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RESEARCH ARTICLE

A Note on Pointwise Semi-Slant Submanifolds in a Para-Kaehler Manifold

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ABSTRACT

In this paper, we study pointwise slant and pointwise semi-slant type-1,2,3 submanifolds in para- Kaehler manifolds. We obtain some theorems, lemmas and examples for pointwise slant and pointwise semi-slant type-1,2,3 submanifolds in para-Kaehler manifolds. We also analyze integrability and foliation properties for distributions of pointwise semi-slant type-1,2,3 submanifolds in para-Kaehler manifolds.

Keywords: *Para-Kaehler manifold, slant submanifold, pointwise semi-slant submanifold.*

AMS Subject Classification (2020): *Primary: 53C43 ; Secondary: 53C15; 53C40.*

1. Introduction

R.K. Rashevski defined para-Kaehler manifold in 1948 [13]. Then, B.A. Rozenfeld studied Para-Kaehler manifold in 1949 [12]. Rozenfeld compared Rashevskij's description with kaehler's description in the complex status and founded the similarity between Kaehler and para- Kaehler ones.

B.Y. Chen explained slant submanifolds of an almost Hermitian manifold in 1990 [6]. He expressed almost complex submanifolds and totaly real submanifolds as a generalization of an almost Hermitian manifold. Then, some geometers studied slant submanifolds [1, 2, 3, 5, 10, 14, 15].

F. Etayo, expressed pointwise slant submanifolds of an almost Hermitian manifold down the noun of quasi-slant submanifolds in 1998 [8]. B.Y. Chen and O.J. Garay studied pointwise slant submanifolds and they get some theorems, lemmas and examples in 2012 [7]. B. Şahin studied pointwise semi-slant submanifolds of a Kaehler manifold in 2013 [16]. Then, some authors studied pointwise semi-slant submanifolds in Riemannian manifolds [4, 7, 9, 11].

This paper is organized as follows. In part 2, we give preliminaries for this article, In part 3, we explain pointwise slant type-1,2,3 submanifolds in a para-Kaehler manifold and In part 4, we diagnose pointwise semi-slant type-1,2,3 submanifolds in a para-Kaehler manifold and we get some examples, theorems, lemmas and results. We also analyze integrability and foliation properties for distributions of pointwise semi-slant type-1,2,3 submanifolds in para-Kaehler manifolds.

2. Preliminaries

Let $(\bar{\mathcal{N}}, C, \breve{g})$ be a 2n-dimensional semi-Riemannian manifold. If there is a tensor field C of type $(1,1)$ on $\bar{\mathcal{N}}$, such that

$$C^2\mathcal{X} = \mathcal{X}, \quad \breve{g}(C\mathcal{X}, \mathcal{Y}) = -\breve{g}(\mathcal{X}, C\mathcal{Y}) \tag{2.1}$$

for any vector fields $\mathcal{X}, \mathcal{Y}$ on $\bar{\mathcal{N}}$, it is called a para-Hermitian manifold. In addition, it is said to be para-Kaehler manifold, if it is satisfies $\bar{\nabla} C = 0$.

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Let $\mathcal{N}$ be a submanifold of $(\bar{\mathcal{N}}, C, \breve{g})$. The Weingarten and Gauss equations are dedicated by

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + \hat{h}(\mathcal{X}, \mathcal{Y}), \tag{2.2}$$

$$\bar{\nabla}_{\mathcal{X}}V = -A_V\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}V, \tag{2.3}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{N})$ and $V \in \Gamma(T\mathcal{N}^{\perp})$. $\hat{h}$ is the fundamental form of $\mathcal{N}$, A_V is te Weingarten endomorphism connected with V and $\nabla^{\perp}$ is the normal connection.
For any tangent vector field X, we define

$$C\mathcal{X} = E\mathcal{X} + S\mathcal{X}, \tag{2.4}$$

where $E\mathcal{X}$ is the tangential component of $C\mathcal{X}$ and $S\mathcal{X}$ is the normal one.
For any normal vector field V, we get

$$CV = eV + sV, \tag{2.5}$$

where eV and sV are the tangential and normal components of CV.

3. Pointwise slant submanifolds of a para-Kaehler manifold

Definition 3.1. We call that a submanifold $\mathcal{N}$ of a para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$ is pointwise slant, if $\breve{g}(E\mathcal{X}, E\mathcal{X})/g(C\mathcal{X}, C\mathcal{X})$ is a function (non-constant) for all time-like or space-like tangent vector field $\mathcal{X}$.

Definition 3.2. If $\mathcal{N}$ is a complex (holomorfik) submanifold, in that case $C\mathcal{X} = E\mathcal{X}$ and above equation is equal to 1. More over if is $\mathcal{N}$ is totaly real (anti-invaryant), then $C = 0$, so $C\mathcal{X} = S\mathcal{X}$ and above equation equals 0. Therefore, both totaly real and complex submanifolds are special sitation of slant submanifolds. A neither totaly real nor complex slant submanifold can be called proper slant.

Definition 3.3. Let $\mathcal{N}$ be a proper pointwise slant of a para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. So we call three different types of proper pointwise slant submanifolds.
type 1; if for any time-like(space-like) vector field $\mathcal{X}$, $E\mathcal{X}$ is space-like(time-like) and $\frac{|E\mathcal{X}|}{|C\mathcal{X}|} > 1$.
type 2; if for any time-like(space-like) vector field $\mathcal{X}$, $E\mathcal{X}$ is space-like(time-like) and $\frac{|E\mathcal{X}|}{|C\mathcal{X}|} < 1$.
type 3; if for any time-like(space-like) vector field $\mathcal{X}$, $E\mathcal{X}$ is time-like(space-like). So, $\breve{g}(E\mathcal{X}, E\mathcal{X}) = 0$.

Theorem 3.1. *Let $\mathcal{N}$ be a proper pointwise slant of a para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. So, there exists three cases.*
***(a)** $\mathcal{N}$ is pointwise slant submanifold type-1 if and only if for any time-like (space-like) vector field $\mathcal{X}$, $E\mathcal{X}$ is space-like (time-like), and $\mu \in (1, +\infty)$ therefore*

$$E^2 = \mu Id, \tag{3.1}$$

we get $\mu = \cosh^2\alpha, \quad with \quad \alpha > 0.$
***(b)** $\mathcal{N}$ is pointwise slant submanifold type-2 if and only if for any time-like (space-like) vector field $\mathcal{X}$, $E\mathcal{X}$ is space-like (time-like), and $\mu \in (0, 1)$ therefore*

$$E^2 = \mu Id, \tag{3.2}$$

we get $\mu = \cos^2\alpha, \quad with \quad 0 < \alpha < 2\pi.$
***(c)** $\mathcal{N}$ is pointwise slant submanifold type 3 if and only if for any time-like (space-like) vector field $\mathcal{X}$, $E\mathcal{X}$ is time-like (space-like), and $\mu \in (-\infty, 0)$ therefore*

$$E^2 = \mu Id, \tag{3.3}$$

we get $\mu = -\sinh^2\alpha, \quad with \quad \alpha > 0.$

Proof. Let $\mathcal{N}$ be a proper pointwise slant type-1 submanifold. For any space like tangent vector field $\mathcal{X}$, $C\mathcal{X}$ is time like. We get $|E\mathcal{X}|/|C\mathcal{X}| > 1$ and $\alpha > 0$.

$$\cosh\alpha = \frac{|E\mathcal{X}|}{|C\mathcal{X}|} = \frac{\sqrt{-\breve{g}(E\mathcal{X}, E\mathcal{X}))}}{\sqrt{-\breve{g}(C\mathcal{X}, C\mathcal{X})}} \tag{3.4}$$

$$-\breve{g}(E\mathcal{X}, E\mathcal{X}) = \breve{g}(E^2\mathcal{X}, \mathcal{X}).$$

From (3.4), we get

$$\breve{g}(E^2\mathcal{X},\mathcal{X}) = -\cosh^2\alpha\breve{g}(C\mathcal{X},C\mathcal{X})$$

Using (2.1), we have

$$\breve{g}(E^2\mathcal{X},\mathcal{X}) = \cosh^2\alpha\breve{g}(\mathcal{X},\mathcal{X})$$
$$E^2 = \cosh^2\alpha.$$

Also, $E^2\mathcal{X}$ and $\mathcal{X}$ are space-like. Because of $E^2\mathcal{X} = \mu X$, we get, $E^2 = \mu = \cosh^2\alpha$.

By using a same way for any time-like tangent vector field Z and EZ, CZ are space-like. Therefore, in place of (3.4) we get

$$\cosh\alpha = \frac{|EZ|}{|CZ|} = \frac{\sqrt{\breve{g}(EZ,EZ))}}{\sqrt{\breve{g}(CZ,CZ)}}.$$

Because of $E^2X = \mu X$, for any space-like or time-like X it further provides for light-like vector fields. Therefore we get $E^2 = \mu Id$.

For second case, if $\mathcal{N}$ is pointwise slant of type 2, for any space-like or time-like vector field $\mathcal{X}$, $|E\mathcal{X}|/|C\mathcal{X}| < 1$. So, we obtain $\alpha > 0$ and

$$\cos\alpha = \frac{|E\mathcal{X}|}{|C\mathcal{X}|} = \frac{\sqrt{-\breve{g}(E\mathcal{X},E\mathcal{X}))}}{\sqrt{-\breve{g}(C\mathcal{X},C\mathcal{X})}}.$$

We can evidence that similar to the solution above. We get $E^2 = \mu Id$ and $E^2 = \mu = \cos^2\alpha$.

Lastly, if $\mathcal{N}$ is pointwise slant of type-3, for any space-like vector field $\mathcal{X}$, $C\mathcal{X}$ is further space-like and $\alpha > 0$. We obtain

$$\sinh\alpha = \frac{|E\mathcal{X}|}{|C\mathcal{X}|} = \frac{\sqrt{\breve{g}(E\mathcal{X},E\mathcal{X}))}}{\sqrt{-\breve{g}(C\mathcal{X},C\mathcal{X})}}.$$

We get $E^2 = \mu Id$ and $E^2 = \mu = \sinh^2\alpha$.

Remark 3.1. α angle is a founction and non-constant. Because $\mathcal{N}$ is pointwise slant submanifold. Lastly, for both pointwise slant submanifolds of type-1 and type-2, if $\mathcal{X}$ is space-like, in that case $C\mathcal{X}$ is time-like. Thus, all pointwise slant submanifold of type-1 or type-2 must be a neutral semi-Riemann manifold.

Proposition 3.1. *Let $\mathcal{N}$ be a proper pointwise slant submanifold of a para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. Then, $\mathcal{N}$ is a proper pointwise slant submanifold of*
**type-1, necessary and sufficent condition $eS\mathcal{X} = -\sinh^2\alpha\mathcal{X}$ for every time-like (space-like) vector field $\mathcal{X}$.*
**type-2, necessary and sufficent condition $eS\mathcal{X} = -\sin^2\alpha\mathcal{X}$ for every time-like (space-like) vector field $\mathcal{X}$.*
**type-3, necessary and sufficent condition $eS\mathcal{X} = \cosh^2\alpha\mathcal{X}$ for every time-like (space-like) vector field $\mathcal{X}$.*

Proof. For all vector field, By using (2.4), (2.5) and by applying C in (2.4), we obtain

$$C^2\mathcal{X} = CE\mathcal{X} + CS\mathcal{X}$$
$$\mathcal{X} = E^2\mathcal{X} + SE\mathcal{X} + eSX + sS\mathcal{X}$$

Equalizing the normal and tangent componenets of the above equation, we get

$$E^2\mathcal{X} + eS\mathcal{X} = \mathcal{X}, \quad SE\mathcal{X} + sS\mathcal{X} = 0. \tag{3.5}$$

Because of a pointwise slant type-1 submanifold, we obtain

$$eS\mathcal{X} = \mathcal{X} - E^2\mathcal{X} = (1-\cosh^2\alpha)\mathcal{X} = -\sinh^2\alpha\mathcal{X}$$

For a pointwise slant type-2 submanifold, we get

$$eS\mathcal{X} = \mathcal{X} - E^2\mathcal{X} = (1-\cos^2\alpha)X = \sin^2\alpha\mathcal{X}$$

and lately, for a pointwise slant type-3 submanifold, we get

$$eS\mathcal{X} = \mathcal{X} - E^2\mathcal{X} = (1+\sinh^2\alpha)X = \cosh^2\alpha\mathcal{X}.$$

For pointwise slant type-3 submanifold $\mathcal{N}^{2y}_{2x}$, we can select an concerted orthonormal slant frame

$$a_1 ...., a_y, a_1^*, ..., a_y^*$$

and $a_m^* = \frac{1}{\sinh\alpha} C a_m$. Recalled that both a_m and a_m^* get the same casual character.
Also, the codimension corresponds with the dimension of the submanifold

$$\frac{1}{\cosh\alpha} S a_1, ..., \frac{1}{\cosh\alpha} S a_y, \frac{1}{\cosh\alpha} S a_1^*, ..., \frac{1}{\cosh\alpha} S a_y^*$$

is a concerted orthonormal frame in $\mathcal{N}^{\perp} E$.

Theorem 3.2. *Let $\mathcal{N}_{2x}^{2y}$ be a pointwise slant type-3 submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}_{2x}^{4y}$. If $\nabla S = 0$, the submanifold is minimal.*

Proof. For a para-Kaehler manifold $\bar{\nabla} C = 0$. By using the normal parts in Weingarten and Gauss formulas, we get $\bar{\nabla} C = 0$, it is be $C\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \bar{\nabla}_{\mathcal{X}} C\mathcal{Y}$.
Then,

$$C\nabla_{\mathcal{X}}\mathcal{Y} + C\hat{h}(\mathcal{X},\mathcal{Y}) = \bar{\nabla}_{\mathcal{X}}(E\mathcal{Y} + S\mathcal{Y})$$

$$E\nabla_{\mathcal{X}}\mathcal{Y} + S\nabla_{\mathcal{X}}\mathcal{Y} + C\hat{h}(\mathcal{X},\mathcal{Y}) = \bar{\nabla}_{\mathcal{X}} E\mathcal{Y} + \bar{\nabla}_{\mathcal{X}} S\mathcal{Y}$$

by using (2.4) and (2.5)

$$E\nabla_{\mathcal{X}}\mathcal{Y} + S\nabla_{\mathcal{X}}\mathcal{Y} + e\hat{h}(\mathcal{X},\mathcal{Y}) + s\hat{h}(\mathcal{X},\mathcal{Y}) = \nabla_{\mathcal{X}} E\mathcal{Y} + \hat{h}(\mathcal{X}, E\mathcal{Y}) - A_{S\mathcal{Y}}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}} S\mathcal{Y}$$

$$\hat{h}(\mathcal{X}, E\mathcal{Y}) + \nabla^{\perp}_{\mathcal{X}} S\mathcal{Y} - S\nabla_{\mathcal{X}}\mathcal{Y} - s\hat{h}(\mathcal{X},\mathcal{Y}) = 0$$

If $\nabla S = 0$, it is be $S\nabla_{\mathcal{X}}\mathcal{Y} = \nabla^{\perp}_{\mathcal{X}} S\mathcal{Y}$, therefore, necessary and sufficient condition:

$$\hat{h}(\mathcal{X}, E\mathcal{Y}) = s\hat{h}(\mathcal{X},\mathcal{Y})$$

By using an adapted orthonormal frame and for type-3, $s^2 V = -\sinh^2\alpha V$ for every normal vector field V.
We get

$$\hat{h}(a_m^*, a_m^*) = \hat{h}(\frac{1}{\sinh\alpha} E a_m, \frac{1}{\sinh\alpha} E a_m) = \frac{1}{\sinh^2\alpha} s^2 \hat{h}(a_m, a_m) = -\hat{h}(a_m, a_m))$$

As a result, $H = \sum_{m=1}^{n} \epsilon_m(\hat{h}(a_m, a_m) + \hat{h}(a_m^*, a_m^*)) = 0$, that $\epsilon_m = \breve{g}(a_m, a_m)$, and $\mathcal{N}$ is minimal.

Theorem 3.3. *Let pointwise slant type-3 surfaces $\mathcal{N}$ of a para-Kaehler manifold $\bar{\mathcal{N}}_2^4$ be minimal. If and only if $\nabla S = 0$.*

Proof. We accept that $\mathcal{N}$ is space-like. and we give orthonormal bases
$a_1, a_2 = \frac{1}{\sinh\alpha} E a_1$ and $a_3 = \frac{1}{\cosh\alpha} S a_1$, $\quad a_4 = \frac{1}{\cosh\alpha} S a_2$ of $T\mathcal{N}$ and $T^{\perp}\mathcal{N}$.

We obtain that $\nabla S = 0$ necessary and sufficient condition $\hat{h}(\mathcal{X}, E\mathcal{Y}) = s\hat{h}(\mathcal{X},\mathcal{Y})$ and for any tangent vector fields $\mathcal{X},\mathcal{Y}$, $A_s V X = -A_V E\mathcal{X}$. For any $\mathcal{X}$ tangent and any V normal and using eguation $SE\mathcal{X} = -sS\mathcal{X}$, we get
$ea_3 = \cosh\alpha a_1, \quad ea_4 = \cosh\alpha a_2$
$sa_3 = \sinh\alpha a_4, \quad sa_4 = \sinh\alpha a_3$

Moreover,

$$A_{sa_3}\mathcal{X} = \frac{1}{\cosh\alpha} A_{sSa_1}\mathcal{X} = -\frac{1}{\cosh\alpha} A_{SEa_1}\mathcal{X} = \frac{1}{\cosh\alpha} e\hat{h}(\mathcal{X}, Ea_1)$$
$$= \frac{\sinh\alpha}{\cosh\alpha} e\hat{h}(\mathcal{X}, a_2) \tag{3.6}$$

We have used $(\nabla_{\mathcal{X}} E)\mathcal{Y} = A_{S\mathcal{Y}}\mathcal{X} + e\hat{h}(\mathcal{X},\mathcal{Y}) = 0$. Also,

$$A_{a_3} E\mathcal{X} = \frac{1}{\cosh\alpha} A_{Sa_1} E\mathcal{X} = -\frac{1}{\cosh\alpha} e\hat{h}(E\mathcal{X}, a_1). \tag{3.7}$$

For $\mathcal{X} = a_1$, from (3.6) and (3.7), it is be;

$$A_{a_3} Ea_1 = -\frac{1}{\cosh\alpha} e\hat{h}(Ra_1, a_1) = -\frac{\sinh\alpha}{\cosh\alpha} e\hat{h}(a_2, a_1) = -A_{sa_3} a_1$$

and for $\mathcal{X} = a_2$,

$$A_{a_3} Ea_2 = -\frac{1}{\cosh\alpha} e\hat{h}(Ea_2, a_1) = -\frac{1}{\cosh\alpha} e\hat{h}(\frac{1}{\sinh\alpha} E^2 a_1, a_1) = \frac{\sinh\alpha}{\cosh\alpha} e\hat{h}(a_1, a_1)$$

and
$$A_{sa_3}a_2 = \frac{\sinh\alpha}{\cosh\alpha}e\hat{h}(a_2, a_2)$$
Because of $\mathcal{N}$ is minimal, we obtain $A_{sa_3}a_2 = -A_{a_3}Ea_2$. So, $A_{sa_3}\mathcal{X} = -A_{a_3}E\mathcal{X}$ for any tangent vector field $\mathcal{X}$. Using in the similar way, it is proved for a_4, then $A_{sV}\mathcal{X} = -A_V E\mathcal{X}$ for V normal and $\mathcal{X}$ tangent to $\mathcal{N}$. Thus, the proof is completed.

Let $\mathcal{R}^{2k}_k$ be a semi-Riemannian submanifold of with the cartesian coordinates $(x_1, ..., x_4)$ and para-complex structure
$$C(\frac{\partial}{\partial x_{2j}}) = \frac{\partial}{\partial x_{2j-1}}, \quad C(\frac{\partial}{\partial x_{2j-1}}) = \frac{\partial}{\partial x_{2j}}$$
that $j = 1, ..., k$. Let $\mathcal{R}^{2k}_k$ be semi-Euclidian space of signature $(+, -, +, -)$ with according to the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{2k}})$.

Now, we can give some examples of pointwise slant submanifolds.

Example 3.1. Let $\mathcal{N}$ be a pointwise slant submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}^4_2$ and for $X : \mathcal{N} \to \mathcal{N}^4_2$,
$$X(u, v) = (u, \sinh v, \cosh v, v)$$
$X_u = \frac{\partial}{\partial x_1}$ and $CX_u = \frac{\partial}{\partial x_2}$
$X_v = \cosh v\frac{\partial}{\partial x_2} + \sinh v\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}$ and $CX_v = \cosh v\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sinh v\frac{\partial}{\partial x_4}$
$$\cosh\alpha = \frac{\breve{g}(X_u, CX_v)}{||X_u|| \cdot ||CX_v||} = \frac{\cosh v}{1 \cdot \sqrt{\cosh^2 v - \sinh^2 v + 1}} = \frac{\cosh v}{\sqrt{2}}$$
$$\cosh\alpha = \frac{\breve{g}(X_v, CX_u)}{||X_v|| \cdot ||CX_u||} = -\frac{\cosh v}{\sqrt{-1} \cdot \sqrt{-2}} = -\frac{\cosh v}{\sqrt{2}}$$
$\mu = \frac{\cosh^2 v}{2}$, if $v > 1$, it is be pointwise slant type-1 submanifold.
Because of type-1, $\mu \in (1, \infty)$.

Example 3.2. Let $\mathcal{N}$ be a pointwise slant submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}^4_2$. For $X : \mathcal{N} \to \mathcal{N}^4_2$,
$$X(u, v) = (\sinh u, \sinh v, \cosh u, \cosh v)$$
$X_u = \cosh u\frac{\partial}{\partial x_1} + \sinh u\frac{\partial}{\partial x3}$ and $CX_u = \cosh u\frac{\partial}{\partial x_2} + \sinh u\frac{\partial}{\partial x4}$
$X_v = \cosh v\frac{\partial}{\partial x_2} + \sinh v\frac{\partial}{\partial x_4}$ and $CX_v = \cosh v\frac{\partial}{\partial x_1} + \sinh v\frac{\partial}{\partial x_3}$
$$\cos\alpha = \frac{\breve{g}(X_u, CX_v)}{||X_u|| \cdot ||CX_v||} = \frac{\cosh u \cdot \cosh v + \sinh u \cdot \sinh V}{\sqrt{\cosh^2 u + \sinh^2 u} \cdot \sqrt{\cosh^2 v - \sinh^2 v}}$$
$$= \frac{\cosh(u + v)}{\sqrt{\cosh 2u}\sqrt{\cosh 2v}}$$
$\mu = \frac{\cosh^2(u+v)}{\cosh 2u \cdot \cosh 2v}$, if $u \neq v$ *and* $u = 0$, $v = 1$, it is be pointwise slant type-2 submanifold. Because of type-2, $\mu \in (0, 1)$.

Example 3.3. Let $\mathcal{N}$ be a pointwise slant submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}^4_2$. For $X : \mathcal{N} \to \mathcal{N}^4_2$,
$$X(a, b) = (a\sin\alpha, ab\sin\alpha, a\cos\alpha, ab\cos\alpha)$$
$X_a = \sin\alpha\frac{\partial}{\partial x_1} + b\sin\alpha\frac{\partial}{\partial x_2} + \cos\alpha\frac{\partial}{\partial x_3} + b\cos\alpha\frac{\partial}{\partial x_4}$ and
$CX_a = b\sin\alpha\frac{\partial}{\partial x_1} + \sin\alpha\frac{\partial}{\partial x_2} + b\cos\alpha\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial x_4}$
$X_b = a\sin\alpha\frac{\partial}{\partial x_2} + a\cos\alpha\frac{\partial}{\partial x_4}$ and
$CX_b = a\sin\alpha\frac{\partial}{\partial x_1} + a\cos\alpha\frac{\partial}{\partial x_3}$
$$\sinh\alpha = \frac{\breve{g}(X_a, CX_b)}{||X_a|| \cdot ||CX_b||}$$

$$= \frac{a\sin^2\alpha + a\cos^2\alpha}{\sqrt{\sin^2\alpha - b^2\sin^2\alpha + \cos^2\alpha + -b^2\cos^2\alpha} \cdot \sqrt{a^2\sin^2\alpha + a^2\cos^2\alpha}}$$
$$= \frac{a}{\sqrt{1-b^2}\sqrt{a^2}} = \frac{1}{\sqrt{1-b^2}}$$

$\mu = \frac{1}{1-b^2}$, if $1-b^2 < 0$, $1 < b^2$. it is be pointwise slant type-3 submanifold. Because of type-3, $\mu \in (-\infty, 0)$.

4. Pointwise semi-slant submanifolds in a para-Kaehler manifold

Definition 4.1. Let $(\bar{\mathcal{N}}, C, \breve{g})$ be a para-Kaehler manifold and $\mathcal{N}$ a semi-Riemannian submanifold of $\bar{\mathcal{N}}$. Thus, we call that $\mathcal{N}$ is a pointwise semi-slant submanifold if there obtain two orthogonal distributions D^T and D^α. Then,
1) $T\mathcal{N} = D^T \oplus D^\alpha$
2) The distribution D^T is an invaryant (holomorphic) distribution, $CD^T = D^T$
3) The distribution D^α is a pointwise slant with slant function α.
Then, we say the angle α the slant function of the pointwise semi-slant submanifold $\mathcal{N}$. The invaryant distribution D^T of a pointwise semi-slant submanifold is a pointwise slant distribution with slant function $\alpha = 0$. If we indicate the dimension of D^T and D^α by k_1 and k_2, therefore we get the following:
a) If $k_2 = 0$, in that case $\mathcal{N}$ is invaryant (holomorphic) submanifold.
b) If $k_1 = 0$, in that case $\mathcal{N}$ is pointwise slant submanifold.
c) If α is constant, in that case $\mathcal{N}$ is proper semi-slant submanifold with slant angle α.
d) If $\alpha = \frac{\pi}{2}$, in that case $\mathcal{N}$ is a CR-submanifold.
If $k_1 \neq 0$ and α is not a constant, we call that a pointwise semi-slant submanifold.

Let $\mathcal{N}$ be a pointwise semi-slant submanifold of a $\bar{\mathcal{N}}$ para-Kaehler manifold. we indicate the projections on the distributions D^T and D^α by C_1 and C_2. Therefore, we obtain

$$\mathcal{X} = C_1\mathcal{X} + C_2\mathcal{X} \tag{4.1}$$

$\mathcal{X} \in \gamma(T\mathcal{N})$. Using C in (4.1), we get

$$C\mathcal{X} = CC_1\mathcal{X} + CC_2\mathcal{X}.$$

Using (2.4)

$$C\mathcal{X} = CC_1\mathcal{X} + EC_2\mathcal{X} + SC_2\mathcal{X}, \tag{4.2}$$
$$CC_1\mathcal{X} \in \Gamma(D^T), \quad SC_1\mathcal{X} = 0, \tag{4.3}$$
$$EC_2\mathcal{X} \in \Gamma(D^\alpha), \quad SC_2\mathcal{X} \in \Gamma(T\mathcal{N}^\perp), \tag{4.4}$$
$$E\mathcal{X} = CC_1\mathcal{X} + EC_2\mathcal{X}. \tag{4.5}$$

It is accepted that $\mathcal{N}$ is pointwise semi-slant type-1,2,3 submanifold of $\bar{\mathcal{N}}$. Necassary and sufficient condition;

For type-1,

$$E^2 = \cosh^2\alpha I. \tag{4.6}$$

For type-2,

$$E^2 = \cos^2\alpha I. \tag{4.7}$$

For type-3,

$$E^2 = -\sinh^2\alpha I. \tag{4.8}$$

Theorem 4.1. *Let $\mathcal{N}$ be a submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}$ and $\mathcal{N}$ is a proper pointwise semi-slant type-1,2,3 submanifold. Then, $D = \{\mathcal{X} \in \Gamma(T\mathcal{N}) \quad | \quad (T_D)^2 = \mu\mathcal{X}\}$ and α is a function.*
For type-1 $\mu = \cosh^2\alpha$,
for type-2 $\mu = \cos^2\alpha$,
for type-3 $\mu = -\sinh^2\alpha$.

Proof. (For type-1) Let $\mathcal{N}$ be a proper pointwise semi-slant submanifold of $\bar{\mathcal{N}}$.

$$\cosh\alpha = \frac{\breve{g}(C\mathcal{X}, E\mathcal{X})}{||CX|| \cdot ||E\mathcal{X}||} = \frac{E\mathcal{X} + S\mathcal{X}, E\mathcal{X}}{\sqrt{\breve{g}(C\mathcal{X}, C\mathcal{X})} \cdot ||E\mathcal{X}||} = \frac{\breve{g}(E\mathcal{X}, E\mathcal{X})}{||\mathcal{X}|| \cdot ||E\mathcal{X}||}$$

$$\cosh\alpha = \frac{\breve{g}(\mathcal{X}, E^2\mathcal{X})}{||\mathcal{X}|| \cdot ||E\mathcal{X}||}$$

and using $\cosh\alpha = \frac{|E\mathcal{X}|}{|S\mathcal{X}|}$,

$$\cosh\alpha = \frac{\breve{g}(\mathcal{X}, E^2\mathcal{X})}{||\mathcal{X}|| \cdot \cosh\alpha ||C\mathcal{X}||}.$$

Using $||C\mathcal{X}|| = ||\mathcal{X}||$

$$\cosh\alpha = \frac{\breve{g}(\mathcal{X}, E^2\mathcal{X})}{||\mathcal{X}^2|| \cdot \cosh\alpha}$$

$$\cosh^2\alpha \cdot \breve{g}(\mathcal{X}, \mathcal{X}) = \breve{g}(\mathcal{X}, E^2\mathcal{X})$$

$$\breve{g}(\mathcal{X}, \cosh^2\alpha\mathcal{X}) = \breve{g}(\mathcal{X}, E^2\mathcal{X})$$

$$E^2\mathcal{X} = \cosh^2\mathcal{X} \quad and \quad E^2 = \cosh^2\alpha I$$

$$E^2 = \mu = \cosh^2\alpha,$$

(For type-2), we get

$$\mu = \cos^2\alpha,$$

(For type-3), we get

$$\mu = -\sinh^2\alpha.$$

Thus, we obtain the following results.

Corollary 4.1. *Let $\mathcal{N}$ be a proper pointwise semi-slant type-1 submanifold of para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. In that case, we get*

$$\breve{g}(E\mathcal{X}, E\mathcal{Y}) = -\cosh^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y}), \tag{4.9}$$

$$\breve{g}(S\mathcal{X}, S\mathcal{Y}) = \sinh^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y}). \tag{4.10}$$

Proof. For $\mathcal{X}, \mathcal{Y} \in D^\alpha$

$$\breve{g}(C\mathcal{X}, \mathcal{Y}) = -\breve{g}(\mathcal{X}, C\mathcal{Y})$$

Using (2.4),

$$\breve{g}(E\mathcal{X} + S\mathcal{X}, \mathcal{Y}) = -\breve{g}(\mathcal{X}, E\mathcal{Y} + S\mathcal{Y})$$

$$\breve{g}(E\mathcal{X}, \mathcal{Y}) = -\breve{g}(\mathcal{X}, E\mathcal{Y})$$

Using $E\mathcal{Y}$ in $\mathcal{Y}$

$$\breve{g}(E\mathcal{X}, E\mathcal{Y}) = -\breve{g}(\mathcal{X}, E^2\mathcal{Y})$$

$$\breve{g}(E\mathcal{X}, E\mathcal{Y}) = -\breve{g}(\mathcal{X}, \cosh^2\alpha\mathcal{Y})$$

$$\breve{g}(E\mathcal{X}, E\mathcal{Y}) = -\cosh^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y})$$

Therefore, we prove (4.9).

$$\begin{aligned}
\breve{g}(C\mathcal{X}, C\mathcal{Y}) &= -\breve{g}(\mathcal{X}, \mathcal{Y}) \\
\breve{g}(E\mathcal{X} + S\mathcal{X}, E\mathcal{Y} + S\mathcal{Y}) &= -\breve{g}(\mathcal{X}, \mathcal{Y}) \\
\breve{g}(EX, E\mathcal{Y}) + \breve{g}(S\mathcal{X}, S\mathcal{Y}) &= -\breve{g}(\mathcal{X}, \mathcal{Y}) \\
\breve{g}(S\mathcal{X}, S\mathcal{Y}) &= -\breve{g}(\mathcal{X}, \mathcal{Y}) + \cosh^2\alpha\breve{g}(\mathcal{X}, \mathcal{Y}) \\
\breve{g}(S\mathcal{X}, S\mathcal{Y}) &= \breve{g}(\mathcal{X}, \mathcal{Y}) \cdot (-1 + \cosh^2\alpha) \\
\breve{g}(S\mathcal{X}, S\mathcal{Y}) &= \sinh^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y})
\end{aligned}$$

In this case, we get (4.10).

Corollary 4.2. *Let $\mathcal{N}$ be a proper pointwise semi-slant type-2 submanifold of para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. In that case, we get*

$$\breve{g}(E\mathcal{X}, E\mathcal{Y}) = -\cos^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y}) \tag{4.11}$$

$$\breve{g}(S\mathcal{X}, S\mathcal{Y}) = -\sin^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y}) \tag{4.12}$$

Proof. For type-2, The proof is obtained, using a method similar to the above proof.

Corollary 4.3. *Let $\mathcal{N}$ be a proper pointwise semi-slant type-3 submanifold of para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. In that case, we get*

$$\breve{g}(E\mathcal{X}, E\mathcal{Y}) = \sinh^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y}), \tag{4.13}$$

$$\breve{g}(S\mathcal{X}, S\mathcal{Y}) = -\cosh^2\alpha \quad \breve{g}(\mathcal{X}, \mathcal{Y}). \tag{4.14}$$

Proof. (For type-3) using similarly method, the proof is obtained.

Theorem 4.2. *Let $\mathcal{N}$ be a proper pointwise type-1,2,3 submanifold of a para-Kaehler manifold.*
1) *The distribution D^T is integrable necessary and sufficient condition,*

$$\breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV) = \breve{g}(\hat{h}(C\mathcal{X}, \mathcal{Y}), SV) \quad \mathcal{X}, \mathcal{Y} \in \Gamma(D^T) \quad and \quad V \in \Gamma(D^\alpha)$$

2) *The distribution D^α is integrable necessary and sufficient condition,*

$$\breve{g}(A_{SEW}Z - A_{SEZ}W, \mathcal{X}) = \breve{g}(A_{SW}Z - A_{SZ}W, C\mathcal{X})$$

$$\mathcal{X} \in \Gamma(D^T), Z, W \in \Gamma(D^\alpha)$$

.

Proof. 1) (For type-1)

$$\begin{aligned} \breve{g}([\mathcal{X}, \mathcal{Y}], V) &= g(\bar{\nabla}_\mathcal{X}\mathcal{Y} - \bar{\nabla}_\mathcal{Y}\mathcal{X}, V) \\ &= \breve{g}(\bar{\nabla}_\mathcal{X}\mathcal{Y}, V) - \breve{g}(\bar{\nabla}_\mathcal{Y}\mathcal{X}, V). \end{aligned}$$

Using (2.1), (2.2), (2.3), (2.4) and (2.5), we get

$$\begin{aligned} \breve{g}([\mathcal{X}, \mathcal{Y}], V) &= -\breve{g}(C\bar{\nabla}_\mathcal{X}\mathcal{Y}, CV) + \breve{g}(C\bar{\nabla}_\mathcal{Y}\mathcal{X}, CV) \\ &= -\breve{g}(C\bar{\nabla}_\mathcal{X}\mathcal{Y}, EV + SV) + \breve{g}(C\bar{\nabla}_\mathcal{Y}\mathcal{X}, EV + SV) \\ &= -\breve{g}(C\bar{\nabla}_\mathcal{X}\mathcal{Y}, EV) - \breve{g}(C\bar{\nabla}_\mathcal{X}\mathcal{Y}, SV) + \breve{g}(C\bar{\nabla}_\mathcal{Y}\mathcal{X}, EV) \\ &+ \breve{g}(C\bar{\nabla}_\mathcal{Y}\mathcal{X}, SV)) \\ &= \breve{g}(\bar{\nabla}_\mathcal{X}\mathcal{Y}, CEV) - \breve{g}(\bar{\nabla}_\mathcal{X}C\mathcal{Y}, SV) - \breve{g}(\bar{\nabla}_\mathcal{Y}\mathcal{X}, CEV) \\ &+ \breve{g}(\bar{\nabla}_\mathcal{Y}C\mathcal{X}, SV)) \\ &= \breve{g}(\bar{\nabla}_\mathcal{X}\mathcal{Y} - \bar{\nabla}_\mathcal{Y}\mathcal{X}, E^2V + SEV) - \breve{g}(\bar{\nabla}_\mathcal{X}CY, SV) \\ &+ \breve{g}(\bar{\nabla}_\mathcal{Y}C\mathcal{X}, SV)) \\ &= \breve{g}([\mathcal{X}, \mathcal{Y}], \cosh^2\alpha V) + \breve{g}([\mathcal{X}, \mathcal{Y}], SEV) - \breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV) \\ &+ \breve{g}(\hat{h}(\mathcal{Y}, C\mathcal{X}), SV)) \end{aligned}$$

$$\begin{aligned} \breve{g}([\mathcal{X}, \mathcal{Y}], V) - \cosh^2\alpha\breve{g}([\mathcal{X}, \mathcal{Y}], V) &= -\breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV) + \breve{g}(\hat{h}(\mathcal{Y}, C\mathcal{X}), SV)) \\ -\sinh^2\alpha\breve{g}([\mathcal{X}, \mathcal{Y}], V) &= \breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV) - \breve{g}(\hat{h}(C\mathcal{X}, \mathcal{Y}), SV)) \end{aligned}$$

In this case,

$$-\sinh^2\alpha \neq 0, \quad V \neq 0, \quad [\mathcal{X}, \mathcal{Y}] = 0$$

Thus, part 1 of the proof is completed.

2) (For type-1)

$$\begin{aligned}
\breve{g}([Z,W],\mathcal{X}) &= -\breve{g}(C[Z,W],C\mathcal{X}) \\
&= -\breve{g}(C\bar{\nabla}_Z W, C\mathcal{X}) + \breve{g}(C\bar{\nabla}_W Z, C\mathcal{X}) \\
&= -\breve{g}(\bar{\nabla}_Z CW, C\mathcal{X}) + \breve{g}(\bar{\nabla}_W CZ, C\mathcal{X}) \\
&= -\breve{g}(\bar{\nabla}_Z (EW+SW), C\mathcal{X}) + \breve{g}(\bar{\nabla}_W (EZ+SZ), C\mathcal{X}) \\
&= -\breve{g}(\bar{\nabla}_Z EW, C\mathcal{X}) - \breve{g}(\bar{\nabla}_Z SW, C\mathcal{X}) + \breve{g}(\bar{\nabla}_W EZ, C\mathcal{X}) \\
&+ \breve{g}(\bar{\nabla}_W SZ, C\mathcal{X}) \\
&= \breve{g}(\bar{\nabla}_Z CEW, \mathcal{X}) - \breve{g}(\bar{\nabla}_Z SW, C\mathcal{X}) - \breve{g}(\bar{\nabla}_W CEZ, \mathcal{X}) \\
&+ \breve{g}(\bar{\nabla}_W SZ, C\mathcal{X}) \\
&= \breve{g}(\bar{\nabla}_Z E^2 W, \mathcal{X} + \breve{g}(\bar{\nabla}_Z SEW, \mathcal{X}) - \breve{g}(\bar{\nabla}_Z SW, C\mathcal{X}) \\
&- \breve{g}(\bar{\nabla}_W E^2 Z, \mathcal{X}) - \breve{g}(\bar{\nabla}_W SEZ, \mathcal{X}) + \breve{g}(\bar{\nabla}_W SZ, C\mathcal{X}) \\
&= \cosh^2\alpha\, \breve{g}(\bar{\nabla}_Z W, \mathcal{X}) - \breve{g}(A_{SEW}Z, \mathcal{X}) + \breve{g}(A_{SW}Z, C\mathcal{X}) \\
&- \cosh^2\alpha\, \breve{g}(\bar{\nabla}_W Z, \mathcal{X}) + \breve{g}(A_{SEZ}W, \mathcal{X}) - \breve{g}(A_{SZ}W, C\mathcal{X})
\end{aligned}$$

$$(1-\cosh^2\alpha)\breve{g}([Z,W],\mathcal{X}) = \breve{g}(A_{SEZ}W - A_{SEW}Z, \mathcal{X}) - \breve{g}(A_{SZ}W - A_{SW}Z, CX)$$

$$-\sinh^2\alpha\, \breve{g}([Z,W],\mathcal{X}) = -\breve{g}(A_{SEW}Z - A_{SEZ}W, \mathcal{X}) + \breve{g}(A_{SW}Z - A_{SZ}W, C\mathcal{X})$$

Because of $[Z,W]=0$, we get part 2) of the proof.

(For type-2 and type-3) using similar method, we get proof.

Theorem 4.3. *Let $\mathcal{N}$ be a proper pointwise semi-slant type-1,2,3 submanifold of a para-Kaehler manifold.*
***1)** The holomorphic distribution D^T describes a totaly geodesic foliation necessary and sufficient conditions*

$$\breve{g}(\hat{h}(\mathcal{X},\mathcal{Y}), SEV) = \breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV) \tag{4.15}$$

$for \mathcal{X},\mathcal{Y} \in \Gamma(D^T) \quad and \quad V \in \Gamma(D^\alpha)$

***2)** The slant distribution (D^α) describes a totaly geodesic foliation on $\mathcal{N}$ necessary and sufficient conditions*

$$\breve{g}(\hat{h}(U,\mathcal{X}), SEV) = \breve{g}(\hat{h}(U, C\mathcal{X}), SV) \tag{4.16}$$

$for \mathcal{X} \in \Gamma(D^T) \quad and \quad U,V \in \Gamma(D^\alpha).$

Proof. (For type-1)
1) $for \mathcal{X},\mathcal{Y} \in \Gamma(D^T) \quad and \quad V \in \Gamma(D^\alpha)$
Let $\mathcal{N}$ be a proper pointwise semi-slant submanifold of a $\bar{\mathcal{N}}$ para-Kaehler manifold. Therefore, we get

$$\breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, V) = -\breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, CV)$$

Using (2.4) and (2.5)

$$\begin{aligned}
\breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, V) &= -\breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, EV + SV) \\
&= -\breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, EV) - \breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, SV) \\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, CEV) - \breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, SV) \\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, E^2V + SEV) - \breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, SV) \\
&= \breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, E^2V) + \breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, SEV) - \breve{g}(\bar{\nabla}_{\mathcal{X}} C\mathcal{Y}, SV)
\end{aligned}$$

Using (2.2) and (4.6)

$$(1-\cosh^2\alpha)\breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, V) = \breve{g}(\hat{h}(\mathcal{X},\mathcal{Y}), SEV) - \breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV)$$

$$(-\sinh^2\alpha)\breve{g}(\bar{\nabla}_{\mathcal{X}}\mathcal{Y}, V) = \breve{g}(\hat{h}(\mathcal{X},\mathcal{Y}), SEV) - \breve{g}(\hat{h}(\mathcal{X}, C\mathcal{Y}), SV)$$

Thus the first part of the theorem is obtained and in a similar way the second part of the theorem is obtained.
(For type-2 and type-3), using similar method, we get proof.

Therefore, the following result is obtained.

Corollary 4.4. *Let $\mathcal{N}$ be a proper pointwise semi-slant type-1,2,3 submanifold of a $\bar{\mathcal{N}}$ para-Kaehler manifold. Therefore $\mathcal{N}$ is a locally Riemannian product manifold $\mathcal{N} = m_T \times m_\alpha$ necessary and sufficient condition;*

$$A_S EVX = A_S VC\mathcal{X}$$

$for \mathcal{X} \in \Gamma(D^\perp)$ and $V \in \Gamma(D^\alpha)$, that m_T is a holomorphic submanifold and m_α is a pointwise semi-slant submanifold of $\bar{\mathcal{N}}$.

Let $\mathcal{N}$ be submanifold of a semi-Riemannian manifold $(\bar{\mathcal{N}}, \breve{g})$. We say $\mathcal{N}$ a totaly umbilic submanifold of $(\bar{\mathcal{N}}, \breve{g})$. If

$$\hat{h}(\mathcal{X}, \mathcal{Y}) = \breve{g}(\mathcal{X}, \mathcal{Y})H \quad for \quad \mathcal{X}, \mathcal{Y} \in \Gamma(T\mathcal{N}) \tag{4.17}$$

that H is the mean curvature vector field of $\mathcal{N}$ in $\bar{\mathcal{N}}$.

Lemma 4.1. *Let $\mathcal{N}$ be a pointwise semi-slant totally umbilic submanifold of para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$. Suppose that $\bar{\mathcal{N}}$ is a para-Kaehler manifold, C is a para-complex structure, $\breve{g}$ is a semi-Riemannian metric. Then for type-1, type-2 and type-3;*

$$H \in \Gamma(SD_2). \tag{4.18}$$

Proof. Let $\bar{\mathcal{N}}$ be a para-Kaehler manifold.
N is β -invariant $(\beta(N) = N)$. For $\mathcal{X}, \mathcal{Y} \in \Gamma(D_1)$, $Z \in \Gamma(N)$

$$\nabla_{\mathcal{X}} \beta Y + \hat{h}(\mathcal{X}, \beta\mathcal{Y}) = \bar{\nabla}_{\mathcal{X}} \beta\mathcal{Y}$$

$$= \beta\bar{\nabla}_{\mathcal{X}}\mathcal{Y}$$

$$= E\nabla_{\mathcal{X}}\mathcal{Y} + S\nabla_{\mathcal{X}}\mathcal{Y} + e\hat{h}(\mathcal{X}, \mathcal{Y}) + s\hat{h}(\mathcal{X}, \mathcal{Y})$$

therefore by taking the inner product of both sides with Z,

$$\breve{g}(\hat{h}(\mathcal{X}, \beta\mathcal{Y}), Z) = \breve{g}(s\hat{h}(\mathcal{X}, \mathcal{Y}), Z). \tag{4.19}$$

From (4.15) and (4.17), we get

$$\breve{g}(\mathcal{X}, \beta\mathcal{Y})\breve{g}(H, Z) = -\breve{g}(\mathcal{X}, \mathcal{Y})\breve{g}(H, \beta Z). \tag{4.20}$$

Interchanging $\mathcal{X}$ and $\mathcal{Y}$

$$\breve{g}(\mathcal{Y}, \beta\mathcal{X})\breve{g}(H, Z) = -\breve{g}(\mathcal{Y}, \mathcal{X})\breve{g}(H, \beta Z). \tag{4.21}$$

Comparing (4.20) with (4.21), we get

$$\breve{g}(\mathcal{X}, \mathcal{Y})\breve{g}(H, \beta Z) = 0 \tag{4.22}$$

that means $H \in \Gamma(SD_2)$

Therefore, the following result is obtained.

Corollary 4.5. *Let $\mathcal{N}$ be a pointwise semi-slant totaly umbilic submanifold of para-Kaehler manifold $(\bar{\mathcal{N}}, C, \breve{g})$ with the semi-slant function α.*
If $\alpha = 0$ on $\mathcal{N}$, therefore $\mathcal{N}$ is a totaly geodesic submanifold of $\bar{\mathcal{N}}$.

Let $\mathcal{R}_k^{2k}$ be a semi-Riemannian submanifold, the cartesian coordinates $(x_1, ..., x_{2j})$ and para-complex structure

$$C(\frac{\partial}{\partial x_{2j}}) = \frac{\partial}{\partial x_{2j-1}}, \quad C(\frac{\partial}{\partial x_{2j-1}}) = \frac{\partial}{\partial x_{2j}}$$

that $j = (1, ..., k)$. Let $\mathcal{R}_k^{2k}$ be semi-Euclidian space of signature $(+, -, +, -, ...)$ with according to the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{2k}})$.

Now, we give some examples of proper pointwise semi-slant submanifolds.

Example 4.1. Let $\mathcal{N}$ be a proper pointwise semi-slant submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}_2^4$, defined by the $X : \mathcal{N}_2^4 \to R_3^6$

$$X(u,v,t,s) = (v, \sinh u, \cosh u, u, t, s)$$

$X_u = \cosh u \frac{\partial}{\partial x_2} + \sinh u \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}$ and $CX_u = \cosh u \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sinh u \frac{\partial}{\partial x_4}$
$X_v = \frac{\partial}{\partial x_1}$ and $CX_v = \frac{\partial}{\partial x_2}$
$X_t = \frac{\partial}{\partial x_5}$ and $CX_t = \frac{\partial}{\partial x_6} = X_s$
$X_s = \frac{\partial}{\partial x_6}$ and $CX_s = \frac{\partial}{\partial x_5} = X_t$

$D^T = span\{X_t, X_s\}$ is a holomorphic distribution and $D^\alpha = span\{X_u, X_v\}$ is a neutral proper pointwise semi-slant distribution with slant function α.

$$\cosh\alpha = \frac{\breve{g}(X_u, CX_v)}{||X_U|| \cdot ||CX_v||} = \frac{\cosh u}{1 \cdot \sqrt{\cosh^2 u - \sinh^2 u + 1}} = \frac{\cosh u}{\sqrt{2}}$$

$$\cosh\alpha = \frac{\breve{g}(X_v, CX_u)}{||X_v|| \cdot ||CX_u||} = -\frac{\cosh u}{\sqrt{-1} \cdot \sqrt{-2}} = -\frac{\cosh u}{\sqrt{2}}$$

semi-slant function is $\alpha = \cosh^{-1}(\frac{\cosh u}{\sqrt{2}})$
$E^2 = \mu = \frac{\cosh^2 u}{2}$, if $(u \geq 1)$, it is be proper pointwise semi-slant (type-1) submanifold. Because of type-1, $\mu \in (1, \infty)$.

Example 4.2. Let $\mathcal{N}$ be a proper pointwise semi-slant submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}_2^4$, defined by the $X : \mathcal{N}_2^4 \to R_3^6$

$$X(t,s,u,v) = (t, s, \sinh u, \sinh v, \cosh u, \cosh v)$$

$X_t = \frac{\partial}{\partial x_1}$ and $CX_t = \frac{\partial}{\partial x_2} = X_s$
$X_s = (\frac{\partial}{\partial x_2})$ and $CX_s = \frac{\partial}{\partial x_1} = X_t$
$X_u = \cosh u \frac{\partial}{\partial x_3} + \sinh u \frac{\partial}{\partial x_5}$ and $CX_u = \cosh u \frac{\partial}{\partial x_4} + \sinh u \frac{\partial}{\partial x_6}$
$X_v = \cosh v \frac{\partial}{\partial x_4} + \sinh v \frac{\partial}{\partial x_6}$ and $CX_v = \cosh v \frac{\partial}{\partial x_3} + \sinh v \frac{\partial}{\partial x_5}$

We find $D^T = span\{X_t, X_s\}$ is a holomorphic distribution and $D^\alpha = span\{X_u, X_v\}$ is a neutral proper pointwise semi-slant distribution type-2 with semi-slant function α.

$$\cos\alpha = \frac{\breve{g}(X_u, CX_v)}{||X_u|| \cdot ||CX_v||} = \frac{\cosh u \cdot \cosh v + \sinh u \cdot \sinh v}{\sqrt{\cosh^2 u + \sinh^2 u} \cdot \sqrt{\cosh^2 v + \sinh^2 v}}$$

$$= \frac{\cosh(u+v)}{\sqrt{\cosh 2u} \cdot \sqrt{\cosh 2v}}$$

semi-slant function is $\alpha = \cos^{-1}(\frac{\cosh(u+v)}{\sqrt{\cosh 2u} \cdot \sqrt{\cosh 2v}})$
$E^2 = \mu = \frac{\cosh^2(u+v)}{\cosh 2u \cdot \cosh 2v}$. if, $(u, v \in R^+)$, it is be proper pointwise semi-slant (type-2) submanifold. Because of type-2, $\mu \in (0, 1)$.

Example 4.3. Let $\mathcal{N}$ be a proper pointwise semi-slant submanifold of a para-Kaehler manifold $\bar{\mathcal{N}}_2^4$, defined by the $X : \mathcal{N}_2^4 \to R_5^{10}$

$$X(a,b,c,d) = (a\sin c, b\sin c, a\sin d, b\sin a, a\cos c, b\cos c, a\cos d, b\cos d, c, d)$$

$X_a = \sin c \frac{\partial}{\partial x_1} + \sin d \frac{\partial}{\partial x_3} + \cos c \frac{\partial}{\partial x_5} + \cos d \frac{\partial}{\partial x_7}$
$X_b = \sin c \frac{\partial}{\partial x_2} + \sin d \frac{\partial}{\partial x_4} + \cos c \frac{\partial}{\partial x_6} + \cos d \frac{\partial}{\partial x_8}$
$X_c = a\cos c \frac{\partial}{\partial x_1} + b\cos c \frac{\partial}{\partial x_2} - a\sin c \frac{\partial}{\partial x_5} - b\sin c \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_9}$
$X_d = a\cos d \frac{\partial}{\partial x_3} + b\cos 4 \frac{\partial}{\partial x_4} - a\sin d \frac{\partial}{\partial x_7} - b\sin d \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}}$
$CX_a = \sin c \frac{\partial}{\partial x_2} + \sin d \frac{\partial}{\partial x_4} + \cos c \frac{\partial}{\partial x_6} + \cos d \frac{\partial}{\partial x_8}$
$CX_b = \sin c \frac{\partial}{\partial x_1} + \sin d \frac{\partial}{\partial x_3} + \cos c \frac{\partial}{\partial x_5} + \cos d \frac{\partial}{\partial x_7}$
$CX_c = b\cos c \frac{\partial}{\partial x_1} + a\cos c \frac{\partial}{\partial x_2} - b\sin c \frac{\partial}{\partial x_5} - a\sin c \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_{10}}$

$CX_d = b\cos d\frac{\partial}{\partial x_3} + a\cos d\frac{\partial}{\partial x_4} - b\sin d\frac{\partial}{\partial x_7} - a\sin d\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}$

We find $D^T = span\{X_a, X_b\}$ is a holomorphic distribution and $D^\alpha = span\{X_c, X_d\}$ is a neutral proper pointwise semi-slant distribution with semi-slant function α.

$$\sinh\alpha = \frac{\breve{g}(X_d, CX_c)}{||X_d|| \cdot ||CX_c||} = \frac{-1}{\sqrt{a^2-b^2-1}\cdot\sqrt{b^2-a^2-1}}$$

semi-slant function is $\alpha = \sinh^{-1}(\frac{-1}{\sqrt{a^2-b^2-1}\cdot\sqrt{b^2-a^2-1}})$
$E^2 = \mu = \frac{1}{(a^2-b^2-1)(b^2-a^2-1)} = \frac{-1}{(a^2-b^2)^2-1}$. if $(a^2-b^2>1)$, it is be proper pointwise semi-slant (type-3) submanifold. Because of (type-3), $\mu \in (-\infty, 0)$.

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