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Stability of Waste Paper Recycling through Graph Theory

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Abstract - The process in which waste paper is collected and then reprocessed for reuse is called paper recycling. Paper recycling is very often routine of human life. Paper recycling is very important to reduce deforestation and pollution. It has been analyzed that scrap dealer plays a vital role in this cycle that collects waste paper from distributor and customer and then sent it to the paper industry, so the effect of scrap dealers is observed here. This waste paper recycling model is divided into four compartments namely paper industry, distributor, customer and scrap dealer. The model is proposed as a system of non-linear differential equation. The basic reproduction number is computed to see the impact of scrap dealer.

Keywords *- Dynamical model, System of non-linear ordinary differential equation, Basic reproduction number, Local stability, Global stability, Weighted graphs, Paper.*

1 Introduction

Recycling is one of the best ways for us to have an optimistic impact on the world in which we want to live healthily. Paper is one of the best materials that we can recycle easily. Recycled paper made from paper and paper products that has already been used. The paper recycling starts with us. It can be learnt at schools, colleges, home, offices, and local communities and even at drop off centers. Recycling of paper helps us in many ways. As the pulp of tree is an only source for producing paper, the recycled fibres from waste paper provides a better alternative. Creating recycled paper pulp, compare to manufacturing pulp from trees to make paper products, and devours less energy and water. By recycling one ton of paper, we save 17 trees, 7000 gallons of water, 463 gallons of oil and more than 3.3 cubic yards of landfill space.

In the case of recycling, many researchers have studied the process of recycling wastepaper. Kleineidam *et al.* [14] obtained optimizing product recycling chains by control theory in 2000. Clement and Marie [4] in 1988 considered method for producing pulp from printed unselected waste paper. In 1994, Nadeau and Allan [18] have studied integrated

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waste paper treatment process. In 1996, production of soft paper products from old newspaper was deliberated by Back and Sangho [1]. Denem and Dennis [5] learned waste paper minimizing paper dispenser in 2004. Kinney and Roland [15] in 1994 wrote book on technology of paper recycling. Bleakley *et al.* [2] in 2000 examined waste paper treatment process. Some of the researchers have analyzed the impact of recycling waste paper on the environment. In 1997, Bystrom *et al.* [3] observed paper recycling: environmental and economic impact. Miranda [17] *et al.* investigated environmental awareness and paper recycling in 2010. Environment impacts of waste paper recycling were deliberated by Virtanen *et al.* [23] in 2013. Different model for the awareness of recycling for waste paper was developed by many researchers. Kara *et al.* [13] studied a stochastic optimization approach for paper recycling reverse logistics network design under uncertainty in 2010. Merrild *et al.* [16] generated life cycle assessment of waste paper management: the importance of technology data and system boundaries in assessing recycling and incineration in 2008. A different model 'A goal programming model for paper recycling system' was formulated in 2008 by Pati *et al.* [19]. Some environment related models like forest model [22] and green belt model [21] was also developed by some researchers to revive the natural resources.

Mathematical modeling of paper recycling is developed in Section 2. Using weighted graph, the stability analysis is carried out in Section 3. In Section 4, sensitivity analysis is analyzed. Numerical analysis is calculated in Section 5 and validated data is given.

2 Mathematical Modeling

We live in the society where deforestation and pollution has taken place. The process of recycled paper from paper industries can reduce it. Paper industries (P) are those which manufacture the paper and even recycle the waste paper. Distributors (D) are those who collect produced paper sold by paper industry. Customer (C) are those who buy paper from distributors. Scrap dealers are those who gather water paper from distributor and customer and give it to the paper industry. Thus, it becomes a cycle. And this process of recycling of paper helps us to save energy, water and resources. Here, scrap dealer works as a control for waste paper recycling model.

Parameters and their notations along with parametric values used to formulate waste paper recycling model are as given in the Table 1.

Table 1. Notation and parametric values

Notation		Parametric value
B	Recruitment rate from wood	0.7
δ	The rate at which distributor buys paper	0.9
η	The rate at which customer buys paper	0.75
α	The rate at which distributor gives paper to scrap dealers	0.02
β	The rate at which customer gives paper to scrap dealers	0.7
γ	The rate at which scrap dealer transports waste paper to paper industry	0.6
μ	Natural waste of paper from each compartment	0.28

To formulate a mathematical model of waste water recycling we have used above notation and necessary assumptions whose transmission diagram is as given in Figure 1.

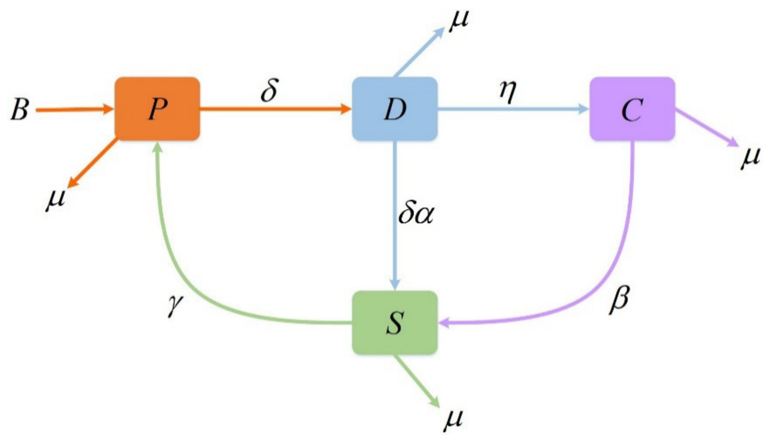

Figure 1. Transmission diagram of waste paper recycling model

The system of non-linear differential equation of transmission of waste paper is as given below:

$$\begin{aligned}
\frac{dP}{dt} &= B - \delta PD + \gamma S - \mu P \\
\frac{dD}{dt} &= \delta PD - \eta D - \delta\alpha DS - \mu D \\
\frac{dC}{dt} &= \eta D - \beta CS - \mu C \\
\frac{dS}{dt} &= \delta\alpha DS + \beta CS - \gamma S - \mu S
\end{aligned} \tag{1}$$

where $P+D+C+S=N$. Also, $P>0; D,C,S \geq 0$.

Adding above system of differential equations, we get

$$\begin{aligned}
\frac{d}{dt}(P+D+C+S) &= B - \delta PD + \gamma S - \mu P + \delta PD - \eta D - \delta\alpha DS - \mu D + \eta D - \beta CS - \mu C \\
&\quad + \delta\alpha DS + \beta CS - \gamma S - \mu S \\
&= B - \mu(P+D+C+S) \geq 0
\end{aligned}$$

which implies that $\limsup_{t\to\infty}(P+D+C+S) \leq \frac{B}{\mu}$.

Therefore, the feasible region of the waste paper recycling model is

$$\Lambda = \left\{ \left(P+D+C+S\right) \Big/ P+D+C+S \le \frac{B}{\mu}, P>0; D,C,S \ge 0 \right\}.$$

Now, solving system of differential equations, we get four equilibrium points:

1) $E_P^* = \left(\frac{B}{\mu},0,0,0\right)$

2) $E_{PCS}^* = \left(\frac{-\mu\gamma + B\beta}{\mu\beta},0,\frac{\gamma+\mu}{\beta},\frac{-\mu}{\beta}\right)$

3) $E_{PDC}^* = \left(\frac{\eta+\mu}{\delta},\frac{B\delta-\mu(\eta+\mu)}{\delta(\eta+\mu)},\frac{\eta\left(B\delta-\mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)},0\right)$

4) $E_{PDCS}^* = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta},\frac{-\beta Y+\gamma+\mu}{\delta\alpha},X,\frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta}\right)$

where

$$\begin{aligned} Y = RootOf\,(&\left(-\beta^2\mu + \beta\delta\alpha\mu\right)x^2 + \left(\mu\beta\gamma - \mu^2\alpha^2\delta - \mu\eta\beta + \mu^2\beta - \beta\delta\alpha B - \delta\alpha\mu^2 + \mu^2\alpha\beta\right)x \\ &+\mu\alpha\gamma\eta + \mu^2\eta + \gamma\eta\mu + \mu^2\alpha\eta) \end{aligned} \tag{2}$$

Next, we compute basic reproduction number R_0 for each equilibrium point E^*, using next generation matrix method.

Let us consider $X' = (P,D,C,S)'$, where derivative is denoted by dash. So,

$$X' = \frac{dX}{dt} = F(X) - V(X)$$

where $F(X)$ is the rate of presence of new individual in compartment and $V(X)$ is the rate of transfer of culture. They are given by

$$F = \begin{bmatrix} \delta PD \\ \delta\alpha DS + \beta CS \\ 0 \\ 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} \eta D + \delta\alpha DS + \mu D \\ \gamma S + \mu S \\ -B + \delta PD - \gamma S + \mu P \\ -\eta D + \beta CS + \mu C \end{bmatrix}$$

Now, $DF(E^*) = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$ and $DV(E^*) = \begin{bmatrix} v & 0 \\ J_1 & J_2 \end{bmatrix}$

where f and v are 4×4 matrices defined as

$$f=\left[\frac{\partial F_i\left(E^*\right)}{\partial X_j}\right] \text{ and } v=\left[\frac{\partial V_i\left(E^*\right)}{\partial X_j}\right].$$

1) Finding f and v for the equilibrium $E_P^*\left(\frac{B}{\mu},0,0,0\right)$, we get

$$f_P=\begin{bmatrix} \frac{\delta B}{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } v_P=\begin{bmatrix} \eta+\mu & 0 & 0 & 0 \\ 0 & \gamma+\mu & 0 & 0 \\ \frac{\delta B}{\mu} & -\gamma & 0 & 0 \\ -\eta & 0 & 0 & \mu \end{bmatrix}$$

Here, v_P is non-singular.

Therefore, the expression of basic reproduction number R_{0_P} is as below:

$$R_{0_P} = \text{ spectral radius of } f_P {v_P}^{-1}$$
$$\Rightarrow R_{0_P}=\frac{\delta B}{\mu(\eta+\mu)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PCS}}=2.1840$ which is grater that 1 that makes model unstable.

2) Finding f and v for the equilibrium $E_{PCS}^*=\left(\frac{-\mu\gamma+B\beta}{\mu\beta},0,\frac{\gamma+\mu}{\beta},\frac{-\mu}{\beta}\right)$, we get

$$f_{PCS}=\begin{bmatrix} \frac{\delta(-\mu\gamma+B\beta)}{\mu\beta} & 0 & 0 & 0 \\ -\frac{\delta\alpha\mu}{\beta} & \frac{\beta(\gamma+\mu)}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } v_{PCS}=\begin{bmatrix} \eta-\frac{\delta\alpha\mu}{\beta}+\mu & 0 & 0 & 0 \\ 0 & \gamma+\mu & 0 & 0 \\ \frac{\delta(-\mu\gamma+B\beta)}{\mu\beta} & -\gamma & \mu & 0 \\ -\eta & 0 & 0 & 0 \end{bmatrix}$$

Here, v_{PCS} is non-singular.

Therefore, the expression of basic reproduction number $R_{0_{PCS}}$ is as below:

$$R_{0_{PCS}} = \text{ spectral radius of } f_{PCS} {v_{PCS}}^{-1}$$
$$\Rightarrow R_{0_{PCS}}=\frac{\delta(-\mu\gamma+B\beta)}{\mu(\eta\beta-\delta\alpha\mu+\mu\beta)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PCS}} = 2.5000$ which is grater that 1 that makes model unstable.

3) Finding f and v for the equilibrium

$$E^{*}_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta-\mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta-\mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0 \right)$$

we get

$$f_{PDC} = \begin{bmatrix} \eta+\mu & 0 & \frac{B\delta-\mu\eta-\mu^2}{\eta+\mu} & 0 \\ 0 & \frac{\beta\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu}+\frac{\beta\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$v_{PDC} = \begin{bmatrix} \eta+\mu & \frac{\alpha\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu} & 0 & 0 \\ 0 & \gamma+\mu & 0 & 0 \\ \eta+\mu & -\gamma & \frac{B\delta-\mu\eta-\mu^2}{\eta+\mu}+\mu & 0 \\ -\eta & \frac{\beta\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)} & 0 & \mu \end{bmatrix}$$

Here, v_{PDC} is non-singular.

Therefore, the expression of basic reproduction number $R_{0_{PDC}}$ is as below:

$$R_{0_{PDC}} = \text{ spectral radius of } f_{PDC}{v_{PDC}}^{-1}$$

$$\Rightarrow R_{0_{PDC}} = \frac{\left(B\delta-\mu(\eta+\mu)\right)(\alpha\delta\mu+\beta\eta)}{\delta\mu(\eta+\mu)(\gamma+\mu)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PDC}} = 0.6456$.

4) Finding f and v for the equilibrium

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$$

where Y is as equation (2) and taking

$$\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta} = P^*, \frac{-\beta Y + \gamma + \mu}{\delta\alpha} = D^*, Y = C^*$$

and

$$\frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} = S^*$$

we get

$$f_{PDCS} = \begin{bmatrix} \delta P^* & 0 & \delta D^* & 0 \\ \delta\alpha S^* & \delta\alpha D^* + \beta C^* & 0 & \beta S^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$v_{PDCS} = \begin{bmatrix} \eta + \delta\alpha S^* + \mu & \delta\alpha D^* & 0 & 0 \\ 0 & \gamma + \mu & 0 & 0 \\ \delta P^* & -\gamma & \delta D^* + \mu & 0 \\ -\eta & \beta C^* & 0 & \beta S^* + \mu \end{bmatrix}$$

Here, v_{PDCS} is non-singular.

Therefore, the expression of basic reproduction number $R_{0_{PDCS}}$ is as below:

$$R_{0_{PDCS}} = \text{ spectral radius of } f_{PDCS} v_{PDCS}^{-1}$$

$$\Rightarrow R_{0_{PDCS}} = \frac{\delta^2\alpha^2 S^* D^*}{(\gamma + \mu)(\eta + \delta\alpha S^* + \mu)} + \frac{\delta\alpha D^* + \beta C^*}{\gamma + \mu} - \frac{\beta S^* \left(\beta\eta C^* + \beta\delta\alpha C^* S^* + \beta\mu C^* + \eta\delta\alpha D^* \right)}{(\eta + \delta\alpha S^* + \mu)(\gamma + \mu)(\beta S^* + \mu)}$$

After putting parametric values given in the Table 1, we get $R_{0_{PDCS}} = 0.7346$.

Above four basic reproduction numbers for distinct equilibrium point of waste paper recycling model shows that equilibrium point E^*_{PDCS} is more appropriate than E^*_{PDC} which indicate that scrap dealers are the most significant factor for the process of waste paper recycling. Therefore, in the further section, we will discuss stability only at two equilibrium point E^{*}_{PDC} and E^{*}_{PDCS}.

3 Equilibrium

In the current section, we will establish local stability and global stability for the waste water recycling model for two equilibrium points.

3.1 Local Stability

The local stability for waste paper recycling model will be discovered here.

First, we will begin with the equilibrium point

$$E^{*}_{PDC}=\left(\frac{\eta+\mu}{\delta},\frac{B\delta-\mu(\eta+\mu)}{\delta(\eta+\mu)},\frac{\eta\left(B\delta-\mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)},0\right)$$

using Jacobian matrix J_{PDC}. The Jacobian matrix J_{PDC} of the waste paper recycling model is as follows:

$$J_{PDC}=\begin{bmatrix} -\frac{B\delta-\mu\eta-\mu^2}{\eta+\mu}-\mu & -\eta-\mu & 0 & \gamma \\ \frac{B\delta-\mu\eta-\mu^2}{\eta+\mu} & 0 & 0 & -\frac{\alpha\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu} \\ 0 & \eta & -\mu & -\frac{B\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)} \\ 0 & 0 & 0 & \frac{\alpha\left(B\delta-\mu\eta-\mu^2\right)}{\eta+\mu}+\frac{B\eta\left(B\delta-\mu\eta-\mu^2\right)}{\delta\mu(\eta+\mu)}-\gamma-\mu \end{bmatrix}$$

Above Jacobian matrix J_{PDC} has four distinct Eigenvalues:

$$\omega_1=-\mu,$$

$$\omega_2=-\frac{1}{2}\left(\frac{B\delta+\sqrt{B^2\delta^2-4\eta^2B\delta+4\eta^3\mu+12\eta^2\mu^2-8\mu\eta B\delta+12\mu^3\eta-4\eta^2B\delta+4\mu^2}}{\eta+\mu}\right),$$

$$\omega_3=-\frac{1}{2}\left(\frac{B\delta+\sqrt{B^2\delta^2-4\eta^2B\delta+4\eta^3\mu+12\eta^2\mu^2-8\mu\eta B\delta+12\mu^3\eta-4\eta^2B\delta+4\mu^2}}{\eta+\mu}\right),$$

$$\omega_4=\frac{\alpha\delta^2\mu B-\alpha\delta\mu^2\eta-\alpha\delta\mu^3+\beta\eta\left(B\delta-\mu\eta-\mu^2\right)-\gamma\delta\mu\eta-\gamma\delta\mu^2-\delta\mu^2\eta-\delta\mu^3}{\delta\mu(\eta+\mu)}$$

One can easily see that ω_1, ω_2 and ω_3 have negative value. And if $\omega_4 < 0$ then one can write

$$\frac{\alpha\delta^2\mu B - \alpha\delta\mu^2\eta - \alpha\delta\mu^3 + \beta\eta\left(B\delta - \mu\eta - \mu^2\right) - \gamma\delta\mu\eta - \gamma\delta\mu^2 - \delta\mu^2\eta - \delta\mu^3}{\delta\mu(\eta+\mu)} < 0$$

$$\Rightarrow \frac{\alpha\delta\mu\left(B\delta - \mu\eta - \mu^2\right) + \beta\eta\left(B\delta - \mu\eta - \mu^2\right) - \delta\mu\left(\gamma\eta + \gamma\mu + \mu\eta + \mu^2\right)}{\delta\mu(\eta+\mu)} < 0$$

$$\Rightarrow \left(B\delta - \mu\eta - \mu^2\right)(\alpha\delta\mu + \beta\eta) - \delta\mu\left(\gamma(\eta+\mu) + \mu(\eta+\mu)\right) < 0$$

$$\Rightarrow \left(B\delta - \mu\eta - \mu^2\right)(\alpha\delta\mu + \beta\eta) - \delta\mu(\eta+\mu)(\gamma+\mu) < 0$$

$$\Rightarrow \frac{\left(B\delta - \mu\eta - \mu^2\right)(\alpha\delta\mu + \beta\eta)}{\delta\mu(\eta+\mu)(\gamma+\mu)} - 1 < 0$$

$$\Rightarrow R_0 - 1 < 0$$

$$\Rightarrow R_0 < 1$$

So, if $R_0 < 1$ then ω_4 has negative value.

Further, we will also discuss the behaviour of the equilibrium point

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta}\right)$$

using Jacobian matrix J_{PDCS}. The Jacobian matrix J_{PDCS} of the waste paper recycling model is as follows:

$$J_{PDCS} = \begin{bmatrix} -\delta D^* - \mu & -\delta P^* & 0 & \gamma \\ \delta D^* & \delta P^* - \eta - \delta\alpha S^* - \mu & 0 & -\delta\alpha D^* \\ 0 & \eta & -\beta S^* - \mu & -\beta C^* \\ 0 & \delta\alpha S^* & \beta S^* & \delta\alpha D^* + \beta C^* - \gamma - \mu \end{bmatrix}$$

where

$$P^* = \frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \; D^* = \frac{-\beta Y + \gamma + \mu}{\delta\alpha},$$

$$C^* = Y, S^* = \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta}$$

and the value of Y is as in the Equation 2. The characteristics equation of the Jacobian matrix J_{PDCS} about the equilibrium point E^*_{PDCS} is as given below

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0$$

where

$$A_1 = -\delta\alpha D^* - \beta C^* + \gamma + 4\mu + \beta S^* - \delta P^* + \eta + \delta\alpha S^* + \delta D^*$$
$$= \delta\alpha\left(S^* - D^*\right) + \beta\left(S^* - C^*\right) + \delta\left(D^* - P^*\right) + \gamma + 4\mu > 0$$

$$A_2 = \delta^2\alpha S D^* + 3\delta\alpha S^*\mu - \beta S^*\delta P^* + \beta S^{*2}\delta\alpha + \beta S^*\delta D^* - \delta\alpha D^*\eta - 3\alpha\delta D^*\mu + \delta^2 P\alpha D^*$$
$$- 3\delta P^*\mu + \eta\delta D^* + 3\mu\delta D^* + \beta S^*\eta + 3\beta S^*\mu + \gamma\delta D^* - 3\beta C^*\mu - \delta^2\alpha D^{*2} - \beta C^*\eta$$
$$+ \gamma\beta S^* - \gamma\delta P^*$$
$$> \delta^2\alpha^2 S^* D^*\left(\beta S^* + \mu\right) - \left(\delta\alpha D^* + \beta C^*\right)\left(\eta + \delta\alpha S^* + \mu\right)\left(\beta S^* + \mu\right)$$
$$+ \beta S^*\left(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*\right) + \left(\eta + \delta\alpha S^* + \mu\right)\left(\gamma + \mu\right)\left(\beta S^* + \mu\right)$$
$$> 1 + \frac{\delta^2\alpha^2 S^* D^*}{\left(\eta + \delta\alpha S^* + \mu\right)\left(\gamma + \mu\right)} - \frac{\delta\alpha D^* + \beta C^*}{\left(\gamma + \mu\right)} + \frac{\beta S^*\left(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*\right)}{\left(\eta + \delta\alpha S^* + \mu\right)\left(\gamma + \mu\right)\left(\beta S^* + \mu\right)}$$
$$> 1 - R_0 > 0$$
$$\Rightarrow R_0 < 1$$

$$A_3 = 2\delta D^*\gamma\mu + \delta^2\alpha D^*\beta S^* P^* - 2\delta\alpha D^*\beta S^*\mu - 2\beta C^*\delta\alpha S^*\mu - \beta C^*\delta^2\alpha S^* D^* + 2\mu\delta^2 P^*\alpha D^*$$
$$- 2\eta\delta\alpha D^*\mu + 2\gamma\eta\mu - 3\beta C^*\mu^2 + 3\gamma\mu^2 + 3\eta\mu^2 - 3\delta\alpha D^*\mu^2 + \eta\gamma\delta D^* - 2\delta^2\alpha D^{*2}\mu$$
$$- \delta^2\alpha D^{*2}\eta - 2\beta C^*\eta\mu + 2\mu\beta S^*\eta + 3\delta\alpha S^*\mu^2 + 2\mu\eta\delta D^* + \gamma\beta S^*\eta + 2\gamma\beta S^*\mu - 2\gamma\delta P^*\mu$$
$$- \delta^2\alpha D^{*2}\beta S^* + 2\beta C^*\delta P^*\mu - 2\beta C^*\mu\delta D^* - \beta C^*\eta\delta D^* - 2\mu\beta S^*\delta P^* + 2\mu\beta S^{*2}\delta\alpha$$
$$+ 2\mu\beta S^*\delta D^* + 2\mu\delta^2\alpha S^* D^* + \beta S^*\eta\delta D^* + \beta S^{*2}\delta^2\alpha D^* - \gamma\beta S^*\delta P^* + \gamma\beta S^{*2}\delta\alpha$$
$$+ \gamma\beta S^*\delta D^* + 2\gamma\delta\alpha S^*\mu + 4\mu^3 + 3\beta S^*\mu - 3\delta P^*\mu^2 + 3\mu^2\delta D^*$$
$$> \mu(\delta^2\alpha^2 S^* D^*(\beta S^* + \mu) - (\delta\alpha D^* + \beta C^*)(\eta + \delta\alpha S^* + \mu)(\beta S^* + \mu)$$
$$+ \beta S^*(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*)) + (\eta + \delta\alpha S^* + \mu)(\gamma + \mu)(\beta S^* + \mu)$$
$$> 1 + \frac{\delta^2\alpha^2 S^* D^*}{\left(\eta + \delta\alpha S^* + \mu\right)\left(\gamma + \mu\right)} - \frac{\delta\alpha D^* + \beta C^*}{\left(\gamma + \mu\right)} + \frac{\beta S^*\left(\beta C^*\eta + \beta C^*\delta\alpha S^* + \beta C^*\mu + \eta\delta\alpha D^*\right)}{\left(\eta + \delta\alpha S^* + \mu\right)\left(\gamma + \mu\right)\left(\beta S^* + \mu\right)}$$
$$> 1 - R_0 > 0$$
$$\Rightarrow R_0 < 1$$

Similarly, one can prove,

$$A_4 = \delta^2\alpha D^*\beta S^* P^*\mu - \beta C^*\mu\delta^2\alpha S^* D^* + \mu^3\eta + \gamma\mu^3 - \delta^2\alpha D^{*2}\mu^2 - \delta\alpha D^*\mu^3 - \beta C^*\mu^2\eta$$
$$+ \beta S^*\mu^2\eta + \mu^2\eta\delta D^* + \mu^3\delta\alpha S^* + \gamma\beta S^*\mu^2 - \gamma\mu^2\delta P^* + \gamma\mu^2\delta D^* + \delta^2\alpha D^*\mu^2 P^*$$
$$- \delta^2\alpha D^{*2}\mu\eta - \delta\alpha D^*\mu^2\eta + \beta C^*\mu^2\delta P^* - \beta C^*\mu^2\delta D^* - \beta S^*\mu^2\delta P^* + \beta S^{*2}\mu^2\delta\alpha$$
$$+ \beta S^*\mu^2\delta D^* + \mu^2\delta^2\alpha S^* D^* + \gamma\beta S^*\eta\mu + \gamma\mu\eta\delta D^* + \gamma\mu^2\delta\alpha S^* - \beta C^*\mu^3 + \beta S^*\mu^3$$
$$- \mu^3\delta P^* + \mu^3\delta D^* + \gamma\mu^2\eta^*$$
$$> 1 - R_0 > 0$$
$$\Rightarrow R_0 < 1$$

It follows that $A_1 > 0, A_2 > 0, A_3 > 0, A_4 > 0$ and also $A_1A_2A_3 > A_3^2 + A_1^2A_4$.

Theorem 1. Using Routh-Hurwitz criterion [20], the equilibrium points

$$E^*_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta - \mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta - \mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0 \right)$$

and

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$$

of the waste water recycling model are locally asymptotically stable with the condition $R_0 < 1$.

3.2 Global Stability

The global stability will be conversed in this section using some graph theoretical results [6, 24] given as below:

- Any graph will consist of the set of vertices and the set of edges.
- (i, j) is called *an edge* from initial vertex i to terminal vertex j.
- *A directed graph* G is the set of vertices and the set of edges where all the edges are directed from one vertex to another [7].
- *The out-degree of a vertex* i is the number of edges whose initial vertex is i denoted as $d^+(i)$.
- *The in-degree of a vertex* i is the number of edges whose terminal vertex is i denoted as $d^-(i)$.
- A directed graph G is called *a weighted directed graph* if each edge is assigned a positive weight.
- *The weight* $w(H)$ of sub-directed graph H is the product of weights on all its edges.
- *A path* in a graph is a finite or infinite sequence of edged which connect the sequence of vertices where all are distinct from others [10].
- *A directed path* in a directed graph is a sequence of edges which connect a sequence of edges which connect a sequence of vertices where all the edges should be directed in the same direction.
- *A cycle graph* is a graph where some number of vertices connected in a closed chain [8].
- *A directed cycle graph* is a directed version of a cycle graph with all the edges being oriented in the same direction.
- *A loop* (or *buckle*) is an edge that connects a vertex i to itself [9].
- *A tree* is any acyclic connected graph [12].
- If tree is directed then it is called *directed tree*.
- *A spanning tree* is a subgraph of a graph G which includes all the vertices of G with minimum number of edges [11].

- If G is a weighted directed graph with n vertices then the weight matrix has order $n \times n$ denoted as $A = \left[a_{ij} \right]$ with entries $a_{ij} > 0$ which is equal to the weight of edge if it exists otherwise it is 0. This kind of weighted directed graph is noted by (G, A).
- G is called *strongly connected directed graph* if for any pair of discrete vertices there exists a directed path.
- (G, A) is called *a strongly connected weighted directed graph* if and only if the weighted matrix A is irreducible.
- The Laplacian matrix $L = \left[l_{ij} \right]$ of (G, A) is defined as $l_{ij} = \begin{cases} -a_{ij} & ; i \neq j \\ \sum\limits_{k \neq i} a_{ik} & ; i = j \end{cases}$.

Proposition 2 (Kirchhoff's matrix tree theorem). Assume $n \geq 2$ and let c_i be the cofactor of l_{ii} in L. Then $c_i = \sum\limits_{\tau \in T_i} w(\tau), i = 1, 2, ..., n$ where T_i is the set of all spanning trees τ of weighted directed graph (G, A) which makes tree at vertex i and $w(\tau)$ is the weight of τ. If the weighted graph (G, A) is strongly connected then $c_i > 0$ for $1 \leq i \leq n$.

Theorem 3. Let c_i be as given in the Kirchhoff's matrix tree theorem. If $a_{ij} > 0$ and $d^+(j) = 1$ for some i, j then $c_i a_{ij} = \sum\limits_{k=1}^{n} c_j a_{jk}$.

Theorem 4. Let c_i be as given in the Kirchhoff's matrix tree theorem. If $a_{ij} > 0$ and $d^-(i) = 1$ for some i, j then $c_i a_{ij} = \sum\limits_{k=1}^{n} c_k a_{ki}$.

Theorem 5. Suppose that the following assumptions are satisfied:

(1) There exists function $V_i : U \to \mathbb{R}, G_{ij} : U \to \mathbb{R}$ and constants $a_{ij} \geq 0$ such that for every $1 \leq i \leq n, V_i' \leq \sum\limits_{j=1}^{n} G_{ij}(z)$ for $z \in U$.

(2) For $A = \left[a_{ij} \right]$, each directed cycle C of (G, A) has $\sum\limits_{(s,r) \in \varepsilon(C)} G_{rs}(z) \leq 0$ for $z \in U$, where $\varepsilon(C)$ denotes the arc set of the directed cycle C.

Then, the function $V(z) = \sum\limits_{i=1}^{n} c_i V_i(z)$, with constant $c_i \geq 0$ as given in the proposition of Kirchhoff's matrix tree theorem, satisfies $V' \leq 0$ then V is a Lyapunov function for the system. First, we will discuss about the global stability for the equilibrium point

$$E^*_{PDC} = \left(\frac{\eta + \mu}{\delta}, \frac{B\delta - \mu(\eta + \mu)}{\delta(\eta + \mu)}, \frac{\eta(B\delta - \mu(\eta + \mu))}{\delta\mu(\eta + \mu)}, 0 \right)$$

using graph theory.

Let $V_{1_{PDC}} = \frac{1}{2}\left(P - P^* + D - D^* + C - C^*\right)^2, V_{2_{PDC}} = D - D^* - D^* \ln\frac{D}{D^*}, V_{3_{PDC}} = \frac{1}{2}\left(C - C^*\right)^2$

Differentiation of $V_{1_{PDC}}, V_{2_{PDC}}, V_{3_{PDC}}$, gives us

$$\begin{aligned} V'_{1_{PDC}} &= \left(\left(P - P^* + C - C^*\right) + \left(D - D^*\right)\right)\left(\mu\left(P - P^* + C - C^*\right) + \mu\left(D - D^*\right)\right) \\ &\le 2\mu\left(P - P^*\right)\left(D - D^*\right) + 2\mu\left(C - C^*\right)\left(D - D^*\right) \\ &= a_{12}G_{12} + a_{23}G_{23} \end{aligned}$$

$$\begin{aligned} V'_{2_{PDC}} &= \beta\left(P - P^*\right)\left(D - D^*\right) \\ &= a_{21}G_{21} \end{aligned}$$

$$\begin{aligned} V'_{3_{PDC}} &= \left(C - C^*\right)\left(\eta D - \eta D^* + \mu C^* - \mu C\right) \\ &\le \eta\left(D - D^*\right)\left(C - C^*\right) \\ &= a_{31}G_{31} \end{aligned}$$

Using above results and the set of three vertices, the weighted graph is created as shown in Figure 2.

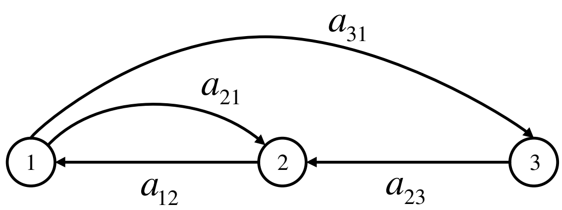

Figure 2. The weighted graph of waste paper model without scrap dealer

With $a_{12} = a_{23} = 2\mu, a_{21} = \beta, a_{31} = \eta$ and others $a_{ij} = 0$.

The related weighted graph has three vertices and two cycles where each cycle $G_{12} + G_{21} = 0$ and $G_{12} + G_{31} + G_{23} = 0$. Then, as assumptions taken in theorem 5, there exists $c_i, 1 \le i \le 3$ such that $V_{PDC} = \sum_{i=1}^{3} c_i V_{i_{PDC}}$ is Lyapunov function. Using theorem 3,

$$d^+(3) = 1 \Rightarrow c_2 a_{23} = c_3 a_{31}$$

Now, taking $c_2 = k$ and putting the values of a_{23} and a_{31}, we get

$$c_3 = \frac{2\mu k}{\eta}$$

Also, $d^+(2) = 1 \Rightarrow c_1 a_{12} = c_2 a_{21} + c_2 a_{23}$

$$\Rightarrow c_1 = \frac{(\beta + 2\mu)k}{2\mu}$$

Therefore,

$$\begin{aligned} V_{PDC} &= \sum_{i=1}^{3} c_i V_{i_{PDC}} \\ &= c_1 V_{1_{PDC}} + c_2 V_{2_{PDC}} + c_3 V_{3_{PDC}} \\ &= \frac{(\beta + 2\mu)k}{2\mu} V_{1_{PDC}} + k V_{2_{PDC}} + \frac{2\mu k}{\eta} V_{3_{PDC}} \end{aligned}$$

where t is an arbitrary constant.

This verifies that E^*_{PDC} is the only invariant set in $\operatorname{int}(\Lambda)$, where $V'_{PDC} = 0$. Hence, E^*_{PDC} is globally asymptotically stable in $\operatorname{int}(\Lambda)$.

Next, we will deliberate the global stability for the equilibrium point

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, Y, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$$

using graph theory. Consider, $V_{1_{PDCS}} = P - P^* \ln\frac{P}{P^*}$, $V_{2_{PDCS}} = D - D^* \ln\frac{D}{D^*}$, $V_{3_{PDCS}} = C - C^* \ln\frac{C}{C^*}$, $V_{4_{PDCS}} = S - S^* \ln\frac{S}{S^*}$

Differentiate $V_{1_{PDCS}}$ and we get,

$$\begin{aligned} V'_{1_{PDCS}} &= \left(1 - \frac{P^*}{P}\right) P' \\ &= \left(1 - \frac{P^*}{P}\right)(B - \delta PD + \gamma S - \mu P) \\ &= \left(1 - \frac{P^*}{P}\right)(\delta P^* D^* - \delta PD - \gamma S^* + \gamma S + \mu P^* - \mu P) \\ &= \delta\left(1 - \frac{P^*}{P}\right)(P^* D^* - PD) + \gamma S^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{S}{S^*}\right) + \mu\left(1 - \frac{P^*}{P}\right)(P^* - P) \\ &= -\delta P^* D^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{PD}{P^* D^*}\right) + \gamma S^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{S}{S^*}\right) + \mu\frac{(P - P^*)^2}{P} \\ &\le \gamma S^* \left(1 - \frac{P^*}{P}\right)\left(1 - \frac{S}{S^*}\right) \\ &= a_{14} G_{14} \end{aligned}$$

Similarly, differentiating $V_{2_{PDCS}}, V_{3_{PDCS}}$ and $V_{4_{PDCS}}$, we get

$$\begin{aligned}
V'_{2_{PDCS}} &= \left(1-\frac{D^*}{D}\right)D' \\
&= \left(1-\frac{D^*}{D}\right)(\delta PD - \eta D - \delta\alpha DS - \mu D) \\
&= \delta P^* D^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{PD}{P^*D^*}\right) - \eta D^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{D}{D^*}\right) - \delta\alpha D^* S^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{DS}{D^*S^*}\right) \\
&\quad - \mu D^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{D}{D^*}\right) \\
&\le \delta P^* D^* \left(1-\frac{D^*}{D}\right)\left(1-\frac{PD}{P^*D^*}\right) \\
&= a_{21}G_{21}
\end{aligned}$$

$$\begin{aligned}
V'_{3_{PDCS}} &= \left(1-\frac{C^*}{C}\right)C' \\
&= \left(1-\frac{C^*}{C}\right)(\eta D - \beta CS - \mu C) \\
&= \eta D^* \left(1-\frac{C^*}{C}\right)\left(1-\frac{D}{D^*}\right) - \beta C^* S^* \left(1-\frac{C^*}{C}\right)\left(1-\frac{CS}{C^*S^*}\right) - \mu C^* \left(1-\frac{C^*}{C}\right)\left(1-\frac{C}{C^*}\right) \\
&\le \eta D^* \left(1-\frac{C^*}{C}\right)\left(1-\frac{D}{D^*}\right) \\
&= a_{32}G_{32}
\end{aligned}$$

$$\begin{aligned}
V'_{4_{PDCS}} &= \left(1-\frac{S^*}{S}\right)S' \\
&= \left(1-\frac{S^*}{S}\right)(\delta\alpha DS + \beta CS - \gamma S - \mu S) \\
&= \delta\alpha D^* S^* \left(1-\frac{S^*}{S}\right)\left(1-\frac{DS}{D^*S^*}\right) + \beta C^* S^* \left(1-\frac{S^*}{S}\right)\left(1-\frac{CS}{C^*S^*}\right) - \gamma S^* \left(1-\frac{S^*}{S}\right)\left(1-\frac{S}{S^*}\right) \\
&\quad - \mu S^* \left(1-\frac{S^*}{S}\right)\left(1-\frac{S}{S^*}\right) \\
&\le \delta\alpha D^* S^* \left(1-\frac{S^*}{S}\right)\left(1-\frac{DS}{D^*S^*}\right) + \beta C^* S^* \left(1-\frac{S^*}{S}\right)\left(1-\frac{CS}{C^*S^*}\right) \\
&= a_{42}G_{42} + a_{43}G_{43}
\end{aligned}$$

Using above results and the set of four vertices, the weighted graph is created as shown in Figure 3.

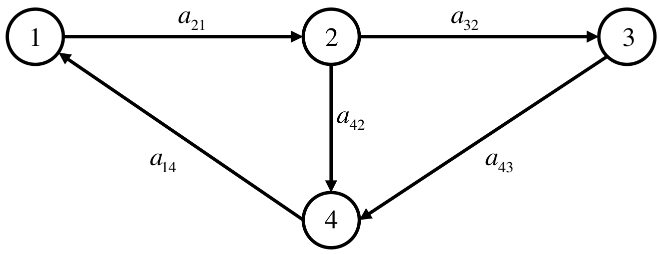

Figure 3. The weighted graph of waste paper model

with $a_{14} = \gamma S^*, a_{21} = \delta P^* D^*, a_{32} = \eta D^*, a_{42} = \delta\alpha D^* S^*, a_{43} = \beta C^* S^*$ and others $a_{ij} = 0$.

The related weighted graph has three vertices and two cycles where each cycle $G_{14} + G_{21} + G_{42} = 0$ and $G_{14} + G_{21} + G_{32} + G_{43} = 0$. Then, as assumptions taken in theorem 5, there exists $c_i, 1 \le i \le 4$ such that $V_{PDCS} = \sum_{i=1}^{4} c_i V_{i_{PDCS}}$ is Lyapunov function. Using theorem 3,

$$d^+(1) = 1 \Rightarrow c_2 a_{21} = c_1 a_{14}$$

Now, taking $c_1 = t$ and putting the values of a_{21} and a_{14}, we get

$$c_2 = \frac{t\gamma S^*}{\delta P^* D^*}$$

Also, $d^+(4) = 1 \Rightarrow c_1 a_{14} = c_4 a_{42} + c_4 a_{43}$

$$\Rightarrow c_4 = \frac{t\gamma S^*}{\delta\alpha D^* S^* + \beta C^* S^*}$$

and

$$d^+(3) = 1 \Rightarrow c_4 a_{43} = c_3 a_{32}$$

$$\Rightarrow \frac{t\gamma S^*}{\delta\alpha D^* S^* + \beta C^* S^*} \beta C^* S^* = c_3 \eta D^*$$

$$\Rightarrow c_3 = \frac{t\beta\gamma C^* S^{*2}}{\left(\delta\alpha D^* S^* + \beta C^* S^*\right)\eta D^*}$$

Therefore,

$$\begin{aligned} V_{PDCS} &= \sum_{i=1}^{4} c_i V_{i_{PDCS}} \\ &= c_1 V_{1_{PDCS}} + c_2 V_{2_{PDCS}} + c_3 V_{3_{PDCS}} + c_4 V_{4_{PDCS}} \\ &= t V_{1_{PDCS}} + \frac{t\gamma S^*}{\delta P^* D^*} V_{2_{PDCS}} + \frac{t\beta\gamma C^* S^{*^2}}{\left(\delta\alpha D^* S^* + \beta C^* S^*\right)\eta D^*} V_{3_{PDCS}} + \frac{t\gamma S^*}{\delta\alpha D^* S^* + \beta C^* S^*} V_{4_{PDCS}} \end{aligned}$$

where t is an arbitrary constant.

This confirms that E^*_{PDCS} is the only invariant set in $\operatorname{int}(\Lambda)$, where $V'_{PDCS} = 0$. Hence, E^*_{PDCS} is globally asymptotically stable in $\operatorname{int}(\Lambda)$.

Theorem 6. The positive equilibrium points

$$E^*_{PDC} = \left(\frac{\eta+\mu}{\delta}, \frac{B\delta - \mu(\eta+\mu)}{\delta(\eta+\mu)}, \frac{\eta\left(B\delta - \mu(\eta+\mu)\right)}{\delta\mu(\eta+\mu)}, 0 \right)$$

and

$$E^*_{PDCS} = \left(\frac{-\delta\alpha Y\mu + \gamma\eta + \mu\eta + \mu\beta Y}{\delta Y\beta}, \frac{-\beta Y + \gamma + \mu}{\delta\alpha}, X, \frac{-\delta\alpha Y\mu - \beta Y\eta + \gamma\eta + \mu\eta}{\delta\alpha Y\beta} \right)$$

of the waste water recycling model are globally asymptotically stable in $\operatorname{int}(\Lambda)$.

4 Sensitivity Analyses

In this section, we will confer about the sensitivity for each parameter used to formulate waste paper recycling model shown in Table 2.

The normalised sensitivity index of the parameters is calculated by using the following formula: $\Upsilon_{\alpha}^{R_0} = \frac{\partial R_0}{\partial \alpha} \cdot \frac{\alpha}{R_0}$ where α is the model parameter.

The rate at which distributor buys paper, the rate at which distributor gives paper to scrap dealers and the rate at which customer gives paper to scrap dealers have positive impact on R_0 which means they are factors which help us to recycle more waste paper. And others have negative effect for the waste paper recycling model and we should not increase them.

Table 2. Sensitivity analysis

Parameter	Value
δ	+
η	-
α	+
β	+
γ	-
μ	-

5 Numerical Simulations

In the current section, some numerical simulation has been done using the parametric values given in the Table 1.

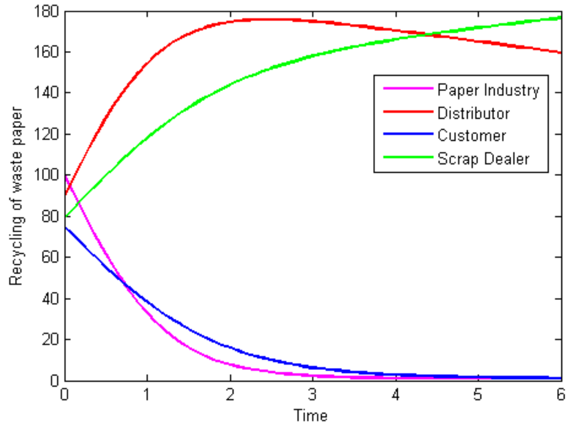

Figure 4. Transmission of waste paper recycling model

Figure 4 depicts that initially paper industry supplies to distributors and which gets maximum at 2 weeks. After buying it from the distributor, the customer runs away with the sticks in almost 6 weeks. When the used papers from the distributor, paper industry and customer increases pile up for scrap dealer and this result recycling of paper again to the paper industry.

Figure 5-7 shows the effect of recycling of waste paper due to scarp dealers. From all figures we can state that as the rate increases the recycling of waste paper is also increases. Which also mean that all the rates mentioned in following three figures have helpful influence on the model?

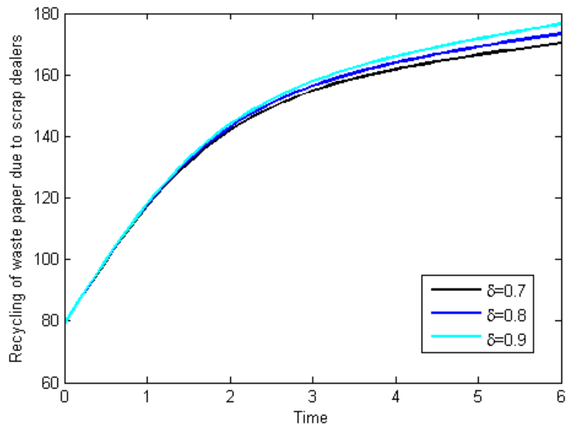

Figure 5. Effect of the rate at which distributor buys paper on the scrap dealer

From Figure 5, it can be determined that as the rate at which distributor buys paper (δ) is varied from 70% to 90%, the recycling of waste paper is increased from 170 to 178. This expressed that we can recycle more waste paper by 4.70% when δ is increased by 20%.

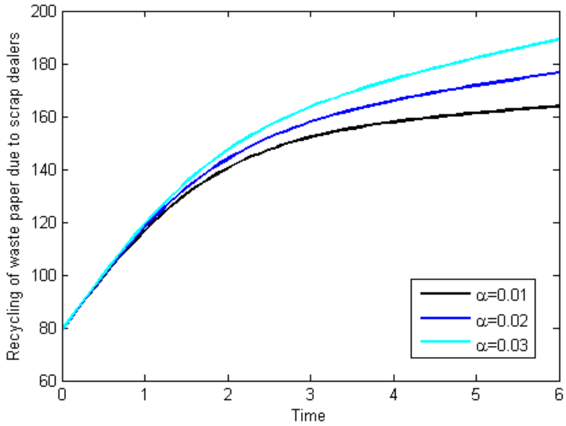

Figure 6. Effect of the rate at which distributor gives paper to scrap dealers on the scrap dealer

From Figure 6 as the rate at which distributor gives paper to scrap dealers (α) is varied from 1% to 3%, the recycling of waste paper is increased from 164 to 189. This shows that when α is increased by 2% there will be an increment of 15.24% in the recycled waste paper.

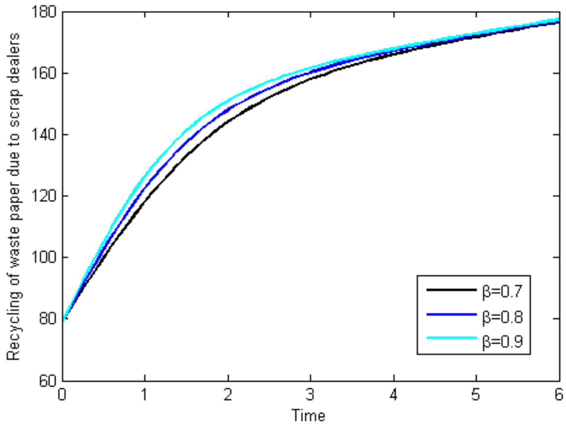

Figure 7. Effect of the rate at which customer gives paper to scrap dealers on the scrap dealer

In Figure 7, the rate at which customer gives paper to scrap dealers (β) is varied from 70% to 90% then the recycling of waste paper is increased from 188 to 189. It indicates that as 20% β is increased only 1% more waste paper is recycled.

Hence, from above three figures it can be concluded that α is more effective factor than δ and β in the recycling of waste paper which directs that distributors should have better awareness than customer and they should not waste leftover paper inappropriately. In fact, they should give as much as waste paper to scrap dealers.

6 Conclusions

In the proposed paper, a mathematical model of the waste paper recycling is formulated to examine the importance of scrap dealers in the process of waste paper recycling. Recycling paper reduces the need for raw material, it also requires much less energy, so it could preserve natural resources like trees, forest, water, fuel, etc. for the future and also condenses greenhouse gases. For that we all need to show our interest in recycling to make it successful. We need to take our time and save the paper products so that it can be recycled. We can minimize our use of paper, too. We can use electronic storage rather than paper storage. If we really need to buy paper, we just could buy a recycled one. Also, we can go digital for the further step in a right direction to protect the environment.

In the section 2, we have found the basic reproduction numbers for all the equilibrium points. From that it can be determined that when only paper industry exists and when distributor doesn't exist, the system will not physically stable. Also, we can conclude that if scrap dealers are not there, the paper recycling increases only by 64.56% but with the existence of scrap dealers, paper recycling will be increased by 73.46%. So, scrap dealers are noteworthy for the waste paper recycling.

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