HELICAL VERSUS OF RECTIFYING CURVES IN LORENTZ SPACES

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Abstract

In this present study we give a characterization of rectifying curves in Lorentz Space IR_1^3. We also consider the helices versus of these curves.

1. Introduction:

Let IR_1^3 denote Lorentz space which is defined as a space to be usual three-dimensional IR-vector space consisting of vectors $\{(x_1, x_2, x_3) | \; x_1, x_2, x_3 \in IR\}$ endowed with the inner product

$$\langle x, \; y\rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 . \tag{1.1}$$

A regular curve $\alpha : I \to IR_1^3$ is called

i. Space-like, if $\langle \alpha', \alpha'\rangle > 0$,

ii. Time-like, if $\langle \alpha', \alpha'\rangle < 0$,

iii. Light-like or null if $\langle \alpha', \alpha'\rangle = 0$, everywhere.

Let α be a space –like or time-like curve,which we assume that it is parameterized by arc length parameter. Let $\vec{t}$ denote α'. We assume $t'(0) = 0$ everywhere. Then we can introduce a unique vector field $\vec{n}$ and function κ so that $t' = \kappa \xi_2 \vec{n}$, where $\xi_2 = \langle \vec{n}, \vec{n}\rangle$.

We call t' the curvature vector field, $\vec{n}$ the principal normal vector field and κ the curvature of the given curve. Since t is a constant length vector field, $\vec{n}$ is orthogonal to $\vec{t}$.

The binormal vector field is defined by

$$\vec{b} = \vec{t} \times \vec{n}$$

which is a unit vector field orthogonal to both $\vec{t}$ and $\vec{n}$. One defines the torsion τ by the equation

$$b' = -\tau \, \varepsilon_2 \vec{n} .$$

The famous Serret-Frenet equations in Lorentz space are given by

Keywords: Curves, Lorentz Spaces, Rectifying Curves

$$t' = \kappa\varepsilon_2 \vec{n}$$
$$n' = -\kappa\varepsilon_1 \vec{t} - \tau\varepsilon_1\varepsilon_2 \vec{b}$$
$$b' = -\tau\varepsilon_2 \vec{n}$$
(1.2)

At each point of the curve, the planes spanned by $\{t, n\}$, $\{t, b\}$ and $\{n, b\}$ are known as the osculating plane, the rectifying plane, and the normal plane respectively. A curve in IR_1^3 is called a twisted curve it is has nonzero curvature and torsion.

A space curve $\alpha: I \to IR_1^3$ whose position vector always lies in its rectifying plane is called a rectifying curve. So for a rectifying curve $\alpha: I \to IR_1^3$, the position vector $\alpha(t)$ satisfies

$$\alpha(s) = \lambda(s)\vec{t}(s) + \mu(s)\vec{b}(s)$$
(1.3)

for some functions λ and μ.

In [5], B,Y. Chen gave a characterization of curves in IR_1^3. He also consider helical versus of these curves. The authors showed that rectifying curves enabled us to interpret rectifying curves kinematically, as these curves whose positon vector field determines the exist of instantenious rotation at each point. These curves are in equilibrium under the action of a force field $F = c\vec{t} - \tau\vec{n}$ for non-zero constant c. The general helices are characterized by the same formula with c=0.

In the present study we adaptate B,Y. Chen's study to Lorentzian case. So we give a full characterization of rectifying curves in Lorentz Space IR_1^3 which are not null. We also consider the helical versus of these curves.

2. Characterization of Rectifying Curves

The following result provides some simple characterization of rectifying curves in IR_1^3.

Theorem 1. Let $\alpha: I \to IR_1^3$ be rectifying curve in IR_1^3 with $\kappa \neq 0$, and let s be its arclength parameter. Then

i. The tangential component of the position vector of the curves given by $\langle \alpha(s), t \rangle = \varepsilon_1 (s + b)$. for some constant b,

ii. The distance $\rho = \|\alpha(s)\|$ satisfies $\rho^2 = (s+b)^2\varepsilon_1 + a^2\varepsilon_3$ for some constants a and b.

iii. The normal component of the position vector of the curve has constant length and the distance function ρ is non constant.

Proof. Let $\alpha\!:\!I \to IR_1^3$ be a curve in IR_1^3 with $\kappa \neq 0$. Without loss of generality, we may assume that α is parameterized by the arc length function s. Suppose that α is a rectifying curve. Then by definition we have

$$\alpha(s) = \lambda(s)f(s) + \mu(s)b(s) \tag{2.1}$$

for some functions $\lambda(s)$ and $\mu(s)$. By taking the derivative of (2.1) with respect to s and applying the Serret-Frenet equations, we find that

$$\lambda'(s) = 1, \quad \lambda\kappa\varepsilon_2 - \tau\mu\varepsilon_2 = 0, \quad \mu'(s) = 0 . \tag{2.2}$$

From the first and third equations of (2.2) we know that $\lambda = s + b$ for some constant b and that μ is constant. Since $\kappa \neq 0$, the constant μ is nonzero. Thus (2.1) implies $\langle \alpha(s), t \rangle = \varepsilon_1 (s + b)$. This proves statement (i).

Since

$$\rho^2 = \langle \alpha(s), \alpha(s) \rangle = \lambda^2 \langle f, f \rangle + \mu^2 \langle b, b \rangle$$
$$= \lambda^2 \varepsilon_1 + \mu^2 \varepsilon_3$$

and $\lambda(s) = s + b$, $\mu(s) = a$ (for some constant a and b).

Then we get

$$\rho^2 = (s + b)^2 \varepsilon_1 + a^2 \varepsilon_3 \ .$$

Equations (2.1) imply that the normal component $\alpha(s)^N$ of the position vector of the curve is given by μb. Since $\varepsilon_3 \mu = \langle \alpha(s), b \rangle$ is constant by the third equation in (2.2), $\alpha(s)^N$ has constant length. This yields statement (ii).

Statement (iii) ollows from the constancy of μ, the fact that $\kappa \neq 0$, the relation $\lambda(s) = s + b$, and the second equation of (2.2) □

The converse of previous theorem is also holds.

3. Helical Versus of Rectifying Curves

It is known that a twisted curve in IR_1^3 is a generalized helix if and only if the ratio $\frac{\tau}{\kappa}$ is a non-zero constant on the curve. On the other hand, for rectifying curve we have the following very simple characterization items of the ratio $\frac{\tau}{\kappa}$.

Theorem 2. Let $\alpha\!:\!I \to IR_1^3$, be a curve with $\kappa \neq 0$. Then α is congruent to a rectifying curve if and only if the ratio of torsion and curvature of the curve is a linear function in arclenght function s, i.e. $\frac{\tau}{\kappa} = c_1 s + c_2$ for some constant c_1 and c_2 with $c_1 \neq 0$.

Proof. Let $\alpha : I \to IR_1^3$ be a unit speed curve with $\kappa \neq 0$. If α is a rectifying curve, then we have (2.2) which implies that

$$\frac{\tau}{\kappa} = \frac{\lambda}{\mu} = \frac{s+b}{a} \tag{3.1}$$

for some constant a and b. Hence the ratio of torsion and curvature of the curve is a non-constant linear function of arclength function s.

Conversely suppose that $\alpha : I \to IR_1^3$ is a curve in IR_1^3 with $\kappa \neq 0$. Such that $\frac{\tau}{\kappa} = c_1 s + c_2$ for some constant c_1 and c_2 with $c_1 \neq 0$. If we put $a = \frac{1}{c_1}$ and $b = ac_2$, then we have $\frac{\lambda}{\mu} = \frac{s+b}{a}$.

Hence by invoking the Serret- Frenet equations we conclude that

$$\frac{d}{ds}\left[\alpha(s) - (s+b)t(s) - ab(s)\right] = 0 \tag{3.2}$$

which implies that α is congruent to a rectifying curve.□

4.Classification of Rectifying Curves in IR_1^3

Let S_1^2 denote time-like or space-like unit sphere in IR_1^3**.** The following result explicitly determines all rectifying curves in IR_1^3.

Theorem 3. Let $\alpha : I \to IR_1^3$ be a rectifying curve in IR_1^3 with $\kappa \neq 0$. Then α is a rectifying curve if and only if up to parametrization, it is given by

$$\alpha(t) = (a \sec t)\, \beta(t) \tag{4.1}$$

where a is a number and $\beta = \beta(t)$ is a time-like or space-like unit speed curve in S_1^2.

Proof. Let $\alpha : I \to IR_1^3$ be a rectifying curve in IR_1^3 with $\kappa \neq 0$.We may assume that $0 \in I$ and $\alpha = \alpha(s)$ is a unit speed curve. According to Theorem1, the distance function $\rho = \left|\alpha(s)\right|$ of the curve satisfying $\rho^2 = s^2\varepsilon_1 + 2bs\varepsilon_1 + b^2\varepsilon_1 + a^2\varepsilon_2$. When t is a space-like end b is time-like vector we get $\rho^2 = s^2 + 2bs + b^2 - a^2$. After making a suitable tranlation in s, we have $\rho^2 = s^2 + c$ for some constant c.Because $0 \in I$, we must have $c>0$. Hence we may put $c = a^2$ for some possible number a.

Now , let us define a curve in S_1^2 by $\beta = \dfrac{\alpha}{\rho}$ thus we have

$$\alpha(s) = \sqrt{s^2+\alpha^2}\,\beta(s)\,. \tag{4.2}$$

By taking the derivative of (4.2) with respect to s we get

$$\alpha'(s) = \frac{s}{\sqrt{s^2+\alpha^2}}\,\beta(s) + \sqrt{s^2+\alpha^2}\,\beta'(s)\,. \tag{4.3}$$

Since $\langle\beta,\beta\rangle = \pm 1$, $\beta'(s)$ is orthogonal to $\beta(s)$ for each s in I. Therefore, equation (4.3) implies that

$$\upsilon = \frac{\alpha}{s^2+\alpha^2} \tag{4.4}$$

where $\upsilon = \|\beta'(s)\|$ is the speed of the spherical curve $\beta = \beta(s)$.
Let us put

$$t = \int_0^s \frac{\alpha}{s^2+\alpha^2}\,du = \arctan\left(\frac{s}{\alpha}\right) \tag{4.5}$$

then s = a tan(t). Substituating this into (4.2) yields (4.1).

Conversely if $\alpha: I \to IR_1^3$ is a curve defined by

$$\alpha(t) = (\alpha \sec t)\ \beta(t) \tag{4.6}$$

for a positive number a and a unit speed space-like or time-like curve in S_1^2, then

$$\alpha'(t) = (\alpha \sec t)\big(\tan t.\beta(t) + \beta'(t)\big)\,. \tag{4.7}$$

Since $\beta(t)$ and $\beta'(t)$ are orthogonal vector fields, (4.7) implies that

$$|\alpha'(t)| = (\alpha \sec^2 t)\,. \tag{4.8}$$

Using the quations (4.6)-(4.8) we get

$$\left\langle \alpha^N, \alpha^N \right\rangle = \rho^2(t) - \frac{\langle \alpha(t), \alpha'(t)\rangle^2}{\|\alpha'(t)\|^2} = \varepsilon_1 \alpha^2 \tag{4.9}$$

where $\varepsilon_1 \in$ {-1, 1}, which shows that the normal component α^N of the position vector has constant length. Because the distance function of α is given by $\rho = \alpha \sec t$, which is nonconstant, Theorem1 implies that α is a rectifying curve.□

Acknowledgement: The authors wish to thank Proffesor K. Arslan for his precious contribution to this paper.

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LORENTZ UZAYLARIDAKİ REKTİFİYAN EĞRİLERİN HELİS OLMA DURUMLARI

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Özet

Bu çalışmada Lorentz uzayı IR_1^3 deki rektifiyan eğrilerin bir karekterizasyonu verilmiştir. Ayrıca bu tür eğrilerin helis olmaları durumunda ortaya çıkan özellikler ele alınmıştır.

Anahtar Kelimeler: Eğriler, Lorenz Uzaylar, Rektifiyan Eğriler

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