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Research Article

Hopf Bifurcation Analysis of a Zika Virus Transmission Model with Two Time Delays

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Abstract

This study focuses on a mathematical model of Zika virus transmission that incorporates multiple time delays. The inclusion of time delays in the model takes into account the incubation period in humans and the latency of disease transmission from mosquitoes. The qualitative behavior of the model was examined in four different conditions by analyzing the characteristic equation corresponding to the endemic equilibrium point. Furthermore, the two distinct time lags were selected as the bifurcation parameter, while the existence of a Hopf bifurcation at the endemic equilibrium point for threshold parameters was confirmed. Subsequently, numerical simulations were used to validate the theoretical analysis for each case using MATLAB.

1. Introduction

Eco-epidemiology is a branch of bio-mathematics that examines infectious disease models to understand the behavior of disease transmission. In recent years, some researchers in this field have focused their attention on models that include time-delay (or memory) terms predator–prey models, considering that the events that occur in real-life predator–prey populations take time to complete. In this manner, natural processes, including gestation times, incubation periods, and transportation delays, were incorporated into the problem and examined as more realistic models.

During the past few decades, numerous researchers have analyzed the local stability and Hopf bifurcation at the equilibrium points of the predator-prey models containing just a single time-lag term, assuming that the other time lags would be unimportant or negligible (see, e.g., [1–7], and reference therein). However, some researcher believe that this assumption is detrimental to the realism of such models and have explored several types of delay terms, and the effects of multiple delays on the stability and the existence of Hopf bifurcation (see, e.g., [8–11], and reference therein).

The system considered in this paper, which models the spread of Zika virus, a vector-borne disease carried by the Aedes mosquito, is an example of a predator-prey system. Zika virus was initially identified in a monkey in Uganda in 1947 and was declared a public health emergency of international concern by the World Health Organization (WHO) in 2016. While the Zika virus was initially assumed to only spread from mosquito bites, it was later found to also be transmitted by sexual intercourse and blood transfer. As a result, Bewick et al.'s model [12], which to our knowledge, is the first mathematical model of the spread of the Zika virus, has attracted the attention of numerous researchers and has been revised multiple times (see, e.g., [13–17] and reference therein).

In the study by Agusto et al. [13], the transmission model of Zika virus considering newly born babies, adult humans, and the mosquito population was built, and local stability of the disease-free equilibrium point was observed as the reproduction number.

In the study by Biswas et al. [15], an ordinary differential equations model was used to study the spread of the Zika virus dynamics that incorporated a nonlinear saturation for host infection and bilinear incidence for vector infection, and analyzed the existence and local stability of a disease-free equilibrium point.

In the study by Ding et al, [16], who are among those who examined the use of the mathematical models in preventing and controlling the spread of infectious diseases, established the following model for the spread of Zika virus disease, with no specific vaccine, drug, or

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treatment, and proposed solution by obtaining basic reproduction number.

$$\frac{dx}{dt} = a_1 - b_1 b_2 v(t) x(t) - d_1 x(t),$$

$$\frac{dy}{dt} = b_1 b_2 v(t) x(t) - d_1 y(t),$$

$$\frac{dz}{dt} = a_2 - b_3 b_4 z(t) y(t) - d_2 z(t),$$

$$\frac{dv}{dt} = b_3 b_4 z(t) y(t) - d_2 v(t).$$

The variables in this problem are x (the number of people who are susceptible), y (the number of people who are infected), z (the number of mosquitoes that are susceptible), and v (the number of mosquitoes that are infected). All the parameters are positive, a_1 is the recruitment rate of x, a_2 is the recruitment rate of z, d_1 is the natural death rate of humans, d_2 is the natural death rate of mosquitoes. b_1 is the contact rate between humans and mosquitoes, b_2 is the transmission rate from an infected mosquito to susceptible humans, b_3 is the contact rate between mosquito and human, and b_4 is the transmission rate from infected humans to susceptible mosquitoes.
Inspired by the aforementioned studies, and to be more closely aligned with real-world phenomena, we analyzed the following compartmental model of the spread of the Zika virus and considered two different time-delay terms, where τ represents the transmission time from the infected mosquitoes into suspected humans and where ρ represents the incubation period of susceptible mosquitoes to become infectious.

$$\frac{dx}{dt} = a_1 - b_1 b_2 v(t-\tau) x(t-\tau) - d_1 x(t), \tag{1.1}$$

$$\frac{dy}{dt} = b_1 b_2 v(t-\tau) x(t-\tau) - d_1 y(t), \tag{1.2}$$

$$\frac{dz}{dt} = a_2 - b_3 b_4 z(t-\rho) y(t-\rho) - d_2 z(t), \tag{1.3}$$

$$\frac{dv}{dt} = b_3 b_4 z(t-\rho) y(t-\rho) - d_2 v(t), \tag{1.4}$$

with the given initial conditions:

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, v(0) \geq 0,$$

and $\tau \geq 0$, $\rho \geq 0$.
The next section of this paper describes how the equilibrium points and reproduction number were obtained. Section3 presents, local stability analysis and the conditions of Hopf bifurcation existence for the endemic equilibrium point of four different situations. In Section 4, numerical simulations for each case of time delays are performed using MATLAB to confirm the theoretical conclusions.

2. Equilibrium Points and Characteristic Equation

First, we found the equilibrium points of the vector-borne system (1.1)-(1.4). The equilibrium points $x(t) = x(t-\tau) = x^*$, $y(t) = y(t-\rho) = y^*$, $z(t) = z(t-\rho) = z^*$, $v(t) = v(t-\tau) = v^*$ satisfy the following system:

$$0 = a_1 - b_1 b_2 v^* x^* - d_1 x^*,$$

$$0 = b_1 b_2 v^* x^* - d_1 y^*,$$

$$0 = a_2 - b_3 b_4 z^* y^* - d_2 z^*,$$

$$0 = b_3 b_4 z^* y^* - d_2 v^*,$$

As such, the time-delayed system (1.1)-(1.4) has two equilibrium points. The first is the disease-free equilibrium point: $P = (\frac{a_1}{d_1}, 0, \frac{a_2}{d_2}, 0)$, in which all vectors are susceptible. The second is the endemic equilibrium point:

$$P_* = \left(\frac{a_1(a_1 b_3 b_4 + d_1 d_2)}{d_1[a_1 b_3 b_4 + d_1 d_2 R_0^2]}, \frac{a_1 d_2 (R_0^2 - 1)}{a_1 b_3 b_4 + d_1 d_2 R_0^2}, \frac{d_1^2 d_2 (a_1 b_3 b_4 + d_1 d_2 R_0^2)}{a_1 b_1 b_2 b_3 b_4 (a_1 b_3 b_4 + d_1 d_2)}, \frac{d_1^2 d_2 (R_0^2 - 1)}{b_1 b_2 (a_1 b_3 b_4 + d_1 d_2)}\right),$$

where the basic reproduction number R_0 was determined by next-generation matrix method reported by Kara [18]:

$$R_0 = \frac{\sqrt{a_1 a_2 b_1 b_2 b_3 b_4}}{d_1 d_2}.$$

These equilibrium points for this vector-borne system are the same for different delay models and those that do not consider time-delays. From the biological perspective, we only consider the endemic equilibrium point P_* and investigate the local stability conditions and conditions for the existence of a Hopf bifurcation.
To obtain the characteristic equation of the system, which consists of two different time-delayed equations described by (1.1)-(1.4), we first linearized the system around the endemic equilibrium point P_* and then obtained the following characteristic equation:

$$(\lambda+d_1)(\lambda+d_2)[\lambda^2+\lambda(d_1+d_2)+d_1d_2+(\lambda a+ad_1)e^{-\lambda\rho}+(\lambda b+bd_2)e^{-\lambda\tau}+me^{-\lambda(\tau+\rho)}]=0, \tag{2.1}$$

where $a=b_3b_4y^*$, $b=b_1b_2v^*$ and $m=ab-d_1d_2$, which was obtained from the determinant:

$$|J_0-\lambda I+J_1e^{-\lambda\tau}+J_2e^{-\lambda\rho}|=0,$$

where the matrices are:

$$J_0=\begin{bmatrix} -d_1 & 0 & 0 & 0\\ 0 & -d_1 & 0 & 0\\ 0 & 0 & -d_2 & 0\\ 0 & 0 & 0 & -d_2 \end{bmatrix},$$

$$J_1=\begin{bmatrix} -b_1b_2v^* & 0 & 0 & -b_1b_2x^*\\ b_1b_2v^* & 0 & 0 & b_1b_2x^*\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$J_2=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -b_3b_4z^* & -b_3b_4y^* & 0\\ 0 & b_3b_4z^* & b_3b_4y^* & 0 \end{bmatrix}.$$

The characteristic equation (2.1) has roots $\lambda_1=-d_1$ and $\lambda_2=-d_2$. Thus, the following second-order transcendental polynomial determines the characteristic of the endemic equilibrium point:

$$[\lambda^2+\lambda(d_1+d_2)+d_1d_2+(\lambda a+ad_1)e^{-\lambda\rho}+(\lambda b+bd_2)e^{-\lambda\tau}+me^{-\lambda(\tau+\rho)}]=0. \tag{2.2}$$

3. Dynamics of Delayed Vector-Borne Model

Here we present the analysis of local stability and the conditions for the existence of a Hopf bifurcation for the endemic equilibrium point, with consideration of the following four situations for time delays τ and ρ.

3.1. Case 1: $\tau=\rho=0$

In this case, the equation (2.2) reduced to:

$$\lambda^2+\lambda(d_1+d_2+a+b)+(ab+ad_1+bd_2)=0.$$

All roots of the above polynomial have negative real parts, since $\lambda_{1,2}=\frac{-(d_1+d_2+a+b)\pm\sqrt{(d_1+d_2+a+b)^2-4(ab+ad_1+bd_2)}}{2}$, as well as $(d_1+d_2+a+b)>0$ and $(ab+ad_1+bd_2)>0$, we concluded that the endemic equilibrium point P_* of the problem in the absence of delays is asymptotically stable.

3.2. Case 2: $\tau>0,\rho=0$

In this case, the equation (2.2) becomes:

$$[\lambda^2+\lambda(d_1+d_2+a)+d_1(d_2+a)+(\lambda b+bd_2+m)e^{-\lambda\tau}]=0. \tag{3.1}$$

The stability of an endemic equilibrium point P_* depends upon the existence of purely imaginary solutions to the above characteristic equation. To demonstrate this, we proceeded with the assumption that $\lambda=i\omega$ $(\omega>0)$ is the root of the equation (3.1) and, we obtained the polynomial:

$$-\omega^2+i\omega(d_1+d_2+a)+d_1(d_2+a)+(i\omega b+bd_2+m)(cos\omega\tau-isin\omega\tau)=0.$$

Separating the real and imaginary parts, we get:

$$-(-\omega^2+d_1(a+d_2))=(m+d_2b)cos\omega\tau+\omega bsin\omega\tau, \tag{3.2}$$

$$-\omega(d_1+d_2+a)=\omega bcos\omega\tau-(m+d_2b)sin\omega\tau. \tag{3.3}$$

The square of equations (3.2) and (3.3) yield the following polynomial:

$$\omega^4-\omega^2[b^2-d_1^2-(d_2+a)^2]+d_1^2(a+d_2)^2-(m+bd_2)^2=0. \tag{3.4}$$

Let $\tilde{B}:=b^2-d_1^2-(d_2+a)^2$, $\tilde{C}:=d_1^2(a+d_2)^2-(m+bd_2)^2$ and $\tilde{\Delta}:=\tilde{B}^2-4\tilde{C}$.
If $\tilde{B}<0$ and $\tilde{C}>0$ or $\tilde{\Delta}>0$, then none of the roots of the equation (3.4) are positive. Unfortunately, the equation (3.1) does not possess purely imaginary roots, but shows that the roots of equation (3.1) have negative real parts. Further analysis is presented in Lemma 3.1.

Lemma 3.1.

i) If $\tilde{B} < 0$ and $\tilde{C} > 0$ or $\tilde{\Delta} < 0$, then equation (3.4) has no positive root. Furthermore, equation (3.1) does not have a purely imaginary root.

ii) If $\tilde{C} < 0$ or $\tilde{B} > 0$ and $\tilde{\Delta} = 0$, then the equation (3.4) has only one positive root, so there is a pair of purely imaginary roots $\pm i\omega_+$ in equation (3.1).

iii) If $\tilde{B} > 0$, $\tilde{C} > 0$ and $\tilde{\Delta} > 0$, then the equation (3.4) has two positive roots. Consequently, equation (3.1) has a pair of imaginary roots $\pm i\omega_+$ and $\pm i\omega_-$.

If we consider Lemma 3.1, either (ii) or (iii) valid, the equations (3.2) and (3.3) can be used to derive the critical values $\tau_j^{\pm}$ of τ, which are associated with the positive roots $\omega_{\pm}$ of the equation (3.4).

$$\tau_j^{\pm} = \frac{1}{\omega_{\pm}}[Arccos(-\frac{\omega_{\pm}^2 d_1(b+d_2)+d_1(a+d_2)(m+d_2b)}{\omega_{\pm}^2 b^2+(m+d_2b)^2})+2\pi j],$$

$\tau_0 := \min\{\tau_0^+, \tau_0^-\}$. If we allow $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$ to be the root of (3.1) near $\tau = \tau_0$ satisfying $\eta(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$, then the condition of the transversality of Lemma 3.2 is true.

Lemma 3.2. *If the transversality condition is satisfied,*

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\lambda=i\omega_0,\tau=\tau_0} > 0,$$

the time-delayed system (1.1)-(1.4) undergoes a Hopf bifurcation at the endemic equilibrium point P_* when $\tau = \tau_0, \rho = 0$ ([9], [19]).

Taking the derivative of equation (3.1) with respect to τ, we obtain:

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda+d_1+d_2+a}{\lambda(\lambda b+d_2b+m)}e^{\lambda\tau} + \frac{b}{\lambda(\lambda b+d_2b+m)} - \frac{\tau}{\lambda}.$$

Using the equations (3.1) and (3.4), the transversality condition is obtained:

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\lambda=i\omega_0,\tau=\tau_0} = \frac{\sqrt{\tilde{\Delta}}}{\omega_0^2 b^2+(m+d_2b)^2} > 0.$$

Similarly, we obtain the same conclusion for other positive roots of equation (3.4).
In accordance with the above analysis, we arrive at Theorem 3.3:

Theorem 3.3. *For the system (1.1)-(1.4) with $\tau > 0$ and $\rho = 0$, the following results are correct:*

a) if the condition i) of the Lemma 3.1 is correct, then the endemic equilibrium point P_* is asymptotically stable for all $\tau > 0$.

b) if the condition ii) or iii) of Lemma 3.1 is correct, then there exists a positive number τ_0 such that the endemic equilibrium point P_* is locally asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Furthermore, the system (1.1)-(1.4) undergoes a Hopf bifurcation at the equilibrium point P_* when $\tau = \tau_0$.

3.3. Case 3: $\tau = 0, \rho > 0$

In this case, the equation (2.2) for equilibrium point P_* becomes the following transcendental equation:

$$\lambda^2 + \lambda(d_1+d_2+b) + d_2(d_1+b) + (\lambda a + ad_1 + m)e^{-\lambda\rho} = 0. \tag{3.5}$$

If $\lambda = i\omega$ $(\omega > 0)$ is a root of the equation (3.5), the following equation is obtained:

$$-\omega^2 + i\omega(d_1+d_2+b) + d_2(d_1+b) + (m+ad_1+i\omega a)(cos\omega\rho - isin\omega\rho) = 0.$$

Separating the real and imaginary parts, we get the following equations:

$$-[-\omega^2 + d_2(d_1+b)] = (m+ad_1)cos\omega\rho + \omega asin\omega\rho, \tag{3.6}$$

$$-\omega(d_1+d_2+b) = \omega acos\omega\rho - (m+ad_1)sin\omega\rho. \tag{3.7}$$

Upon summing the squares of the equations (3.6) and (3.7), we get the following equation:

$$\omega^4 - \omega^2[a^2 - d_2^2 - (d_1+b)^2] + d_2^2(b+d_1)^2 - (m+ad_1)^2 = 0. \tag{3.8}$$

If $B := a^2 - d_2^2 - (d_1+b)^2$ and $C := d_2^2(b+d_1)^2 - (m+ad_1)^2$ and $\Delta := B^2 - 4C$ following a similar procedure to case 2, we reach Lemma 3.4.

Lemma 3.4.

i) If $C > 0$ and $B < 0$ or $\Delta < 0$, then equation (3.8) does not have any positive root. Similarly, equation (3.5) does not possess a purely imaginary root.

ii) If $C < 0$ or $B > 0$, and $\Delta = 0$, then the equation (3.8) has only one positive root. Therefore, equation (3.5) has a pair of imaginary roots $\pm i\omega_1$.

iii) If $B > 0$, $C > 0$, and $\Delta > 0$, then the equation (3.8) has two positive roots. And equation (3.5) has a pair of imaginary roots $\pm i\omega_1$ and $\pm i\omega_2$.

When condition ii) or iii) of the lemma (3.4) is met, the critical values ρ_{kj}'s of ρ with respect to ω_k, $(k = 1,2)$ are as follows:

$$\rho_{kj} = \frac{1}{\omega_k}[Arccos\Big(-\frac{d_2(a+d_1)\omega_k^2 + d_2(b+d_1)(m+ad_1)}{\omega_k^2 a^2 + (m+ad_1)^2}\Big) + 2\pi j], \quad k = 1,2, j = 0,1,2,...$$

If $\lambda(\rho_0) = i\omega(\rho_0)$ where $\rho_0 := \min\{\rho_{k0} : k = 1,2\}$ is a root of the equation (3.5) near $\rho = \rho_0$ with $\omega(\rho_0) = \omega_*$, then we have the following transversality condition by differentiating the equation (3.5) with respect to ρ:

$$\Big(\frac{d\lambda}{d\rho}\Big)^{-1} = \frac{2\lambda + d_1 + d_2 + b}{\lambda(\lambda a + d_1 a + m)} e^{\lambda\rho} + \frac{a}{\lambda(\lambda a + d_1 a + m)} - \frac{\rho}{\lambda},$$

By combining equations (3.5) and (3.8), we deduce the following conclusion for which the transversality condition holds:

$$Re\Big(\frac{d\lambda}{d\rho}\Big)^{-1}\Big|_{\lambda = i\omega_*, \rho = \rho_0} = \frac{\sqrt{\Delta}}{\omega_*^2 a^2 + (m+d_1 a)^2} > 0.$$

Theorem 3.5. *For the system (1.1)-(1.4) with the case $\tau = 0$ and $\rho > 0$, similar conclusions to Theorem 3.3 can be obtained according to $\rho = \rho_0$ critical value. The following results hold:*

a) if the condition i) of Lemma 3.4 holds, then the endemic equilibrium point P_* is asymptotically stable for all $\rho > 0$.

b) if the condition ii) or iii) of Lemma 3.4 is correct, then there exists a positive number ρ_0 such that the endemic equilibrium point P_* is locally asymptotically stable when $\rho \in [0, \rho_0)$ and unstable when $\rho > \rho_0$. Furthermore, the system (1.1)-(1.4) undergoes a Hopf bifurcation at the equilibrium point P_* when $\rho = \rho_0$.

3.4. Case 4: $\tau > 0, \rho > 0$

In this case, we suppose that τ is a parameter and $\rho \in [0, \rho_0)$ is a constant in its stability interval. If $\lambda = i\bar{\omega}$ is a root of equation (2.2), then equation (2.2) reduces to:

$$-\bar{\omega}^2 + i\bar{\omega}(d_1 + d_2) + d_1 d_2 + (i\bar{\omega}a + ad_1)e^{-i\bar{\omega}\rho} + (i\bar{\omega}b + d_2 b)e^{-i\bar{\omega}\tau} + me^{-i\bar{\omega}(\tau+\rho)} = 0.$$

Where τ is a parameter, separating the real and imaginary parts gives:

$$-[-\bar{\omega}^2 + d_1 d_2 + ad_1 cos\bar{\omega}\rho + \bar{\omega} asin\bar{\omega}\rho] = (d_2 b + mcos\bar{\omega}\rho)cos\bar{\omega}\tau + (\bar{\omega}b - msin\bar{\omega}\rho)sin\bar{\omega}\tau, \tag{3.9}$$

$$-[\bar{\omega}(d_1 + d_2) + \bar{\omega}acos\bar{\omega}\rho - ad_1 sin\bar{\omega}\rho] = (\bar{\omega}b - msin\bar{\omega}\rho)cos\bar{\omega}\tau - (d_2 b + mcos\bar{\omega}\rho)sin\bar{\omega}\tau. \tag{3.10}$$

By eliminating $cos\bar{\omega}\tau$ and $sin\bar{\omega}\tau$, we get the equation:

$$\Psi(\bar{\omega}) = \bar{\omega}^4 - P_1\bar{\omega}^2 + P_2 + P_3 cos\bar{\omega}\rho + P_4 sin\bar{\omega}\rho = 0, \tag{3.11}$$

where

$$\theta = a(\bar{\omega}^2 + d_1^2) - mb,$$

$$P_1 = \quad b^2 - a^2 - d_1^2 - d_2^2,$$

$$P_2 = d_1^2(a + d_2^2) - m^2 - b^2 d_2^2,$$

$$P_3 = 2d_2\theta,$$

$$P_4 = -2\bar{\omega}\theta.$$

If $\Psi(0) = ad_1^2(1+2d_2) - b^2(a^2 + d_2^2 + 2ad_2) - 2abd_1 d_2 < 0$, then because of $\lim_{\bar{\omega}\to\infty}\Psi(\bar{\omega}) \to \infty$ equation (3.11) has at least one positive root. Not loose of generality, we assume that it has $\bar{\omega}_1, ...\bar{\omega}_4$ four positive roots, and for each $\bar{\omega}_k$, $(k = 1,..,4)$ there exists a sequence $\{\tau_{kj} : j = 0,...,k = 1,..,4\}$ so that equation (3.11) satisfies the criteria.
Using the equations (3.9) and (3.10), we get the critical time delays τ_{kj}'s of τ for each $\bar{\omega}_k$:

$$\tau_{kj} = \frac{1}{\bar{\omega}_k}\Bigg(Arccos\Bigg[-d_1\bar{\omega}_k{}^2[b + d_2 cos\bar{\omega}_k\bar{\rho}] - d_1(ma + d_2^2 b) + \bar{\omega}_k d_1[2ab - d_2(d_1 + d_2)]sin\bar{\omega}_k\bar{\rho}$$
$$-d_1 d_2(2ab - d_1 d_2)cos\bar{\omega}_k\bar{\rho} \Big/ \Big(\bar{\omega}_k{}^2 b^2 + m^2 + d_2^2 b^2 + 2mb(d_2 cos\bar{\omega}_k\bar{\rho} - \bar{\omega}_k sin\bar{\omega}_k\bar{\rho})\Big)\Bigg] + 2\pi j\Bigg), j = 0,1,...$$

whereby $\bar{\tau}_0 := min(\tau_{kj} : k = 1,..,4, j = 0,1,...)$ and $\bar{\omega} := \bar{\omega}_0$ are the corresponding root of the equations (3.9) and (3.10) with $\rho = \bar{\rho} \in [0,\rho_0)$.

To verify the transversality condition, if $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$ is the root of equation (2.2) at $\tau = \bar{\tau}_0$ to satisfy $\eta(\bar{\tau}_0) = 0$, $\omega(\bar{\tau}_0) = \bar{\omega}$ with $\rho = \bar{\rho} \in [0,\rho_0)$, the derivation of (2.2) with respect to τ is:

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda + d_1 + d_2}{-\lambda[\lambda^2 + \lambda(d_1+d_2) + d_1 d_2 + a(\lambda + d_1)e^{-\lambda\bar{\rho}}]} - \frac{\tau}{\lambda} + \frac{[a - \bar{\rho}a(\lambda + d_1)]e^{-\lambda\bar{\rho}} + be^{-\lambda\tau} - m\bar{\rho}e^{-\lambda(\tau+\bar{\rho})}}{\lambda[b(\lambda + d_2)e^{-\lambda\tau} + me^{-\lambda(\tau+\bar{\rho})}]},$$

and

$$\begin{aligned}\bar{\omega}Re\Big(\frac{d\lambda}{d\tau}\Big)^{-1}\Big|_{\lambda=i\bar{\omega},\tau=\bar{\tau}_0} =& \left(\frac{\bar{\omega}(2\bar{\omega}^2 + d_1^2 + d_2^2) + \bar{\omega}a(d_2 - d_1)cos\bar{\omega}\bar{\rho} - a(2\bar{\omega}^2 + d_1(d_1 + d_2))sin\bar{\omega}\bar{\rho}}{\alpha^2 + \beta^2}\right) \\ &+ \Bigg([\bar{\omega}a\bar{\rho}d_2 bcos\bar{\omega}(\bar{\rho} + \bar{\tau}_0) - b^2\bar{\omega} - a(1 - \bar{\rho}d_1)d_2 bsin\bar{\omega}(\bar{\rho} - \bar{\tau}_0) - \bar{\omega}ab(1 - d_1)cos\bar{\omega}(\bar{\rho} - \bar{\tau}_0) \\ &- \bar{\omega}^2 a\bar{\rho}bsin\bar{\omega}(\bar{\rho} + \bar{\tau}_0) - ma(1 - \bar{\rho}d_1)sin\bar{\omega}\bar{\tau}_0 - m\bar{\omega}\bar{\rho}acos\bar{\omega}\bar{\tau}_0 + mb(1 + \bar{\rho}d_2)sin\bar{\omega}\bar{\rho} \\ &+ mb\bar{\omega}\bar{\rho}cos\bar{\omega}(\bar{\rho} + 2\bar{\tau}_0))]/(\gamma^2 + \eta^2)\Bigg) \\ &\neq 0,\end{aligned}$$

where

$$\alpha^2 + \beta^2 = \bar{\omega}^2 b^2 + m^2 + b^2 d_2^2 + 2mb(d_2 cos\bar{\omega}\bar{\rho} - \bar{\omega}sin\bar{\omega}\bar{\rho}),$$

$$\gamma^2 + \eta^2 = \Big(d_2 bcos\bar{\omega}\bar{\tau}_0 + \bar{\omega}bsin\bar{\omega}\bar{\tau}_0 + mcos\bar{\omega}(\bar{\tau}_0 + \bar{\rho})\Big)^2 + \Big(\bar{\omega}bcos\bar{\omega}\bar{\tau}_0 - d_2 bsin\bar{\omega}\bar{\tau}_0 - msin\bar{\omega}(\bar{\tau}_0 + \bar{\rho})\Big)^2.$$

Therefore, the transversality condition is satisfied ([9], [19], [20]).

Theorem 3.6. *If we consider the system (1.1)-(1.4) with $\tau > 0$ and $\rho \in [0,\rho_0)$, and suppose $\psi(0) < 0$ is satisfied, then we reach the following results:*

a) *the endemic equilibrium point P_* is asymptotically stable in which $\tau \in [0,\bar{\tau}_0)$.*

b) *the endemic equilibrium point P_* is unstable for $\tau > \bar{\tau}_0$. Moreover, the system (1.1)-(1.4) undergoes a Hopf bifurcation at the endemic equilibrium point P_* when $\tau = \bar{\tau}_0$.*

4. Numerical Demonstrations

To validate the theoretical conclusions drawn in section 3, we refer to numerical simulations performed using MATLAB.
To reach our objective, the following set of parametric values of the system (1.1)-(1.4) is taken, so that the endemic equilibrium point $P_* = (0.2204, 2.7796, 0.9643, 3.1534)$ is obtained.

$$a_1 = 0.3, a_2 = 0.7, b_1 = 0.8, b_2 = 0.5, b_3 = 0.4, b_4 = 0.5, d_1 = 0.1, d_2 = 0.17.$$

The demonstrations in this section are arranged according to the cases in the previous section.

Case 1: In this situation, in the absence of time delays, endemic equilibrium point P_* is asymptotically stable.

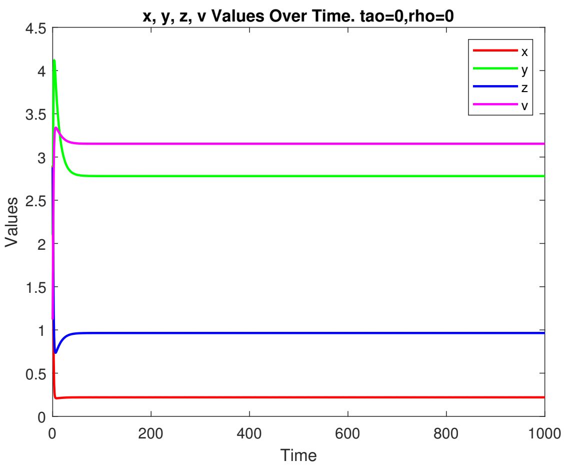

Figure 4.1: When $\tau = 0$ and $\rho = 0$, then the system (1.1)-(1.4) is asymptotically stable.

Case 2: In this situation, the system (1.1)-(1.4) satisfies Theorem 3.3, so the critical value of time delay $\tau_0 = 1.3253$ for $\omega_0 = 1.2515$ is obtained. In Figure 4.2, Figure 4.3, Figure 4.4, the change in behavior of the endemic equilibrium point P_* for $\rho = 0$ and different values of τ and the Hopf bifurcation phase diagram are simulated.

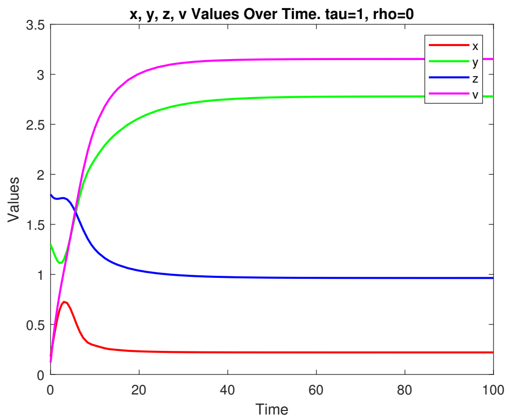

Figure 4.2: When $\tau = 1 < \tau_0$ and $\rho = 0$, then the endemic equilibrium point of the system (1.1)-(1.4) is stable

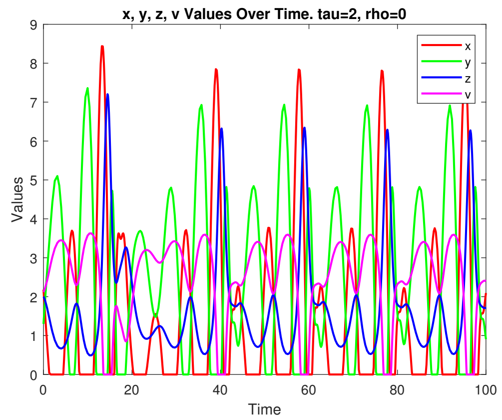

Figure 4.3: When $\tau = 2 > \tau_0$ and $\rho = 0$, then the endemic equilibrium point of the system (1.1)-(1.4) is unstable

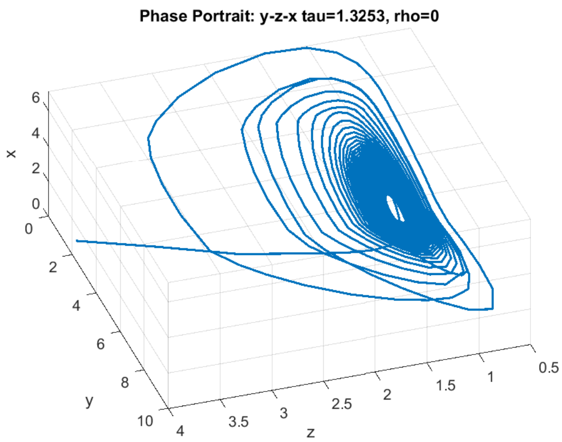

Figure 4.4: When $\tau = \tau_0 = 1.3253$ and $\rho = 0$, then Hopf bifurcation occurs

Case 3: In this situation, the system (1.1)-(1.4) satisfies Theorem 3.5: therefore, the critical value of time delay $\rho_0 = 3.4088$ for $\omega_1 = 0.5178$ is obtained. In Figure 4.5, Figure 4.6, Figure 4.7, the change in behavior of the endemic equilibrium point P_* for $\tau = 0$ and differing values of ρ and the Hopf bifurcation diagram are simulated.

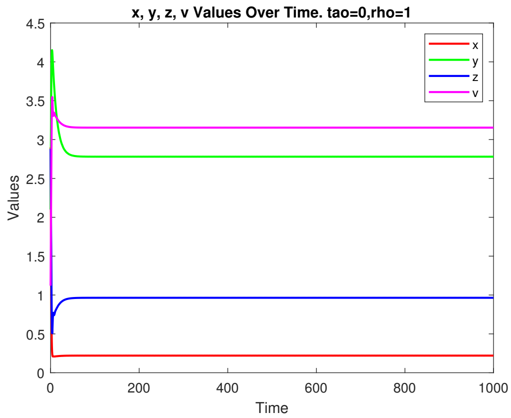

Figure 4.5: When $\tau = 0$ and $\rho = 1 < \rho_0$, then the endemic equilibrium point of the system (1.1)-(1.4) is stable

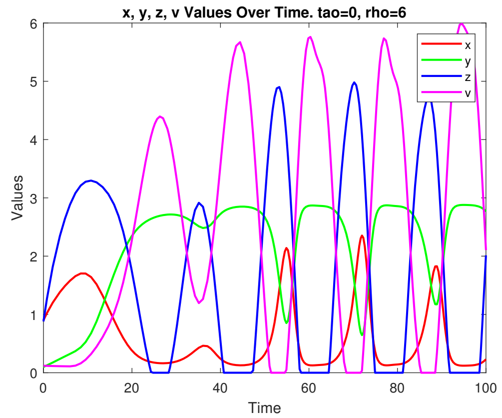

Figure 4.6: When $\tau = 0$ and $\rho = 6 > \rho_0$, then the endemic equilibrium point of the system (1.1)-(1.4) is unstable

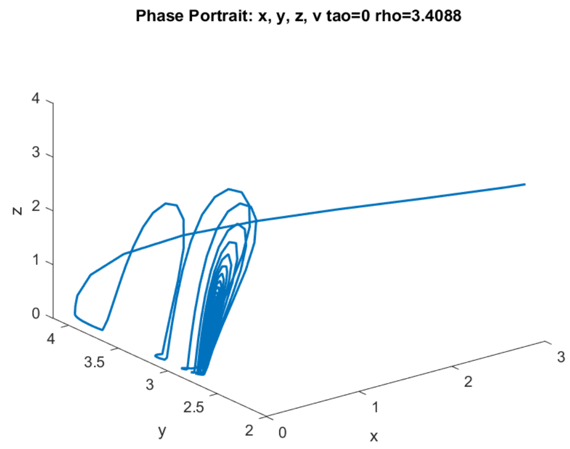

Figure 4.7: When $\tau = 0$ and $\rho = \rho_0 = 3.4088$, then Hopf bifurcation occurs

Case 4: Here, the system (1.1)-(1.4) satisfies the Theorem 3.6; therefore, the critical value of the time delay $\tau_0 = 1.6837$ is obtained for $\bar{\omega} = 1.008$, and $\bar{\rho} = 2$ is the chosen in stability interval. Figure 4.8, Figure 4.9, Figure 4.10 show the simulated change in behavior of the endemic equilibrium point P_* and the Hopf bifurcation.

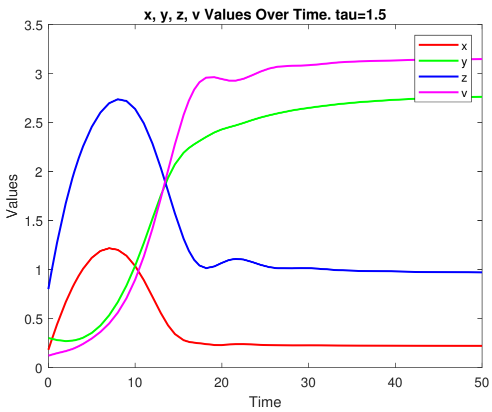

Figure 4.8: When $\tau = 1 < \tilde{\tau}_0$ and $\rho = \tilde{\rho} = 2$ are met, then the equilibrium point of the system (1.1)-(1.4) is stable.

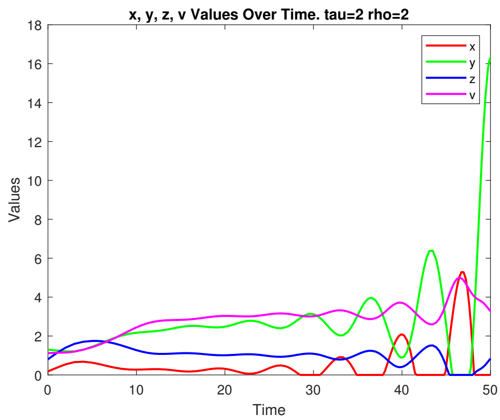

Figure 4.9: When $\tau = 1 > \tilde{\tau}_0$ and $\rho = \tilde{\rho} = 2$ are met, then the equilibrium point of the system (1.1)-(1.4) is unstable.

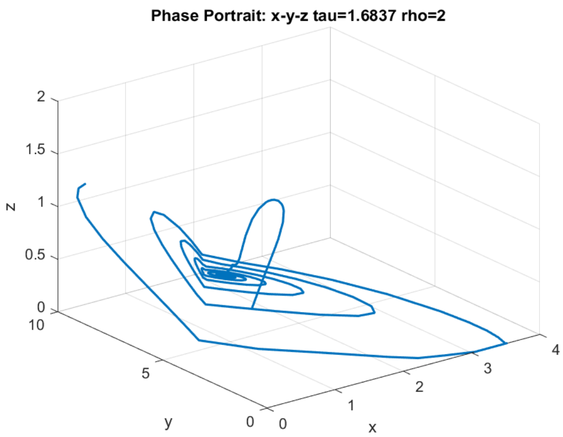

Figure 4.10: When $\tau = \tilde{\tau}_0 = 1.6837$ and $\rho = \tilde{\rho} = 2$ are met, then Hopf bifurcation occurs.

5. Conclusion

In this paper, we analyzed a mathematical model of Zika virus transmission. Despite previous studies reviewed in the introduction, our model includes two different time delays τ, which represents the transmission of infection from infected mosquitoes to susceptible humans, and ρ, which represents the incubation period of susceptible mosquitoes to become infected. These time delays make our model more realistic. We obtained the conditions for asymptotic stability and Hopf bifurcation at the endemic equilibrium point, and the critical values of time delays (threshold parameters). Using numerical simulations, we observed visually that the time delay depends on the threshold parameters similar to those of the obtained theoretical results and when the time delays exceed these values, the disease dynamics change.
From an eco-epidemiological perspective, these results are important for understanding the life cycle of the Zika virus and analyzing the incubation periods and transmission time to control the spread of the disease.
This multiple-time model can be extended to include sexual transmission, blood transfer situations, and numbers of asymptomatic humans for future studies.

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References

[1] P. Baydemir, H. Merdan, E. Karaoglu, G. Sucu, *Complex dynamics of a discrete-time prey–predator system with Leslie type*, International Journal of Bifurcation and Chaos, **30**(10) (2020), Article ID 2050149, https://doi.org/10.1142/S0218127420501497
[2] S. Chomcheon, Y. Lenbury, W. Sarika, *Stability, Hopf bifurcation and effects of impulsive antibiotic treatments in a model of drug resistance with conversion delay*, Adv Differ Equ., **274** (2019), 1-18.
[3] M. E. Hajji, M. F. S. Aloufi, M. H. Alharbi, *Influence of seasonality on Zika virus transmission[J]*, AIMS Mathematics, **9**(7) (2024), 19361-19384.
[4] A. F. Nindjin, M. A. Aziz-Alaoui, M. Cadivel, *Analysis of a predator–prey model with modified Leslie–Gower and Holling-type II schemes with time delay*, Nonlinear Analysis: Real World Applications, **7**(5) (2006), 1104-1118.
[5] S. Olaniyi, *Dynamics of Zika virus model with nonlinear incidence and optimal control strategies*, Appl. Math. Inf. Sci., **12** (5) (2018), 969-982.
[6] R. P. Singh, *Bifurcation and stability analysis of delayed SIR model*, J. Phys.: Conf. Ser., **2267** (2022), https://doi.org/10.1088/1742-6596/2267/1/012011
[7] L. Wang, X. Wu, *Stability and Hopf bifurcation for an SEIR epidemic model with delay*, Advances in the Theory of Nonl. Anal. and its Appl., **2**(3) (2018), 113-127.
[8] C. Çelik, H. Merdan, *Hopf bifurcation analysis of a system of coupled delayed-differential equations*, Appl. Math. Comput., **219**(12) (2013), 6605-6617.
[9] E. Karaoğlu, H. Merdan, *Hopf bifurcations of a ratio-dependent predator–prey model involving two discrete maturation time delays*, Chaos Solitons Fract., **68** (2014), 159-168.
[10] J. Li, Y. Kuang, *Analysis of a model of the glucose-insulin regulatory system with two delays*, SIAM J. Appl. Math., **67**(3) (2007), 757-776.
[11] X. Lin, H. Wang, *Stability analysis of delay differential equations with two discrete delays*, Can. Appl. Math. Q., **20** (2012), 519-533.
[12] S. Bewick, W. F. Fagan, J. Calabrese, F. Agusto, *Zika virus: Endemic versus epidemic dynamics and implications for disease spread in the Americas*, bioRxiv 2016:041897. https://doi.org/10.1101/041897
[13] F. B. Agusto, S. Bewick, W. F. Fagan, *Mathematical model of Zika virus with vertical transmission*, Infect. Dis. Model., **2**(2) (2017), 244-267.
[14] E. Bonyah, K. Okosun, *Mathematical modeling of Zika virus*, Asian Pacif. J. Trop. Dis., **6** (2016), 673-679.
[15] S. K. Biswas, U. Ghosh, S. Sarkar, *A mathematical model of Zika virus transmission with saturated incidence and optimal control: A case study of 2016 Zika outbreak in Puerto Rico*, International Journal of Modelling and Simulation, **44**(3) (2023), 172–189.
[16] C. Ding, N. Tao, Y. Zhu, *A mathematical model of Zika virus and its optimal control*, 35th Chinese Control Conference (CCC), (2016), 2642-2645.
[17] A. J. Kucharski, S. Funk, R. M. Eggo, H. P. Mallet, W. J. Edmunds, *Transmission dynamics of Zika virus in island populations: A modelling analysis of the 2013–14 French Polynesia outbreak*, PLoS Negl. Trop. Dis., **10**(5) (2016), Article ID e0004726, https://doi.org/10.1371/journal.pntd.0004726
[18] R. Kara, M. Meyvacı, *Hopf bifurcation analysis of time-delay Zika virus model*, Proceeding of e International Hybrid Conference on Mathematical Development and Applications (ICOMAA-24), (2024), 106-114.
[19] Y. Yang, G. Huang, Y. Dong, *Stability and Hopf bifurcation of an HIV infection model with two time delays*, Math Biosci Eng., **20**(2) (2022), 1938-1959.
[20] B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, UK, 1981.