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Isosceles Orthogonal Geometric Constants for Morrey Spaces

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Abstract. In this paper, we calculate the value of new geometric constants for the Morrey spaces and small Morrey spaces. The new geometric constants which were investigated are generalizations of the other new constants $\Omega(X)$ and $\overline{\Omega}(X)$ for Banach spaces X. The two constants are related to isosceles orthogonal type and introduced by Liu *et al* in 2022. We introduce the generalizations of the constants which are denoted by $\Omega^{(s)}(X)$ and $\overline{\Omega}^{(s)}(X)$ for $s \geq 1$. We calculate the value of each of the constants for Morrey spaces $\mathcal{M}_q^p$ and small Morrey spaces $m_{q,\lambda}^p$. The results show that $\Omega^{(s)}(\mathcal{M}_q^p) = \frac{2^{s+1}}{5^{s-1}}$ and $\overline{\Omega}^{(s)}(m_{q,\lambda}^p) = \frac{2^{s+1}}{5^{s-1}}$.

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1. Introduction

One of the important function spaces used in functional analysis is Morrey space. The Morrey spaces were first introduced by Charles Bradfield Morrey in 1938 [9]. For $1 \leq p \leq q < \infty$, the Morrey space $\mathcal{M}_q^p$ is the set of any measurable function f such that the norm

$$\|f\|_{\mathcal{M}_q^p} = \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}}$$

is finite, where $|B(a,r)|$ denotes the Lebesgue measure of the open ball $B(a,r)$ centered at $a \in \mathbb{R}^n$ and radius of $r > 0$. Throughout this paper, we shall denote $B(r) = B(0,r)$ for simplicity. If $p = q$, then $\mathcal{M}_q^p$ is the Lebesgue space L^p. Therefore, the Morrey Spaces can be considered as one of the extensions of the Lebesgue spaces. Meanwhile, one of the modifications of Morrey spaces is small Morrey space m_q^p which is defined as the collection of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{m_q^p} = \sup_{a\in\mathbb{R}^n, 0<r<1} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}} \tag{1.1}$$

is finite. Here, the norm in (1.1) is generalized become the form

$$\|f\|_{m_{q,\lambda}^p} = \sup_{a\in\mathbb{R}^n, 0<r<\lambda} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}}, \tag{1.2}$$

where $\lambda > 0$ is fixed. It can be seen that for $\lambda = 1$, (1.2) reduces to (1.1). For $\lambda > 0$ and $1 \leq p \leq q < \infty$, we then defined a new space $m_{q,\lambda}^p$ as the collection of the measurable function f such that $\|f\|_{m_{q,\lambda}^p}$ is finite. It can be seen that the

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Morrey spaces $\mathcal{M}_q^p$ and the small Morrey spaces $m_{q,\lambda}^p$ are Banach spaces. If we define the space $\mathcal{M}_{q,\lambda}^p$ for $0 < \lambda \leq +\infty$ and $1 \leq p \leq q < \infty$ to be the set of all functions f for which

$$\|f\|_{\mathcal{M}_{q,\lambda}^p} = \sup_{a\in\mathbb{R}^n, 0<r<\lambda} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

then $\mathcal{M}_{q,\lambda}^p$ generalizes $\mathcal{M}_q^p$ and $m_{q,\lambda}^p$.

Let X be a Banach space. In order to investigate the geometric properties of X, some geometric constants had been defined. Some of them are Von Neumann-Jordan constant $C_{NJ}(X)$ (see [4]), James constant $C_J(X)$ (see [7]), the Dunkl-William constant C_{DW} (see [3]), and Zbáganu contant C_Z (see [12]). Moreover, some generalizations and modifications of Von Nueman-Jordan constant also have been defined, for example: generalized Von-Neumann Jordan constant $C_{NJ}^{(s)}(X)$ (see [2]), modified Von Neumann-Jordan constant $C'_{NJ}(X)$ (see [1, 5]), and generalized modified Von Neumann-Jordan constant $\bar{C}_{NJ}^{(s)}(X)$ (see [11]).

In [6], three geometric constants for Morrey spaces, namely the von Neumann-Jordan constant, the James constant, and the Dunkl-Williams contant, have been computed to obtain that $C_{NJ}(\mathcal{M}_q^p) = C_J(\mathcal{M}_q^p) = 2$ and $C_{DW}(\mathcal{M}_q^p) = 4$. Moreover, in [10], it has been obtained that

$$C_{NJ}^{(s)}(\mathcal{M}_q^p) = C'_{NJ}(\mathcal{M}_q^p) = \bar{C}_{NJ}^{s}(\mathcal{M}_q^p) = C_Z(\mathcal{M}_q^p) = 2$$

and

$$C_{NJ}^{(s)}(m_{1,q}^p) = C'_{NJ}(m_{1,q}^p) = \bar{C}_{NJ}^{s}(m_{1,q}^p) = C_Z(m_{1,q}^p) = 2.$$

Recently, some new constants for a Banach space X have been defined related to isosceles orthogonal type. The new constants are $\Omega(X)$ and $\Omega'(X)$ which have been introduced by Liu *et al* [8].

Related to the definitions of the constants, an element $x \in X$ is said to be isosceles orthogonal to $y \in X$, denoted by $x\perp_I y$, if $\|x + y\| = \|x - y\|$. By the notations, the new constants $\Omega(X)$ and $\Omega'(X)$ are defined by

$$\Omega(X) = \sup\left\{ \frac{\|x + 2y\|^2 + \|2x + y\|^2}{5\|x + y\|^2} : x, y \in S_X, x\perp_I y \right\}$$

and

$$\Omega'(X) = \sup\left\{ \frac{\|x + 2y\|^2 + \|2x + y\|^2}{5\|x + y\|^2} : x\perp_I y \right\},$$

where $S_X = \{x \in X : \|x\| = 1\}$. For more details about the constants $\Omega(X)$ and $\Omega'(X)$, one may refer to [8].

For $1 \leq s < \infty$, we extend the constant $\Omega(X)$ become $\Omega^{(s)}(X)$ and the constant $\Omega'(X)$ become $\bar{\Omega}^{(s)}(X)$ which have been defined by

$$\Omega^{(s)}(X) = \sup\left\{ \frac{\|x + 2y\|^s + \|2x + y\|^s}{5^{s-1}\|x + y\|^2} : x, y \in S_X, x\perp_I y \right\}$$

and

$$\bar{\Omega}^{(s)}(X) = \sup\left\{ \frac{\|x + 2y\|^s + \|2x + y\|^s}{5^{s-1}\|x + y\|^s} : x\perp_I y \right\}.$$

It is clear that,

$$\Omega^{(s)}(X) \leq \bar{\Omega}^{(s)}(X), \quad s \geq 1. \tag{1.3}$$

In this paper, we calculate the geometric constants $\Omega^{(s)}$ and $\bar{\Omega}^{(s)}(X)$ for the Morrey spaces $\mathcal{M}_q^p$, where $1 \leq p < q < \infty$. Moreover, we also calculate the constants for small Morrey space $m_{q,\lambda}^p$, where $\lambda > 0$ and $1 \leq p < q < \infty$.

2. Main results

The following theorems are our main results on the new geometric constants for the Morrey space $\mathcal{M}_q^p$ and the small Morrey space $m_{q,\lambda}^p$.

Theorem 2.1. *Let $s \geq 1$. If $1 \leq p < q < \infty$, then $\Omega^{(s)}(\mathcal{M}_q^p) = \frac{2^{s+1}}{5^{s-1}} = \bar{\Omega}^{(s)}(\mathcal{M}_q^p)$.*

Corollary 2.2. *If $1 \leq p < q < \infty$, then $\Omega(\mathcal{M}_q^p) = \frac{8}{5} = \Omega'(\mathcal{M}_q^p)$.*

Theorem 2.3. *Let $s \geq 1$ and $\lambda > 0$. If $1 \leq p < q < \infty$, then $\Omega^{(s)}(m_{q,\lambda}^p) = \frac{2^{s+1}}{5^{s-1}} = \bar{\Omega}^{(s)}(m_{q,\lambda}^p)$.*

Corollary 2.4. *Let $\lambda > 0$. If $1 \leq p < q < \infty$, then $\Omega(m_{q,\lambda}^p) = \frac{8}{5} = \Omega'(m_{q,\lambda}^p)$.*

Before proving the main results, we first state and prove the following lemmas and theorems which will be used to prove our main results.

Lemma 2.5. *Let $a \in \mathbb{R}^n$ and $r > 0$. Then,*

$$|B(a,r)| = |B(r)| = \frac{\omega_{n-1}}{n} r^n,$$

where ω_{n-1} is a constant that does not depend on a and r.

Theorem 2.6. *Let X be a Banach space, then $\frac{1+2^s}{5^{s-1}} \leq \bar{\Omega}^{(s)}(X) \leq \frac{2^{s+1}}{5^{s-1}}$ for $s \geq 1$.*

Proof. Let $s \geq 1$. Suppose that $x \perp_I y$. Then $\|x+y\| = \|x-y\|$, and by using triangle inequality, we have that

$$\begin{aligned}
\frac{\|x+2y\|^s + \|2x+y\|^s}{5^{s-1}\|x+y\|^s} &= \frac{\left(\frac{1}{2}\right)^s \|x-y-3(x+y)\|^s + \left(\frac{1}{2}\right)^s \|3(x+y)+(x-y)\|^s}{5^{s-1}\|x+y\|^s} \\
&\leq \left(\frac{1}{2}\right)^s \frac{(\|x-y\| + 3\|x+y\|)^s + (3\|x+y\| + \|x-y\|)^s}{5^{s-1}\|x+y\|^s} \\
&= \frac{2^{s+1}}{5^{s-1}}.
\end{aligned}$$

In the other hand, for $y = 0$ and $x \neq 0$, we have that

$$\bar{\Omega}^{(s)}(X) \geq \frac{\|x\|^s + \|2x\|^s}{5^{s-1}\|x\|^s} = \frac{1+2^s}{5^{s-1}},$$

and this completes the proof of Theorem 2.6. □

Theorem 2.7. *Let X be a Banach space, then $\Omega^{(s)}(X) \leq \frac{2^{s+1}}{5^{s-1}}$ for $s \geq 1$.*

Proof. This is a direct consequence of the inequality (1.3) and Theorem 2.6. □

Theorem 2.6 and 2.7 obviously extend the results in [8]. Next, we provide some functions belonging to $\mathcal{M}^p_q$ and some functions belonging to $m^p_{q,\lambda}$ for $1 \leq p < q < \infty$ and $\lambda > 0$.

Lemma 2.8. *[6] Let $1 \leq p < q < \infty$. Define $f(x) := |x|^{-n/q}$ for $x \in \mathbb{R}^n$. If $g(x) = \mathcal{X}_{(0,1)}(|x|) \cdot f(x)$ and $h(x) = \mathcal{X}_{[1,\infty)}(|x|) \cdot f(x)$ for all $x \in \mathbb{R}^n$, then*

$$\|f\|_{\mathcal{M}^p_q} = \|g\|_{\mathcal{M}^p_q} = \|h\|_{\mathcal{M}^p_q} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}}.$$

Lemma 2.9. *Let $1 \leq p < q < \infty$ and $\lambda > 0$. Define $f(x) := \mathcal{X}_{(0,\lambda)}(|x|)|x|^{-n/q}$ for $x \in \mathbb{R}^n$. For $0 < \delta < \lambda$, define $g(x) = \mathcal{X}_{(0,\delta)}(|x|) \cdot f(x)$ and $h(x) = \mathcal{X}_{[\delta,\lambda)}(|x|) \cdot f(x)$ for all $x \in \mathbb{R}^n$. Then,*

$$\|f\|_{m^p_{q,\lambda}} = \|g\|_{m^p_{q,\lambda}} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \text{ and } \|h\|_{m^p_{q,\lambda}} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \left(1 - \left(\frac{\delta}{\lambda}\right)^{-\frac{np}{q}+n}\right)^{\frac{1}{p}}.$$

Proof. We see that f, g, and h can be considered as radial functions. We first check that for any $r > 0$ and $s > 0$, the integral $\int_{B(a,r)} |x|^{-s} dx$ is maximized when $a = 0$ by using rearrangement inequality argument. Specifically, for the measurable set $A \subseteq \mathbb{R}^n$ and any function $f : \mathbb{R}^n \to [0,\infty)$, we let A^* be the symmetric rearrangement of A and f^* be the symmetric decreasing rearrangement of f, i.e.

$$A^* = \{x \in \mathbb{R}^n, \omega_n |x|^n < |A|\}, \text{ and } f^*(x) = \int_0^\infty \mathcal{X}_{\{y: f(y)>t\}*}(x) dt,$$

where $\omega_n = |B(1)|$ denotes the volume of the unit ball in $\mathbb{R}^n$. It can be checked that for the ball $B(a,r)$ and the function $f(x) = |x|^{-s}$, then $B(a,r)^* = B(r)$, $f^* = f$, and $[\mathcal{X}_{B(a,r)}]^* = \mathcal{X}_{B(r)}$. Hence, by Hardy-Littlewood inequality, we have that

$$\int_{B(a,r)} |x|^{-s} dx \leq \int_{\mathbb{R}^n} f(x)\mathcal{X}_{B(a,r)}(x) dx \leq \int_{\mathbb{R}^n} f^*(x)\mathcal{X}_{B(r)}(x) dx = \int_{B(r)} |x|^{-s} dx.$$

Therefore, the integral $\int_{B(a,r)} |x|^{-s} dx$ is maximized when $a = 0$. Thus, Lemma 2.5 yields

$$\|f\|_{m^p_{q,\lambda}} = \sup_{a\in\mathbb{R}^n, 0<r<\lambda} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)\cap B(0,\lambda)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}}$$

$$= \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\omega_{n-1} \int_0^r s^{-\frac{np}{q}+n-1} ds\right)^{\frac{1}{p}} = \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\frac{\omega_{n-1}}{n} \frac{q}{q-p} r^{-\frac{np}{q}+n}\right)^{\frac{1}{p}} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}}.$$

For g, we may also obtain that

$$\|g\|_{m^p_{q,\lambda}} = \sup_{a\in\mathbb{R}^n, 0<r<\lambda} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)\cap B(0,\lambda)\cap B(0,\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\cap B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}}.$$

We consider two cases as follows.

(i) If $0 < r < \delta$, then

$$\frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\cap B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}}.$$

(ii) If $\delta \le r < \lambda$, then

$$\frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\cap B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \left(\frac{\delta}{r}\right)^{n\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

The assumption $1 \le p < q < \infty$ then implies that

$$\sup_{\delta\le r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\cap B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \sup_{\delta\le r<\lambda} \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \left(\frac{\delta}{r}\right)^{n\left(\frac{1}{p}-\frac{1}{q}\right)} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}}.$$

From the two cases, we obtain

$$\|g\|_{m^p_{q,\lambda}} = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}}.$$

Finally, for h,

$$\begin{aligned}
\|h\|_{m^p_{q,\lambda}} &= \sup_{a\in\mathbb{R}^n, 0<r<\lambda} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)\cap B(0,\lambda)\cap(B(0,\lambda)\setminus B(0,\delta))} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\
&= \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\cap(B(\lambda)\setminus B(\delta))} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\
&= \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\setminus B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\
&= \sup_{\delta\le r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\setminus B(\delta)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\
&= \sup_{\delta\le r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\frac{\omega_{n-1}}{n-1} \frac{q}{q-p}\right)^{\frac{1}{p}} \left(r^{-\frac{np}{q}+n} - \delta^{-\frac{np}{q}+n}\right)^{\frac{1}{p}} \\
&= \sup_{\delta\le r<\lambda} \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \left(1 - \left(\frac{\delta}{r}\right)^{-\frac{np}{q}+n}\right)^{\frac{1}{p}} \\
&= \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}} \left(\frac{q}{q-p}\right)^{\frac{1}{p}} \left(1 - \left(\frac{\delta}{\lambda}\right)^{-\frac{np}{q}+n}\right)^{\frac{1}{p}}.
\end{aligned}$$

It completes the proof of Lemma 2.9. □

In proving our main results, we define functions f_q, g_q, and h_q, where $f_q = |x|^{-n/q}, g_q(x) = \mathcal{X}_{(0,1)}(|x|)f_q(x), h_q(x) = \mathcal{X}_{[1,\infty)}(|x|)f(x)$ for $x \in \mathbb{R}^n$. Hence, by Theorem 2.8, $\|f_q\|_{\mathcal{M}^p_q} = \|g_q\|_{\mathcal{M}^p_q} = \|h_q\|_{\mathcal{M}^p_q}$. We also define the function $\tilde{g}_q = g_q/\|f_q\|_{\mathcal{M}^p_q}$ and $\tilde{h}_q = h_q/\|f_q\|_{\mathcal{M}^p_q}$. Then, $\tilde{g} + \tilde{h} = (g_q + h_q)/\|f_q\|_{\mathcal{M}^p_q} = f_q/\|f_q\|_{\mathcal{M}^p_q}$. Moreover, $\tilde{g}, \tilde{h} \in S_{\mathcal{M}^p_q}$.

For $0 < \delta < \lambda$, we define the functions $f_{q,\delta}, g_{q,\delta}$, and $h_{q,\delta}$ where $f_{q,\delta} = \mathcal{X}_{(0,\lambda)}|x|^{-n/q}, g_{q,\delta}(x) = \mathcal{X}_{(0,\delta)}(|x|)f_{q,\delta}(x), h_{q,\delta}(x) = \mathcal{X}_{[\delta,\lambda)}(|x|)f_{q,\delta}(x)$ for $x \in \mathbb{R}^n$. By Theorem 2.9, $\|f_{q,\delta}\|_{m^p_{q,\lambda}} = \|g_{q,\delta}\|_{m^p_{q,\lambda}}$ and

$$\|h_{q,\delta}\|_{m^p_{q,\lambda}} = \|h_{q,\delta}\|_{m^p_{q,\lambda}}\left(1 - \left(\frac{\delta}{\lambda}\right)^{-\frac{np}{q}+n}\right)^{\frac{1}{p}}.$$

Next, we consider the function $\tilde{g}_{q,\delta} = g_{q,\delta}/\|g_{q,\delta}\|_{m^p_{q,\lambda}}$ and $\tilde{h}_{q,\delta} = h_{q,\delta}/\|h_{q,\delta}\|_{m^p_{q,\lambda}}$. Then, $\tilde{g}, \tilde{h} \in S_{m^p_{q,\lambda}}$. By using these functions, we are now ready to prove our main results.

Proof of Theorem 2.1. It is clear that $\tilde{g}_q \perp_I \tilde{h}_q$. We shall compute the form

$$\|\tilde{g}_q + 2\tilde{h}_q\|_{\mathcal{M}^p_q} = \frac{1}{\|f\|_{\mathcal{M}^p_q}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(a,r)} (\mathcal{X}_{(0,1)}(|x| + 2\cdot\mathcal{X}_{[1,\infty)})(|x|))^p\,|x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}.$$

(i) If $0 < r < 1$, then

$$\begin{aligned}
&\sup_{a\in\mathbb{R}^n, 0<r<1} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(a,r)} (\mathcal{X}_{(0,1)}(|x| + 2\cdot\mathcal{X}_{[1,\infty)})(|x|))^p\,|x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}\\
&= \sup_{0<r<1} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(r)} \mathcal{X}_{(0,1)}(|x|)|x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}\\
&= \sup_{0<r<1} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(r)} |x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}\\
&= \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}}\left(\frac{q}{q-p}\right)^{\frac{1}{p}} = \|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

(ii) If $1 \le r < \infty$, then

$$\begin{aligned}
&\sup_{a\in\mathbb{R}^n, r\ge 1} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(a,r)} (\mathcal{X}_{(0,1)}(|x| + 2\cdot\mathcal{X}_{[1,\infty)})(|x|))^p\,|x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}\\
&= 2\sup_{r\ge 1} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(r)} \mathcal{X}_{[1,\infty)}(|x|)|x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}\\
&= 2\sup_{r\ge 1} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}}\left(\int_{B(r)\setminus B(1)} |x|^{-\frac{np}{q}}\,dx\right)^{\frac{1}{p}}\\
&= 2\sup_{r\ge 1}\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}}\left(\frac{q}{q-p}\right)^{\frac{1}{p}}\left(1 - r^{\frac{np}{q}-n}\right)^{\frac{1}{p}}\\
&= 2\left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{q}}\left(\frac{q}{q-p}\right)^{\frac{1}{p}} = 2\|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Hence, $\|\tilde{g}_q + 2\tilde{h}_q\|_{\mathcal{M}^p_q} = 2$. By a similar way, $\|2\tilde{g}_q + \tilde{h}_q\|_{\mathcal{M}^p_q} = 2$. Since $\tilde{g}_q + \tilde{h}_q = \tilde{f}_q$, we have that

$$\bar{\Omega}^{(s)}(\mathcal{M}^p_q) \ge \frac{\|g_q + 2h_q\|^s + \|2g_q + h_q\|^s}{5^{s-1}\|g_q + h_q\|} = \frac{2^s + 2^s}{5^{s-1}} = \frac{2^{s+1}}{5^{s-1}}.$$

Theorem 2.6 then implies that

$$\bar{\Omega}^{(s)}(\mathcal{M}^p_q) = \frac{2^{s+1}}{5^{s-1}}.$$

In the other hand, by the choice of the functions and Theorem 2.7, we obtain that

$$\Omega^{(s)}(\mathcal{M}^p_q) = \frac{2^{s+1}}{5^{s-1}}.$$

These prove Theorem 2.1. □

Proof of Theorem 2.3. We consider the functions $\tilde{g}_{q,\delta}$ and $\tilde{h}_{q,\delta}$. We first need to check if the functions are isometrically orthogonal or not. Note that by properties of radial function,

$$\begin{aligned}\|\tilde{g}_{q,\delta} + \tilde{h}_{q,\delta}\|_{m^p_{q,\lambda}} &= \frac{1}{\|f_{q,\lambda}\|_{m^p_{q,\lambda}}} \sup_{a\in\mathbb{R}^n, 0<r<\lambda} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)} (\mathcal{X}_{(0,\delta)}(|x|) + \mathcal{X}_{[\delta,\lambda)}(|x|))^p \, |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\ &= \frac{1}{\|f_{q,\lambda}\|_{m^p_{q,\lambda}}} \sup_{0<r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} (\mathcal{X}_{(0,\delta)}(|x|) + \mathcal{X}_{[\delta,\lambda)}(|x|))^p \, |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}}.\end{aligned}$$

We consider two cases as follows.

(i) For $0 < r < \delta$,

$$\sup_{0<r<\delta} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} (\mathcal{X}_{(0,\delta)}(|x|) + \mathcal{X}_{[\delta,\lambda)}(|x|))^p \, |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \sup_{0<r<\delta} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = \|f_{q,\lambda}\|_{m^p_{q,\lambda}}.$$

(ii) For $\delta \le r < \lambda$,

$$\begin{aligned}\sup_{\delta\le r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} (\mathcal{X}_{(0,\delta)}(|x|) + \mathcal{X}_{[\delta,\lambda)}(|x|))^p \, |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} &= \sup_{\delta\le r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} \mathcal{X}_{[\delta,\lambda)}(|x|)|x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\ &= \sup_{\delta\le r<\lambda} \frac{1}{|B(r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)\setminus B(\delta)} \mathcal{X}_{[\delta,\lambda)}(|x|)|x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} \\ &= \|h_{q,\delta}\|_{m^p_{q,\lambda}} = \|f_{q,\lambda}\|_{m^p_{q,\lambda}}.\end{aligned}$$

From the two cases, we can conclude that

$$\frac{1}{\|f_{q,\lambda}\|_{m^p_{q,\lambda}}} \sup_{0<r<\lambda} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(r)} (\mathcal{X}_{(0,\delta)}(|x|) + \mathcal{X}_{[\delta,\lambda)}(|x|))^p \, |x|^{-\frac{np}{q}} dx\right)^{\frac{1}{p}} = 1$$

and $\|\tilde{g}_{q,\delta} + \tilde{h}_{q,\delta}\|_{m^p_{q,\lambda}} = 1$. By the same way, we may obtain $\|\tilde{g}_{q,\delta} - \tilde{h}_{q,\delta}\|_{m^p_{q,\lambda}} = 1$. Hence, $\tilde{g}_{q,\delta} \perp_I \tilde{h}_{q,\delta}$. By following the technique as in calculating the norm $\|\tilde{g}_{q,\delta} + \tilde{h}_{q,\delta}\|_{m^p_{q,\lambda}}$, we obtain

$$\|\tilde{g}_{q,\delta} + 2\tilde{h}_{q,\delta}\|_{m^p_{q,\lambda}} = 2$$

and

$$\|2\tilde{g}_{q,\delta} + \tilde{h}_{q,\delta}\|_{m^p_{q,\lambda}} = 2.$$

Therefore,

$$\bar{\Omega}^{(s)}(m^p_{q,\lambda}) \ge \frac{\|\tilde{g}_{q,\lambda} + 2\tilde{h}_{q,\lambda}\|^s_{m^p_{q,\lambda}} + \|2\tilde{g}_{q,\lambda} + \tilde{h}_{q,\lambda}\|^s_{m^p_{q,\lambda}}}{5^{s-1}\|\tilde{g}_{q,\lambda} + \tilde{h}_{q,\lambda}\|_{m^p_{q,\lambda}}} = \frac{2^s + 2^s}{5^{s-1}} = \frac{2^{s+1}}{5^{s-1}}.$$

By Theorem 2.6,

$$\bar{\Omega}^{(s)}(m^p_{q,\lambda}) = \frac{2^{s+1}}{5^{s-1}}.$$

Moreover, the choice of functions and Theorem 2.7 implies that

$$\Omega^{(s)}(m^p_{q,\lambda}) = \frac{2^{s+1}}{5^{s-1}}.$$

These prove Theorem 2.3. □

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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