda)$ are solutions of the boundary value problem (1.1)-(1.3), we get

$$y(x,\lambda)=-\frac{1}{\lambda^2\Gamma(\lambda)}\left\{\vartheta(x,\lambda)\int_0^x\left(-\phi''(t,\lambda)+q(t)\phi(t,\lambda)\right)f(t)dt\ +\phi(x,\lambda)\int_x^{\pi}\left(-\vartheta''(t,\lambda)+q(t)\vartheta(t,\lambda)\right)f(t)dt\right\}.$$

Integration of the terms containing second derivatives by parts yields in view of (1.2), (1.3)

$$y(x,\lambda)=-\frac{f(x)}{\lambda^2}-\frac{1}{\lambda^2}\left(Z_1(x,\lambda)+Z_2(x,\lambda)\right), \tag{4.5}$$

where

$$Z_1(x,\lambda)=\frac{1}{\Gamma(\lambda)}\left[\vartheta(x,\lambda)\int_0^x\phi'(t,\lambda)f'(t)dt+\phi(x,\lambda)\int_x^{\pi}\vartheta'(t,\lambda)f'(t)dt\right],$$

$$Z_2(x,\lambda)=\frac{1}{\Gamma(\lambda)}\left[\vartheta(x,\lambda)f(0)-\phi(x,\lambda)f(\pi)\right.$$
$$\left.+\vartheta(x,\lambda)\int_0^x\phi(t,\lambda)q(t)f(t)dt+\phi(x,\lambda)\int_x^{\pi}\vartheta(t,\lambda)q(t)f(t)dt\right].$$

Now consider the integral

$$I_n(x)=\frac{1}{2\pi i}\oint_{\Omega_n}\lambda y(x,\lambda)d\lambda,$$

where $\Omega_n=\left\{\lambda:|\lambda|=|\lambda_n^0|+\dfrac{\tau}{2}\right\}$ is a clockwise directed curve and n is a sufficiently large natural number. From (4.5), we get

$$\frac{1}{2\pi i}\oint_{\Omega_n}\lambda y(x,\lambda)d\lambda=\frac{1}{2\pi i}\oint_{\Omega_n}\frac{f(x)}{\lambda}d\lambda-\frac{1}{2\pi i}\oint_{\Omega_n}\frac{\{Z_1(x,\lambda)+Z_2(x,\lambda)\}}{\lambda}d\lambda. \tag{4.6}$$

Thus, we obtain

$$I_n(x)=2\sum_{n=1}^{N}Res_{\lambda=\lambda_n}\left(\lambda y(x,\lambda)\right).$$

From the equations (4.3) and (4.6), we get

$$-f(x)+\varepsilon_n(x) = -\sum_{n=1}^{N}\frac{\phi(x,\lambda_n)}{\alpha_n}\left\{\int_0^{\pi}\phi(t,\lambda_n)f(t)\upsilon(t)dt\right\},$$

where

$$\varepsilon_n(x) = -\frac{1}{2\pi i}\oint_{\Omega_n}\{Z_1(x,\lambda)+Z_2(x,\lambda)\}d\lambda.$$

For fixed $\sigma > 0$ and sufficiently large $\lambda^* > 0$, the following relations hold:

$$\max_{x\in[0,\pi]}\|Z_2(x,\lambda)\| \le \frac{C_2}{|\lambda|}, \quad \lambda\in G_\sigma, \quad |\lambda|\ge\lambda^*, \tag{4.7}$$

$$\max_{x\in[0,\pi]}\|Z_1(x,\lambda)\| \le \frac{C_1}{|\lambda|}, \quad \lambda\in G_\sigma, \quad |\lambda|\ge\lambda^*. \tag{4.8}$$

From expressions (4.7) and (4.8) it follows that the equality

$$\lim_{n\to\infty}\max_{x\in[0,\pi]}|\varepsilon_n(x)| = 0$$

is satisfied. The last equation gives the following expansion formula:

$$f(x) = \sum_{n=1}^{\infty} a_n\phi(x,\lambda_n),$$

where

$$a_n = \sum_{n=1}^{\infty}\frac{1}{\alpha_n}\left\{\int_0^{\pi}\phi(t,\lambda_n)f(t)\upsilon(t)dt\right\}.$$

Since the system $\{\phi(x,\lambda_n)\}$ forms an orthogonal base at $L_{2,\upsilon}[0,\pi]$, Parseval equality is satisfied. □

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

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