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Volatility Modelling of Cryptocurrencies According to Different Investment Horizons: The Case of Bitcoin *

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Abstract

In this study, the fractal structure, efficiency, and long memory features of Bitcoin are investigated according to different investment horizons. The study utilized daily returns from 01.01.2017 to 22.11.2023, applying the maximum overlap discrete wavelet transform, Rescaled Range (R/S) analysis, and volatility models. The analysis results revealed a deviation of Bitcoin returns from the average and a negative correlation, indicating a lack of permanent behaviour in the series. The analysis demonstrates the rejection of the efficient market hypothesis and reveals a chaotic structure in the Bitcoin market. Furthermore, we observed a hyperbolic rate of decrease in returns at long-term investment horizons due to information shocks. This indicates that past returns can predict future returns. This suggests that instead of the efficient market hypothesis, the fractal market hypothesis is valid due to the existence of recurring trends. Finally, we determined the most appropriate volatility models for Bitcoin. The analysis shows that information shocks in Bitcoin returns at medium- and long-term investment horizons decrease over time, and past returns can predict future returns. However, volatility and information shocks are transitory at short- and medium-term investment horizons but can vary. All analysis methods yield consistent and compatible results, suggesting their potential extension to other cryptocurrency markets beyond the Bitcoin market.

Keywords: Cryptocurrency, Investment Horizon, Wavelet Analysis, Efficient Market Hypothesis, Fractal Market Hypothesis, Volatility, Bitcoin.

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1. INTRODUCTION

Accelerated advancements in information technologies have profoundly influenced economic systems, particularly the progression of currencies. Historically, there has been a progression from barter systems to precious metals, representational currencies, and paper money, culminating in a notable transformation with the advent of digital currencies in the 21st century. Nakamoto introduced Bitcoin in 2008, and subsequently, over 22,000 cryptocurrencies have arisen, collectively valued at more than $1 trillion (CoinMarketCap, 2023). Cryptocurrencies like Ethereum, Ripple, and Binance utilize blockchain and cryptographic technologies to facilitate decentralized peer-to-peer transactions without intermediaries like banking institutions. Their digital characteristics, autonomy from centralized entities, and divisibility differentiate these currencies from conventional financial assets, enabling them to obtain value from secure algorithms and transaction records rather than physical assets. The reduction of transaction costs and the principle of decentralization have facilitated the extensive acceptance of cryptocurrencies, culminating in significant price volatility and heightened transaction volumes. Initially utilized in online gaming, cryptocurrencies garner considerable attention from investors, regulators, and the media. With daily trade volumes over $3 billion, these assets have gained significant prominence in global financial markets. Nonetheless, the considerable volatility of cryptocurrencies and the absence of complete regulatory frameworks indicate that they remain in a developmental stage as financial instruments. Consequently, cryptocurrencies have emerged as a significant subject of scholarly inquiry, underscoring the necessity for additional examination to comprehend their potential as a modern asset class and their ramifications for countries and investors. This field contains substantial information that can be enhanced through analysis utilizing the Efficient Market Hypothesis (EMH), the Fractal Market Hypothesis (FMH), and volatility models.

Louis Bachelier's work, published in 1900, has opened the debate on whether price movements in financial markets are random for many years. In this context, the Random Walk Hypothesis argues that there is no discernible trend in price movements, that past price data cannot predict future prices, and that it is impossible to beat the market. In other words, this hypothesis suggests that it is unlikely to achieve a consistent advantage in the market. The Random Walk Hypothesis has been tested for many years and largely parallels the EMH. The EMH posits that prices move randomly and that financial asset prices fully reflect all available information. In line with this assumption, it is accepted that asset prices exhibit a normal distribution. Additionally, The EPH argues that prices move randomly, and financial asset prices reflect all available information. Therefore, according to the EPH, asset prices are assumed to follow a normal distribution. Thus, the EPH can be characterised by linearity, continuity and stationarity. However, research and observations have revealed that financial asset prices often exhibit sharp and heavy-tailed distributions and significant anomalies in price movements. These findings indicate that financial asset prices have a nonlinear, discontinuous, and complex structure. Therefore, it has been understood that new hypotheses need to be developed to explain these price movements. In this context, the FMH developed by Peters (1994) argues that financial asset prices exhibit Fractional Brownian Motion (FBM), and returns have a fractal distribution characterised by self-similarity and extended memory (Liu et al., 2022). FPH aims to explain the dynamics of financial markets more realistically by considering the irregular, volatile, and nonlinear nature of market movements. The fundamental assumptions of EPH fall short of explaining the crises, imbalances, chaos, distinct trends, speculative bubbles, and other anomalies in financial markets. In contrast, FPH provides a framework that can address such phenomena. While EPH generally focuses on a single investment horizon, FPH presents a broader perspective by acknowledging that investors may have different investment horizons. Thanks to these features, FPH allows for a more comprehensive analysis of market dynamics. As a result, FPH offers a more flexible and realistic approach to explaining crises, irregularities, distinct trends, and speculative bubbles in financial markets as an alternative to the unrealistic rigid assumptions of EPH. This different perspective has led to a growing interest in FPH. The FMH argues that financial instruments exhibit fractal characteristics over time and that financial time series display a self-similar structure when analyzed across different time scales. FPH also suggests that long-term memory and recurring structures at different investment horizons are present in price movements.

In this context, the Hurst exponent is the most commonly used method in long-term memory analysis. According to Hurst exponent coefficients, three different states of financial time series can be explained: a series having a long memory, meaning a trend; the series returning to its old average value; or the series exhibiting a completely random walk. Many methods are used to analyze different investment horizons in financial time series. Among these, the Maximum Overlapping Discrete Wavelet Transform (MODWT) is often preferred due to its lack of requirement for dyadic (powers of 2) properties and its more flexible structure (Gençay et al., 2002). The combined use of the Hurst exponent and wavelet analysis are effective methods in examining financial time series. These two analysis methods are of great importance in understanding behaviours in financial markets and developing investment strategies. The Hurst exponent is used to determine long-term trends in the price movements of financial assets and the tendency of the series to revert to its old mean. This information can help investors develop risk management strategies and make investment decisions focused on trend following or mean reversion. Wavelet analysis, on the other hand, can detect short-, medium-, and long-term cycles, volatility changes, and market anomalies in financial time series. This method allows for predicting market cycles and potential financial developments or crises in advance. Additionally, it contributes to adjusting risk levels according to market conditions. These two methods help investors and portfolio managers create balanced, risk-optimized portfolios, adapt to market dynamics, and make informed investment decisions. Hurst exponent and wavelet analysis have become fundamental tools of modern portfolio management. These methods can contribute significantly to understanding the cryptocurrency markets characterized by high volatility and complex structures. These methods are indispensable for developing a risk management and portfolio optimization strategy sensitive to the cryptocurrency markets' rapid changes and unexpected price movements. The study aims to determine appropriate volatility models for cryptocurrencies according to different investment horizons and examine them within the EMH and the FMH framework. The study focuses on Bitcoin, the cryptocurrency with the highest market value (CoinMarketCap, 2023). The analysis of Bitcoin contributes to the consistency and significance of the findings, allowing for generalizable results for the cryptocurrency market. Additionally, the results of the analysis allowed the findings related to Bitcoin to be evaluated within the scope of EPH and FPH.

A review of the existing literature reveals that studies on the price movements of cryptocurrencies do not comprehensively address the fractal dynamics of prices, the existence of extended memory, wavelet analysis and volatility models. Moreover, existing studies generally focus on a single time scale and ignore market dynamics and investor behaviour over different investment horizons. This study performs a filtering process for different investment horizons using multiscaling analysis. The EPH argues that there is no link between past and future market prices and that earning above the market return is impossible. However, long memory in financial markets suggests that past price data can be used to predict future prices. While this situation questions the validity of the EPH, it allows investors to earn returns above the market average. In this framework, there is a contrast between EPH and FPH. FPH provides an important framework for understanding the complex structure of the markets to explain the dynamics of financial markets more comprehensively and realistically. The primary motivation of this study is that studies on the cryptocurrency market generally ignore different investment horizons. In this context, the study is expected to contribute to the literature. The research findings are expected to contribute significantly to a better understanding of the cryptocurrency market dynamics, developing effective investment strategies, optimal portfolio diversification and improving risk management practices. It is also expected to provide a more in-depth perspective on the fractal structure of the cryptocurrency market and the effects of extended memory. The structure of the study is as follows: The "Literature Review" section provides a general evaluation of relevant studies. The "Methodology" section defines the data used and summarizes the methods applied in the analysis. The "Findings" section presents the empirical results, while the "Conclusion" part summarizes the key insights and discusses policy recommendations.

2. LITERATURE REVIEW

The financial literature has frequently analyzed cryptocurrency markets under both the Efficient market hypothesis and the Fractal market hypothesis, with numerous studies examining the level of market efficiency and price dynamics in these markets. Studies conducted within the context of the Efficient market hypothesis suggest that cryptocurrency markets are generally not efficient in a weak form but may exhibit efficiency during specific periods. For instance, Urquhart (2016) analyzed Bitcoin's market efficiency using the BDS and R/S Hurst Exponent tests. The findings suggested that Bitcoin does not operate as an efficient market; however, the division of data into sub-periods revealed periods of increased efficiency, indicating the potential for Bitcoin to achieve market efficiency in the future. Similarly, Nadarajah and Chu (2017), through BDS and Variance Ratio tests, observed that Bitcoin's market aligns with the efficient market hypothesis over time, asserting that no information loss accompanies this evolution. In a comparable analysis, Açıkalın and Sakınç (2022) employed Run Test and Variance Ratio Test analyses, concluding that the broader cryptocurrency market does not exhibit weak efficiency. Several studies focused on the persistence and long-memory properties of cryptocurrency markets. Bariviera (2017), using R/S and DFA analyses, determined that Bitcoin returns do not exhibit long memory, though its volatility does. Al-Yahyaee et al. (2018) employed MF-DFA methods to demonstrate multifractality and long memory traits across Bitcoin, gold, and other indices. They highlighted Bitcoin's more pronounced multifractality, implying lower market efficiency. Celeste et al. (2020) validated the fractal market hypothesis for Bitcoin, Ethereum, and Ripple, revealing that Bitcoin shows long memory and progresses toward maturity, while Ethereum and Ripple display speculative tendencies. Çelik (2020) indicated that the fractal market hypothesis was valid in the Bitcoin market in 2013-2019 and that financial bubbles and regime changes strengthen the long-memory structure of the market. Similarly, Sağlam Bezgin (2023), through R/S Analysis, confirmed the Fractal Market Hypothesis across various indices, with Bitcoin showing the highest persistence among cryptocurrencies and indices studied. A subset of research incorporated advanced econometric models to explore volatility and structural dynamics. Dyhrberg (2016) utilized GARCH and EGARCH models, revealing Bitcoin's high volatility persistence and symmetric market responses, emphasizing its value in financial markets. Mensi, Al-Yahyaee & Kang (2019) using FIGARCH, FIAPARCH, and HYGARCH models, found long memory and structural breaks in Bitcoin and Ethereum returns and volatility. Also, Mensi, Lee et. al. (2019) found that Bitcoin is less efficient than Ethereum and that market efficiency varies by time scale and sub-periods. Their findings were consistent with Eteman and Işığıçok (2022), who applied FIGARCH to Bitcoin, Ethereum, and Cardano, affirming long memory through statistically significant parameters. The impact of external shocks on cryptocurrency market dynamics was another theme in the literature. Mnif and Jarboui (2021) observed that Bitcoin's fractal structure diminished post-pandemic, indicating improved efficiency. Similarly, Assaf et al. (2022) noted shifts in long-term memory during and after COVID-19, with DASH exhibiting persistent traits, while Ethereum, Litecoin, and Ripple showed reduced market efficiency. These results are similar to those found by Özdemir Yazgan (2022), who used ARFIMA-LSTM hybrid models to see Bitcoin's extended memory. This shows how advanced forecasting techniques can be helpful in changing market conditions. The comparative analysis of cryptocurrencies with traditional assets has also enriched the understanding of market efficiency. Moralı and Uyar (2018), through R/S Analysis, validated the fractal market hypothesis across precious metals such as gold and platinum. Their findings resonate with those of Fakhfekh and Jeribi (2020), who used advanced GARCH models to highlight asymmetry in cryptocurrency responses to market information, drawing parallels to other asset classes. Similarly, Caporale et al. (2018) noted persistence in Bitcoin, Litecoin, Ripple, and DASH markets, emphasizing the variability of persistence across periods. In conclusion, the reviewed studies collectively highlight the evolving nature of cryptocurrency markets, characterized by long memory, multifractality, and structural breaks. These traits challenge the weak form of the Efficient Market Hypothesis, offering a nuanced perspective on cryptocurrencies as an emerging asset class.

3. METHODOLOGY

3.1. Aim of the Study and Data Set

This research aims to model the volatility of cryptocurrencies across different investment horizons. To achieve this, wavelet coefficients were calculated for the returns of Bitcoin cryptocurrency based on the Maximum overlap discrete wavelet transform. Following this, the return series were filtered into seven different time scales (investment horizons): D1 (2-4 days), D2 (4-8 days), D3 (8-16 days), D4 (16-32 days), D5 (32-64 days), D6 (64-128 days), and D7 (128-256 days), along with S7. Furthermore, we computed the Hurst exponent, representing the fractal structure, for each horizon by analyzing the returns generated from filtering the Bitcoin cryptocurrency across different investment horizons. Additionally, we utilized short-term and long-term memory models to capture the dynamics of time-varying volatility and the persistence of shocks affecting it. The study used daily closing prices of Bitcoin cryptocurrency sourced from the Bitstamp platform, calculated based on a global volume-weighted average of selected cryptocurrency exchanges worldwide and processed using the Python programming language. Data from January 1, 2017, to November 22, 2023 (totaling 2,517 days) were analyzed. The focus on Bitcoin's daily closing prices starting in 2017 stems from the substantial surge in interest from individual and institutional investors during this period, accompanied by increased liquidity and heightened volatility. In addition, focusing on Bitcoin, the cryptocurrency with the highest market capitalization, in the research and analysis is anticipated to improve the reliability and relevance of the findings, thus aiding in the broader application of the results to the overall cryptocurrency market. Bitcoin returns are calculated with the help of $r_{i,t} = ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right)$ formula. $r_{i,t}$, price returns of the i'th variable at time t and $P_{i,t}$ and $P_{i,t-1}$ denote the closing prices of the variables at time t and t-1, respectively.

3.2. Research Methodology

The analysis steps are depicted in Figure 1.

Figure 1. Steps of Analysis

Step 1	Maximum Overlap Discrete Wavelet Transform (MODWT) Analysis
Step 2	Rescaled Range (R/S) Analysis
Step 3	Volatility Models

3.2.1. Maximum Overlap Discrete Wavelet Transform (MODWT)

The Maximum Overlap Discrete Wavelet Transform (MODWT) is often used in economic and financial forecasting because it is flexible and can work around the problems with the Discrete Wavelet Transform (DWT). Like DWT, MODWT uses wavelet filters to decompose data into various scales and calculate wavelet coefficients. MODWT, on the other hand, has some significant benefits, like working with data sets of any size and having very accurate timing, which keeps the sample size and makes scale-specific wavelet coefficients. This enhances the ability to capture the underlying features of time series more effectively. The zero-phase property of MODWT makes it easier to understand wavelet details in the context of the original time series. This makes it a more asymptotically helpful tool for estimating wavelet variance. Unlike DWT, MODWT is non-orthogonal and does not require filtered output sampling but supports multi-scale analysis (In & Kim, 2012, p. 24). The pyramid algorithm (Mallat, 1989) is used to compute detail and approximation coefficients for efficient decomposition. These MODWT coefficients are represented as indicated in equation (1) (Gençay et al., 2002, p. 135):

$$\widetilde{\boldsymbol{w}} = \widetilde{W}_{\boldsymbol{x}} \tag{1}$$

In this equation (1), $\boldsymbol{x}$ represents the $Nx1$ observation vector. $\widetilde{W}$, represents the $(J+1) \times N \times N(J+1) \times N \times N$ matrix that defines the MODWT in this equation. For $N > 2^J$, the discrete wavelet coefficients are expressed as follows:

$$\widetilde{\boldsymbol{w}} = \left[\widetilde{w}_1, \widetilde{w}_2, \dots \dots \dots \dots \widetilde{w}_J, \widetilde{v}_J\right]^T \tag{2}$$

Here, as in the DWT, $\widetilde{\boldsymbol{w}}_{\boldsymbol{i}} = N/2^i$ is expressed as dimension of a vector of wavelet coefficients and is associated with changes in the $\lambda_i = 2^{i-1}$dimension of the scale. In addition, $\widetilde{\boldsymbol{v}}_J, N/2^J$ is a vector of scaling coefficients and is related to changes in the $2^J = 2\lambda_J$ sional scale (Gençay et al., 2002, p. 136).

The pyramid algorithm used by the Maximum overlap discrete wavelet transform is similar to that of DWT. However, MODWT applies scaled versions of wavelet and scaling filters. Here $\boldsymbol{x}$ represents a vector of observations with NN dimensions**,** while $\widetilde{h_l} = \widetilde{h_l}/2^l, l = 0,1,2, \dots \dots \dots \dots .., L-1,$ denotes the rescaled wavelet filter (high-pass filter), and represents the rescaled scaling filter (low-pass filter). $\widetilde{g_l} = \widetilde{g}_l/2^l, l = 0,1,2, \dots \dots .., L-1$ represents the rescaled scaling filter (low-pass filter) (Gençay et al., 2002, p. 136). The pyramid algorithm of MODWT is as follows: The MODWT pyramid algorithm requires three key components for each iteration: x data vector, h_l wavelet filter, and g_l scaling filter. Using wavelet and scaling filters, the observation vector x_t $\forall_t = 0,1,2 \dots .., N-1$ s filtered. The first iteration begins by applying each filter to obtain the following wavelet and scaling coefficients (Gençay et al., 2002, p. 136):

$$\tilde{w}_{1,t} = \sum_{l=0}^{L-1} \tilde{h}_l \tilde{x}_{t-l \bmod N} \tag{3}$$

and

$$\tilde{v}_{1,t} = \sum_{l=0}^{L-1} \tilde{g}_l \tilde{x}_{t-l \bmod N} \tag{4}$$

Here, the obtained $\tilde{v}_{1,t}$ values are assigned as a new observation vector. Subsequently, $\tilde{v}_{1,t}$ s filtered using wavelet and scaling filters to derive the second level for x_t $\forall_t = 0,1,2 \ldots\ldots, N-1$.

$$\tilde{w}_{2,t} = \sum_{l=0}^{L-1} \tilde{h}_l \tilde{x}_{1,t-l \bmod N} \tag{5}$$

and

$$\tilde{v}_{2,t} = \sum_{l=0}^{L-1} \tilde{g}_l \tilde{v}_{1,t-l \bmod N} \tag{6}$$

After obtaining the second level, the wavelet coefficient vector is designated as $\tilde{w} = [\tilde{w}_1, \tilde{w}_2, \tilde{v}_2]^T$ (Gençay et al., 2002, p. 137). The $v_{2,t}$ values obtained at the second level are set as the new observation vector. Then, $v_{2,t}$ is filtered for $\forall_t = 0,1,2 \ldots\ldots, N-1$ using wavelet and scaling filters. Here, the wavelet coefficient vector is assigned as $\tilde{w} = [\tilde{w}_1, \tilde{w}_{2,} \tilde{w}_3, \tilde{v}_3]^T$. This procedure can be repeated up to $J = log_2(N)$ iterations, and the final wavelet coefficient vector is given as $\tilde{w} = [\tilde{w}_1, \tilde{w}_{2,} \ldots\ldots\ldots.. \tilde{w}_J, \tilde{v}_J]^T$ (Gençay et al., 2002, p. 137). The $\tilde{w}_J, \tilde{v}_J$ series obtained above are filtered using wavelet and scaling filter coefficients for $\forall_t = 0,1, \ldots\ldots, N-1$ resulting in the $J-1$ dimensioned $\tilde{v}_{J-1}$ vector as shown below:

$$\tilde{v}_{J-1,t} = \sum_{l=0}^{L-1} \tilde{h}_{l,t+l \, modN} + \sum_{l=0}^{L-1} \tilde{g}_l \tilde{v}_{J,t+l modN} \tag{7}$$

The J-2 dimensioned $\tilde{v}_{j-2}$ vector is obtained as shown below (Gençay et al., 2002, p. 137):

$$\tilde{v}_{j-2,t} = \sum_{l=0}^{L-1} \tilde{h}_l \tilde{w}_{J-1,t+l \, modN} + \sum_{l=0}^{L-1} \tilde{g}_l \tilde{v}_{J-1,t+l \, modN} \tag{8}$$

This procedure can be repeated until the first wavelet level and scaling coefficients are combined to produce the original observation vector. The procedure continues until the first wavelet level, and the initial observation vector is obtained for $\forall_t = 0,1,2,3, \ldots\ldots\ldots\ldots\ldots. N-1$ as shown in the following Equation (9) (Gençay et al., 2002, p. 137):

$$x_t = \sum_{l=0}^{L-1} \tilde{h}_l \widetilde{w}_{1,t+l\ modN} + \sum_{l=0}^{L-1} \tilde{g}_l \tilde{v}_{1,t+l\ modN} \quad (9)$$

Wavelet analysis is a robust tool for analyzing time series data's time and frequency components, offering insights into short- and long-term trends, abrupt changes, and irregular patterns (Uyar, 2019, p. 141). It is particularly effective for non-stationary and complex time series, revealing how relationships between variables evolve across scales. In financial applications, wavelet analysis decomposes asset returns into various components, enabling an in-depth examination of volatility and risk transformation over time (Uyar & Kangallı Uyar, 2021, p. 316). This method is highly valuable in economics and finance due to its multiscaling capability, which is crucial in markets where participants operate on diverse investment horizons. Investors and portfolio managers often make decisions at different time scales, adding complexity to variable dynamics. By addressing these multi-scale dynamics, wavelet analysis provides critical insights into the temporal evolution of market behaviour (In & Kim, 2012, p. 5). Key techniques in wavelet analysis include General Wavelet Transform, stationary process-long memory analysis, denoising, and multiscaling. Multiscaling is particularly effective in understanding the time-varying nature of investments. Table 1 categorizes the cycle times corresponding to different horizons across annual, monthly, and daily scales.

Table 1. Wavelet Analysis Time Horizons by Multiscale Method

Scales (2^j)		Annual Frequency	Monthly Frequency	Daily Frequency
1	2^1	2-4	2-4	2-4
2	2^2	4-8	4-8	4-8
3	2^3	8-16	8-16 (8 months-1 year 4 months)	8-16
4	2^4	16-32	16-32 (1 year 4 months-2 years 8 months)	16-32(3 weeks 1 days-6 weeks 2 days)
5	2^5	32-64	32-64 (2 years 8 months-5 years 4 months)	32-64(6 weeks 2 days-12 weeks 4 days)
6	2^6	64-128	64-128 (5 years 4 months-10 years 8 months)	64-128 (12 weeks 4 days-25 weeks 3 days)
7	2^7	128-256	128-256 (10 years 8 months-21 years 4 months)	128-256 (25 weeks 3 days - 51 weeks 1 days)
8	2^8	256-512	…..	….

Source: Crowley (2007, p. 214). The theoretical maximum number of scales is expressed as 9. When the number of scales is denoted by j (j=9), frequencies are calculated using 2^j notation.

After conducting a wavelet analysis on the scale frequencies in Table 1, we will predict the specified number of scales and estimate coefficients for different investment horizons. In Table 1, we categorize the time scales as follows: The grouping is as follows: short-term, medium-term, and long-term. This categorization examines how investors' movements with short-, medium-, and long-term investment horizons evolve across different time scales. Short-term investment horizons represent short-term changes caused by shocks occurring in the 2–16-day time scales and encompass daily and weekly variations. Medium-term investment horizons represent medium-term changes occurring in the 32–128-

day time scales and encompass monthly and quarterly variations. Long-term investment horizons represent long-term changes in time scales of 256 days or more and correspond to periods that encompass annual variations (Uyar & Kangallı Uyar, 2021, p. 319).

3.2.2. Rescaled Range (R/S) Analysis

This method was developed by Hurst (1951) in a study on the Nile River flow rate, and the 'Rescaled Range (R/S)' test was proposed by Mandelbrot (1972, 1975) for use as a range or R/S statistic on the standard deviation. This method is the best-known non-parametric method for analysing long-term memory. The R/S statistic is an interval obtained by rescaling a time series using its standard deviation and partial sums of deviations from its mean. Taking $X_{1,}X_{2,..............}X_n$ as the return series of the sample and $\overline{X_n} = \frac{1}{n}\sum_j X_j$ as the mean, the classical R/S statistic is calculated using the following formula:

$$\tilde{Q}_n \equiv \frac{1}{S_n}\left[\underset{1 \le k \le n}{\max}\sum_{j=1}^{k}(X_j - \bar{X}_n) - \underset{1 \le k \le n}{\min}\sum_{j=1}^{k}(X_j - \bar{X}_n)\right] \tag{10}$$

where S_n denotes the usual (maximum likelihood). The standard deviation estimator is formulated as follows:

$$S_n \equiv \left[\frac{1}{n}\sum_{j}(X_j - \bar{X}_n)^2\right]^{1/2} \tag{11}$$

It is formulated as follows. The first term in brackets in equation (10) represents the maximum value of the first k deviations of the absolute values of the deviations of the X_j values from the sample mean (over k). Since the deviations of X_1 from the mean are 0 (zero), this maximum value does not always take a negative value. On the other hand, the second term in brackets represents the minimum value (over k) of the same set of partial sums of X_j. However, this minimum value is always positive. Therefore, the difference between these two terms, called the range, takes a value greater than or equal to zero. In short, the difference between the two terms never takes a negative value ($Qn \geq 0$) (Lillo & Farmer, 2004, p. 6; Lo, 1991, pp. 1286-1287). Although the usability of this method depends on the highest and lowest values in the dataset, the effect of outliers should be taken into account. Due to this effect, the technique can only be applied to stable and trend-free time series (Cao et al., 2018, p. 2). If a time series is stationary and does not have the long memory property, the R/S statistic tends to approach a random variable at a rate of $T^{1/2}$ If a stationary time series y_t has the long memory property, Mandelbrot (1975) showed that the R/S statistic converges to a random variable at the rate H^T (where H is the Hurst parameter). Based on this result, if a time series exhibits short memory, depending on the sample size, the log-log plot of the R/S statistic should be distributed around a straight line with a slope of ½. However, in the case of a time series with long memory, if the sample size is large enough, the log-log plot should be distributed around a straight line with slope $H > 1/2$ (Zivot & Wang, 2006, p. 280). Mandelbrot and Wallis (1969), addressed issues such as autocorrelations, variance ratios and spectral decompositions to demonstrate the superiority of R/S analysis over conventional methods for detecting long-run dependence. For example, Mandelbrot and Wallis (1969) showed through Monte Carlo simulations that the R/S statistic can detect long-run dependence in non-Gaussian time series with skewness and kurtosis. Mandelbrot (1972, 1975) also emphasised that for stochastic processes with infinite variance, the R/S statistic converges almost surely and provides a distinct advantage over autocorrelations and variance ratios in such processes (Lo, 1991, p. 12867). In their 1969 study, Mandelbrot and Wallis defined metrics called 'Hurst Exponent' to determine the long memory feature. They also developed a graphical technique to assess these metrics by utilising R/S analysis.

3.2.3. Short and Long Memory Volatility Models

3.2.3.1. Generalized Autoregressive Conditional Variance (GARCH) Model

ARCH models serve as the basis for volatility modelling but lack a definitive method for selecting lag order, leading to potentially complex models and parameter estimation issues. These limitations prompted Bollerslev (1986) to extend Engle's ARCH model by incorporating ARMA processes, resulting in the Generalized Autoregressive Conditional Heteroskedasticity (GARCH, p, q) model. Unlike ARCH, GARCH models account for both past errors and previous conditional variances, providing a more comprehensive framework for modelling volatility (Rachev et al., 2007, p. 284). This extension addresses the shortcomings of ARCH models, offering greater flexibility in capturing the dynamics of conditional variance in financial time series (Rachev et al., 2007, p. 284). Bollerslev's (1986) simplest GARCH model is:

$$h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_p \varepsilon_{t-p}^2 + \beta_1 h_{t-1}^2 + \cdots + \beta_q h_{t-q}^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_i h_{t-j}^2 \qquad (12)$$

In this equation, the coefficient α_1 indicates the short-term effect and the coefficient β_i indicates the long memory effect, i.e. the memory of the shocks to the series (Engle, 2002, p. 342). GARCH(p,q) condition; If $\omega > 0$, $\alpha_1 \geq 0$, $\beta_i \geq 0$, $\alpha_1 + \beta_1 < 1$, the effect of shocks on the conditional variance is not permanent (terminated) (Davidson & Mackinnon, 2004, p. 579; Tsay, 2010, p. 133). If $\alpha_1 + \beta_1 < 1$, the effect of shocks is permanent, and if $\alpha_1 + \beta_1 > 1$, shocks have a long memory feature (Brooks, 2008, p. 423).

3.2.3.2. GARCH-M Model (GARCH-MEAN)

The return of a financial asset may depend on its volatility in the financial market. In order to model such a phenomenon, the GARCH-M model, where M (Mean) is GARCH on average, can be used (Tsay, 2010, p. 142). GARCH models are arguably more popular than ARCH models. A GARCH-M model takes the following form (Brooks, 2019, pp. 525-526):

$$y_t = \mu + \delta\sigma_{t-1} + \mu_t, \qquad \mu_t \sim N(0, \sigma_t^2 \qquad (13)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \mu t_{t-1}^2 + \beta\sigma_{t-1}^2 \qquad (14)$$

In equation (13), y_t indicates that the return series is serially correlated. When δ is positive and statistically significant, the average return rises as a result of the increase in conditional variance. Therefore, it may be possible to interpret δ as a risk premium (Brooks, 2019, pp. 525-526).

3.2.3.3.C-GARCH Model (Component GARCH)

$$\sigma_t^2 = q_1 + \alpha_1(\varepsilon_{t-1}^2 - q_{t-1}) + b_1(\sigma_{t-1}^2 - q_{t-1}) \qquad (15)$$

$$q_t = \alpha_0 + pq_{t-1} + \theta(\varepsilon_{t-1}^2 - \sigma_{t-1}^2) \qquad (16)$$

q_t: is the long memory volatility, whereas the first equation is the non-permanent component (short-term volatility movements) that approaches 0 with the power of $(\alpha + \beta)$ and the second equation is the long-run component that approaches $q_t$with the power of p (Ding & Granger, 1996).

3.2.3.4. FIGARCH Model

In the context of financial market volatility, the Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH) model, developed by Baillie et al. (1996), has the potential to provide more accurate and suitable results for conditional variance processes. Compared to the GARCH model, FIGARCH can reflect long-range dependencies more precisely and flexibly. The key difference between the two models is that while older shocks in the time series tend to decay rapidly, the impact of recent shocks persists hyperbolically for an extended period (Kasman & Torun, 2007, p. 17). The FIGARCH(p, d, q) model can be formulated using the following equation:

$$[1 - \alpha(L) - \beta(L)](1 - L)^d \varepsilon_t^2 = \omega + [1 - \beta(L)]v_t \qquad (17)$$

Here ω represents the constant value, α represents the coefficient of the ARCH model,β represents the coefficient of the GARCH model, and d represents the fractional differencing parameter, which indicates the degree of long memory. In this context, d is a value within the range $0 < d < 1$. Such a model exhibits a much more flexible structure compared to GARCH-type models used to explain the transient dependencies observed in financial market volatility (Davidson, 2004, p. 20). The FIGARCH model can be briefly explained as follows:

$$\phi(L)\ (1 - L)^d \varepsilon_t^2 = \omega + [1 - \beta(L)]v_t \qquad (18)$$

ε_t^2, represents the squared errors of the GARCH process. It is assumed that all roots of $\phi(L)$ and $[1 - \beta(L)]$ lie outside the unit circle to ensure the stationarity of the process.

When $d = 0$ FIGARCH (p, d, q) process becomes GARCH (p, q).

When $d = 1$ FIGARCH (p, d, q) process becomes IGARCH (p, q). This situation implies that shocks have an infinite impact on future volatility.

If $0 < d < 1$ FIGARCH (p, d, q) process indicates that the impact of shocks on variance decreases hyperbolically.

3.2.3.5. HYGARCH Model

The Hyperbolic GARCH (HYGARCH) model was developed by Davidson in 2004 to test whether the Fractionally Integrated GARCH (FIGARCH) model exhibits a stationary structure. This model is used to detect long-term memory in conditional variance. Proposing the inclusion of weights in the difference operator to expand the conditional variance of the FIGARCH model, Davidson (2004) states that the HYGARCH model should be as follows:

$$h_t^2 = \omega[1-\beta(L)]^{-1} + [1-(1-\beta(L))^{-1}\rho(L)\{1+\alpha(1-L)^d\}]\varepsilon_t^2 \qquad (19)$$

In equation (17) the Hyperbolic GARCH model is obtained by replacing $(1-L)^d$ with $\{1+\alpha[(1-L)^d]\}$ in the FIGARCH model. The coefficient α represents the weight factor's coefficient. Generally, in model estimation, the coefficient α is used as $\ln(\alpha)$ or log(α) When α =1 or $\ln(\alpha)$=0 the model reduces to the FIGARCH model, and this process becomes stationary when $\ln(\alpha) < 0$ (Davidson, 2004). While FIGARCH and HYGARCH models effectively capture volatility clustering and long memory in financial time series, they fail to detect asymmetry in return volatility (Eyüboğlu & Eyüboğlu, 2022, p. 25).

4. FINDINGS

Table 2. Descriptive Statistics of Bitcoin Return Series

	Mean	Maximum	Minimum	Standard Deviation	Skewness	kurtosis	Jarque-Bera	P. Value	Number of Observations
Bitcoin	0.0014	0.2383	-0.4939	0.0398	-0.7788	15.194	15850.9	[0.0000]	2517

Note: ***, ** and * represent statistical significance at the 1%, 5%, and 10% confidence intervals, respectively.

Table 2 presents the descriptive statistics for the Bitcoin return series. Examining Table 2, it can be stated that while the average return shows a positive trend with 0.14%, the maximum return of 23.83% and the minimum return of -49.39% indicate a wide fluctuation range and high volatility (3.98%). The skewness value is expected to be 0 (zero) in a normal distribution. However, the negative skewness value (-0.7788) indicates that the distribution is left-skewed, and the kurtosis value (15.194) is observed to be positive and greater than 3. This shows a more peaked (leptokurtic) structure than a normal distribution. Finally, the Jarque-Bera statistic rejects the hypothesis "H0: the series is consistent with a normal distribution. Due to the very high value of the Bitcoin return series, it is observed that the series does not exhibit a normal distribution. The statistical results for the ARCH-LM and Ljung-Box test statistics for the Bitcoin return series are presented in Table 3.

Table 3. ARCH-LM and Ljung-Box Test Statistics for Bitcoin Return Series

	ARCH (2)	ARCH (5)	ARCH (10)	Q (5)	Q (10)	Q (20)	Q^2 (5)	Q^2 (10)	Q^2 (20)
Bitcoin	13.817***	8.299***	5.116***	14.224**	19.929**	26.561	48.419***	63.996***	79.793***

Note: ***, **, and *represent statistical significance at the 1%, 5%, and 10% confidence levels. [ ] indicates p-values, and Q and Q2 represent the Ljung-Box test for independence of return and squared return series, respectively. ARCH(2,5,10) refers to the ARCH-LM test for order (2, 5, and 10).

Examining the ARCH-LM and Ljung-Box (Q and Qsquare) statistics in Table 3, variance issues in the residuals of the included Bitcoin return series are evident up to lag 2, 5, and 10. Ljung-Box (Q and Qsquare) test statistics for different lag orders were used to assess the independence of the return error and squared return error series for the variables. Analysis of the statistical results reveals an autocorrelation problem in Bitcoin's logarithmic return series. In other words, the analysis relates the logarithmic return series of the Bitcoin cryptocurrency to its past values.

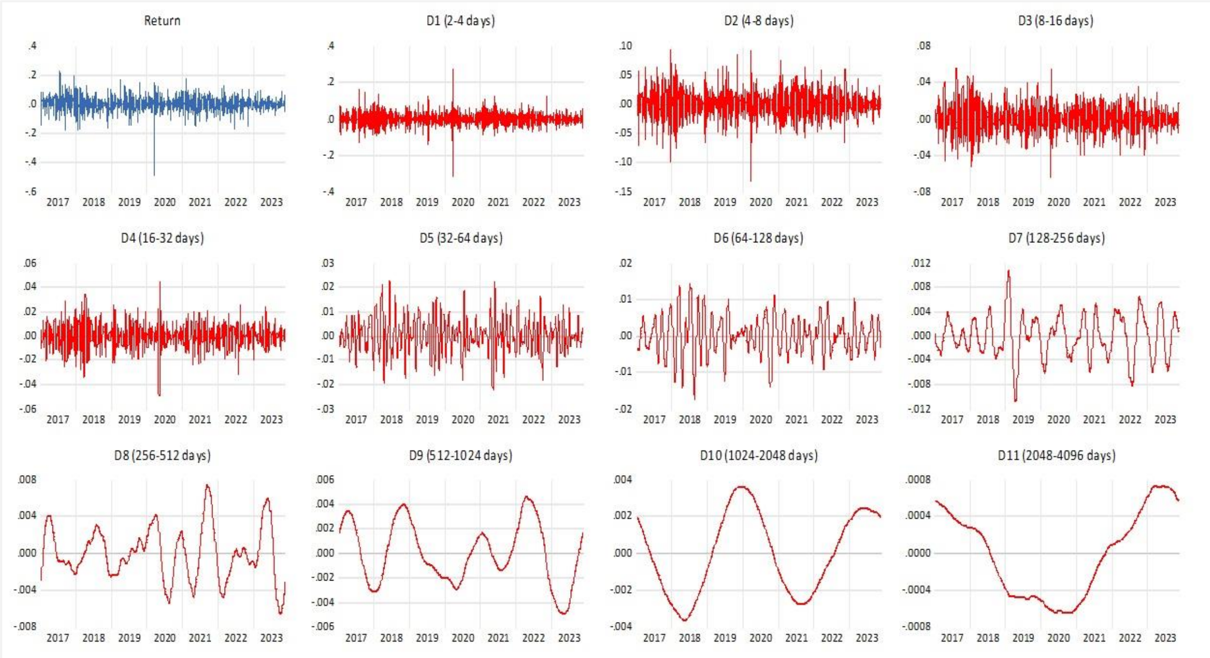
Table 4. Unit Root Test Results for Bitcoin

		Level			Return		
	Model	ADF	PP	KPSS	ADF	PP	KPSS
Bitcoin	With constant	-1.4024	-1.4124	3.4584	52.8097***	-52.7415***	0.2168
	With constant and Trend	-1.9326	-1.9350	0.4477	52.8317***	-52.7613***	0.0822

Note: In the ADF test, the maximum lag was set to 26, and the optimal lag was determined using the Schwarz Information Criterion. For the PP and KPSS tests, the long-run variance was obtained using the Bartlett kernel estimator, and the bandwidth was determined using the Newey-West method. For the ADF and PP tests, the critical values are -3.433122 (1%), -2.862651 (5%), and -2.567407 (10%) for the model with a constant; and -3.962212 (1%), -3.411849 (5%), and -3.127817 (10%) for the model with a constant and a trend. For the KPSS test, the critical values are 0.739000 (1%), 0.463000 (5%), and 0.347000 (10%) for the model with a constant, and 0.216000 (1%), 0.146000 (5%), and 0.119000 (10%) for the model with a constant and a trend. The symbols ***, **, and * indicate statistical significance at the 1%, 5%, and 10% levels, respectively.

Analyzing how a variable's previous values influence its current state provides insight into a time series' long-term characteristics. Regression analysis across periods helps reveal the series' evolution. To check for stationarity, unit root tests like the Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) tests are very important (Tarı, 2014, p. 387). Table 4 presents the results of these tests for the Bitcoin price and return series analyzed in this study. For the ADF and PP tests, the null hypothesis (H0) assumes non-stationarity (I(1)), while the KPSS test assumes stationarity (I(0)). All tests found the Bitcoin price series to be non-stationary. However, the return series produced significantly negative ADF and PP test statistics, thereby rejecting the null hypothesis of non-stationarity. Conversely, the KPSS test confirmed stationarity, failing to reject its null hypothesis. These results indicate that the Bitcoin return series exhibits a stationary structure. The MODWT was used to break down Bitcoin returns, as shown in Figure 2. The scale was set to the seventh level to balance information richness and detail preservation. Scaling could theoretically go up to the 11th level (log2(2517) = 11.3), but keeping it at the 7th level ensures the best insights without filtering too much. The wavelet-based decomposition highlights the underlying structure and dynamics of Bitcoin returns at different scales.

Figure 2. Graphs of Returns and Different Time Scale Return Series for Bitcoin



Note: This figure presents the multiresolution analysis of BTC's daily series using the Maximum overlap discrete wavelet transform data from 01/01/2017-22/11/2023. The Y-axis shows the multiscale movement of returns. The series is decomposed using a Daubechies wavelet filter of length seven (7) at the scale level J = 1 to 7 (or 2-128 days). The periodic boundary condition is applied to solve the extreme wavelet coefficients.

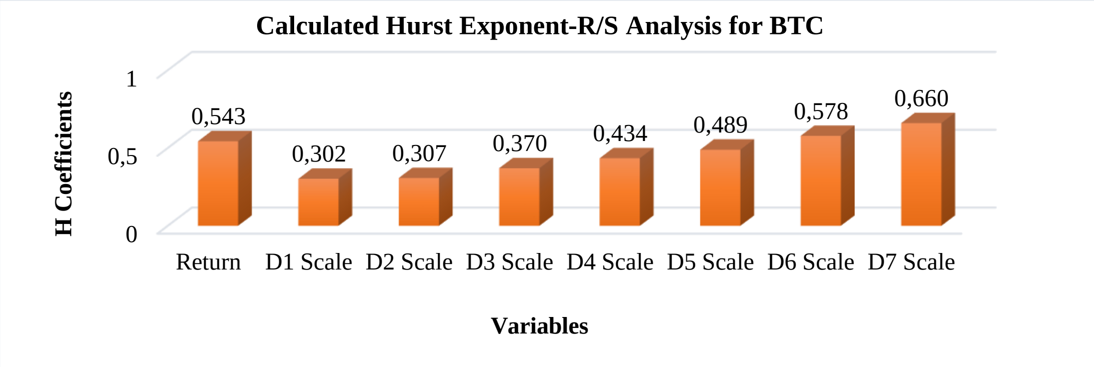
This study applied the Hurst exponent and R/S analysis, which Hurst (1951) developed to identify long-term memory properties in time series data. Detailed analyses of Bitcoin were conducted, calculating Hurst exponent (*H*) coefficients for return series across various investment horizons. This method was employed to understand better the complexity and dynamics of the Bitcoin cryptocurrency market. The results obtained provide both quantitative and qualitative insights. Table 5 presents the calculated Hurst exponent for Bitcoin in detail, while Figure 3 further clarifies it graphically.

Table 5. Hurst Exponent - R/S Analysis Results Calculated for Bitcoin

	Calculated Hurst Exponent (H) R/S Analysis	Confidence Intervals
Bitcoin Return	0.543	0.3979-0.5966[b]
D1 Scale	0.302***	0.3684-0.6262[a]
D2 Scale	0.307***	0.3684-0.6262[a]
D3 Scale	0.370	0.3979-0.5966[b]
D4 Scale	0.434	0.3979-0.5966[b]
D5 Scale	0.489	0.3979-0.5966[b]
D6 Scale	0.578	0.3979-0.5966[b]
D7 Scale	0.660***	0.3979-0.5966[a]

Note: The returns to scale obtained according to different investment horizons in the table are categorized as (D1 = 2-4 days, D2 = 4-8 days, D3 = 8-16 days, D4 = 16-32 days, D5 = 32-64 days, D6 = 64-128 days, D7, 128-256 days). The symbols ***, **, and * indicate statistical significance at the 1% (0.3684-0.6262)a, 5% (0.3979-0.5966)b, and 10% (0.4137-0.5811)c confidence levels, respectively. The Hurst exponent confidence interval values at the 1%, 5%, and 10% levels were obtained from Weron (2002).

Figure 3. Hurst Exponent Graph Calculated for Bitcoin



When examining the results presented in Table 5 and Figure 3, the Hurst exponent coefficients calculated for the returns of the Bitcoin cryptocurrency according to short-, medium-, and long-term investment horizons have been determined. It has been determined that the lowest Hurst exponent coefficient, 0.302, is found in the 2–4-day data of the D1 short-term investment horizon. The highest Hurst exponent value was observed at 0.660 in the D7 long-term investment horizon with 128–256-day data. Theoretically, when the Hurst exponent approaches the value of 1, the level of noise in the series decreases, and the expected values deviate from the Random Walk Hypothesis (Mulligan, 2004). In line with this information, when evaluating the findings for the Bitcoin cryptocurrency, it is observed that the calculated Hurst exponent for Bitcoin returns is 0.543. Therefore, this finding differs from the H = 0.50 value assumed by the efficient market hypothesis. According to the fractal market hypothesis, a Hurst exponent coefficient of less than 0.50 indicates that the series lacks continuity, deviates from the mean, possesses negative autocorrelation, and exhibits a mean-reversion process. In this case, it is stated that increases/decreases are likely to be followed by movements in the opposite direction. On the other hand, the Hurst exponent greater than 0.50 indicates that the series exhibits persistent and trend-reinforcing behaviour, meaning the presence of long memory and the validity of the fractal market hypothesis. The Hurst exponent calculated for Bitcoin returns being 0.543, which is greater than the value of H = 0.50, indicates that Bitcoin returns do not conform to the assumptions of the efficient market hypothesis and do not exhibit a random walk; instead, they show a discrete Brownian motion as specified by the fractal market hypothesis. Additionally, this finding supports that Bitcoin returns exhibit a continuous, trend-reinforcing persistent behaviour and the presence of long memory. When looking at the long-term investment horizons D6 and D7, it is observed that the Hurst exponents are 0.578 and 0.660, respectively. These values are more significant than 0.50, indicating that information shocks reaching the market are eliminated at a hyperbolic rate on long-term investment horizons, and future returns can be predicted based on past returns. This situation indicates that the efficient market hypothesis is invalid in long-term investment horizons and that the market creates repeating trends. Thus, the fractal market hypothesis is valid. The fact that the Hurst coefficients in long-term investment horizons take similar values across the entire sample provides evidence of self-similarity, that is, fractality. On the other hand, when the short-term investment horizons (D1, D2, and D3) were examined, the Hurst exponent values were calculated as 0.302, 0.307, and 0.370, respectively. Since these values are below 0.50, it has been concluded that there is no behaviour consistent with the assumptions of the efficient market hypothesis in Bitcoin returns and that the series does not exhibit a random walk. Similarly, for medium-term investment horizons (D4 and D5), the Hurst exponent values are 0.434 and 0.489, respectively. Since these values are also below 0.50, it has been determined that the assumptions of the efficient market hypothesis do not hold for Bitcoin returns and that the series does not exhibit a random walk. These findings, obtained in both short- and medium-term investment horizons, indicate that Bitcoin returns exhibit a chaotic structure and that investors may react to incoming market information more than they would in an efficient market, causing sharp and high volatility.

Short and long-memory models play a critical role in examining the effects of volatility and shocks that change over time in the cryptocurrency market. Short-memory models are successful in understanding the short-term effects of volatility dynamics. For example, the GARCH model is frequently preferred to model the current period's volatility based on the volatility of past periods. On the other hand, long-memory models are used to analyze long-term dependencies and trends in financial time series. The FIGARCH model is widely used to examine the long-memory characteristics of cryptocurrency price movements. In analyzing cryptocurrency markets, combining short and long-term models allows for a better understanding of the market's volatility structure, the effects of shocks, and how market behaviours change over time. It should be noted that cryptocurrency markets have different dynamics compared to traditional financial markets. Therefore, the development of risk management and investment strategies specific to cryptocurrency markets is of great importance. The volatility modelling results for Bitcoin according to different investment horizons are provided in Table 6. To determine the most suitable model for Bitcoin, both short-memory and long-memory presence and the validity of the EMH and the FMH were tested. ARCH, GARCH, EGARCH, T-GARCH, GJR-GARCH, C-GARCH, APARCH, GARCH-M, HYGARCH, FIGARCH, FIAPARCH, and FIEGARCH models (p, q = 0, 1, 2) were estimated in the p and q ranges. These models were evaluated based on the log-likelihood value and AIC, SIC, and HQ information criteria, and the most suitable volatility models were selected. The α and β parameters in Table 6 represent the coefficients of the ARCH and GARCH terms, respectively. Here, α represents the ARCH effect, which predicts the response to any shock or news in the cryptocurrency market. β, on the other hand, represents the GARCH effect that defines the persistence of volatility. A high α coefficient indicates that the volatility is more sensitive to incoming news; a high β coefficient, on the other hand, suggests that the volatility is more persistent and that this effect dissipates more slowly. In the GARCH model, the sum of α and β is expected to be less than 1. The Component GARCH (1,1) model, on the other hand, is a method used to model the volatility of financial time series and separates volatility into two components: persistent (long-term) and transitory (short-term) components. The parameters theta (θ) and rho (ρ) represent the components of the C-GARCH (Component GARCH) model, while *"d"* denotes the long memory parameter. The long memory parameter *"d"* has been interpreted according to the value ranges presented in the studies by Hosking (1981) and Tkacz (2001, p. 23). This parameter is used to measure the presence of long-term dependencies in the series. Table 6 presents the findings obtained from the analyses for Bitcoin.

Table 6. Volatility Modeling Results for Bitcoin According to Different Investment Horizons

	BTC- Return	D1	D2	D3	D4	D5	D6	D7
Optimal Volatility Model	HYGARCH (1,d,1)	AR(1) MA(1) C-GARCH (1,1)	C-GARCH (1,1)	AR(1) GARCH-M (1,1)	AR(1) MA(1) GARCH (1,1)	AR(1) MA(1) GARCH(1,1)	AR(1) FIGARCH (1,d,1)	AR (1) MA (1) FIGARCH (1,d,1)
Mean Equation								
Constant (c)	0.001162 (0.00045224)**	9.06e-09 (2.07e-07)	1.52e-05 (0.000220)	-0.020387 (0.002012)***	7.96e-05 (0.000651	-0.000888 (0.001043)	0.118386 (0.74723)	0.113983 (0.007925)***
AR	-	-0.493505 (0.0119438)***	-	0.826489 (0.010019)***	0.927651 (0.007298)***	0.986310 (0.001890)***	0.999069 (0.0060204)***	0.9996008 (0.9478e-00)***
MA	-	-.0998978 (0.000476)***	-	-	0.999275 (0.000443)***	0.998898 (0.000299)***	-	0.967708 (0.003214)***
Variance Equation								
Constant (ω)	-0.060869 (0.18434)	0.011282 (0.023788)	0.036116 (0.026355)***	5.94e-06 (5.01e-07)***	7.10e-08 (1.28e-08)***	2.81e-08 (2.37e-09)***	0.085982 (0.049328)*	0.347665 (0.139680)**
d-Figarch	0.614177 (0.13830)***	-	-	-	-	-	0.733670 (0.050443)***	0.832334 (0.025514)***
ARCH (α)	0.344742 (0.085068)***	0.113867 (0.023601)***	0.112865 (0.042401)***	0.648055 (0.037057)***	0.248612 (0.020794)***	0.593614 (0.036564)***	0.533356 (0.12439)***	0.102953 (0.017328)***
GARCH(β)	0.826760 (0.066608)***	0.552241 (0.071324)***	0.646333 (0.046807)***	0.379450 (0.018769)***	0.739983 (0.017795)***	0.309394 (0.025707)***	0.277347 (0.16687)*	0.040838 (0.021817)*
C-GARCH(ρ)	-	0.999847 (0.000329)***	0.999950 (5.10e-15)***	-	-	-	-	-
C-GARCH(θ)	-	0.225272 (0.034579)***	0.267109 (0.041138)***	-	-	-	-	-
GARCH-M	-	-	-	-1.172661 (0.025294)***	-	-	-	-
Log (α)	0.126340 (0.070794)*	-	-	-	-	-	-	-
AIC	-3.978168	-5.967514	-5.466327	-7.255935	-10.25342	-12.65830	-13.602058	-17.289134
SIC	-3.961952	-5.946658	-5.450111	-7.239714	-10.23720	-12.64208	-13.585842	-17.268284
HQ	-3.972283	-5.959945	-5.460442	-7.250048	-10.24753	-12.65241	-13.596172	-17.281567
Log-Likelihood	5013.525	7516.133	7886.373	9134.966	12905.80	15931.140	17125.189	21767.375

Note: The symbols ***, **, and * represent statistical significance at the 1%, 5%, and 10% levels, respectively, while () indicates the standard errors**.**

The analysis identifies the HYGARCH (1,*d*,1) model as the most suitable for Bitcoin returns, with the parameter *"d"* representing long memory, valued at 0.614177 and statistically significant at the 1% level. This indicates that Bitcoin returns exhibit a substantial long-memory property, where past trends influence current price movements. Long memory implies that volatility arises from the long-term effects of prior informational shocks, decaying at a hyperbolic rate. Consequently, forecasting Bitcoin returns using historical data is viable. The α coefficient of 0.344742 indicates that approximately 34% of the informational shocks are derived from past periods, while the β coefficient of 0.826760 suggests that shocks persist across consecutive periods. These findings reveal Bitcoin's volatile structure, where periods of high or low volatility tend to follow similar periods. The Log(α) value of 0.126340<1 implies that short-term volatility effects are limited and diminish over time, while long-memory effects dominate. The analysis supports the fractal market hypothesis over the efficient market hypothesis, highlighting Bitcoin's chaotic and non-stationary market dynamics. For short-term investment horizons, models such as AR(1), MA(1), and C-GARCH(1,1) are most suitable, particularly for 2-4 day and 4-8 day periods. The θ coefficients of 0.225272 and 0.267109 suggest a permanent but moderate volatility component, where short-term market disruptions amplify long-term volatility. This heightens sensitivity to short-term fluctuations, requiring market participants to account for cumulative risk impacts in their strategies. Additionally, the negative θ coefficient of the GARCH-M model for the 8-16 day horizon suggests that higher risk (volatility) negatively impacts average returns, contradicting the conventional risk-return tradeoff. This implies investors anticipate lower returns during heightened volatility, diverging from traditional market expectations. For medium-term investment horizons, AR(1), MA(1), and GARCH(1,1) models demonstrate strong predictive capacity for 16-32 day and 32-64 day periods. The α and β coefficients, statistically significant at the 1% level, indicate that informational shocks are carried over from previous periods, with cumulative sums of these coefficients close to 1, signifying persistent but eventually diminishing volatility effects. These results show that Bitcoin doesn't behave like itself in the short and medium term. This suggests that the market is more sensitive to new information and doesn't behave as the efficient market hypothesis says it should. For long-term investment horizons of 64-128 days and 128-256 days, the most appropriate models are AR(1) FIGARCH(1,*d*,1) and AR(1) MA(1) FIGARCH(1,*d*,1). The d coefficients, 0.733670 and 0.832334, respectively, confirm the strong long-memory properties of Bitcoin returns. This indicates that historical trends significantly influence current price movements, with the long-term effects of prior informational shocks fading gradually. The α coefficients of 0.533356 and 0.102953 show that 53% and 10% of informational shocks for these periods come from past volatility, respectively. The β coefficients of 0.277347 and 0.040838 show that shocks from the past are still affecting volatility today. Findings validate the fractal market hypothesis, indicating that Bitcoin's volatility exhibits short- and long-term memory components. For adequate portfolio diversification, risk-seeking investors should focus on short- and medium-term horizons (2-4 days to 32-64 days), where market volatility is more pronounced. On the other hand, we advise risk-averse or neutral investors to concentrate on long-term horizons (64-

256 days), where volatility effects diminish over time, thereby enabling more stable returns. Incorporating Bitcoin into portfolios, particularly within these identified horizons, can enhance diversification benefits and mitigate the risks associated with its volatile nature. In conclusion, understanding Bitcoin's volatility dynamics and employing appropriate models for different investment horizons provide valuable insights for portfolio managers, researchers, and investors. Strategies tailored to Bitcoin's unique market behaviour can improve decision-making, manage risks effectively, and optimize diversification in the cryptocurrency market.

5. CONCLUSION

The cryptocurrency market, originating with Bitcoin and proliferating with several cryptocurrencies, has emerged as a significant alternative asset class. It provides investors with chances for portfolio diversification and enhanced risk-return tradeoffs. The expanding market, characterized by rising value and volume, needs comprehensive comprehension from investors, scholars, and regulators to traverse its distinct dynamics adeptly. This study aims to model the volatility of cryptocurrencies across different investment horizons. We calculated the Hurst exponent, which quantifies the fractal characteristics of the series, using the filtered returns across these time scales. Lastly, we applied short- and long-memory models to capture the dynamics of time-varying volatility and the persistence of shocks affecting it.

The MODWT analysis reveals that overall returns and those across various investment horizons are volatile. Moreover, similar periods of low volatility typically follow low-volatility phases, while high-volatility phases tend to precede subsequent high-volatility periods. Additionally, the cryptocurrency exhibits fractal properties, including self-similarity and self-affinity. We also find that Bitcoin returns exhibit fractal dynamics across various investment horizons. In the short term, entering positive or negative information into the Bitcoin market significantly influences BTC returns and intensifies volatility. However, we observe that the impact of these information shocks is transient. In medium-term investment horizons, information shocks affecting the Bitcoin market have shown a certain level of persistence, with transitions between market regimes potentially lasting. The persistence of these shocks appears to grow within each regime. In long-term investment horizons, the impact of information shocks gradually weakens, with annual cycles becoming more apparent over time. These cycles reflect low-frequency trends and long-memory traits in Bitcoin returns. As a result, the MODWT analysis has shown that BTC returns across different investment horizons exhibit enduring fluctuations under varying market conditions and display fractal volatility characteristics.

The Hurst exponent coefficients reveal that both Bitcoin and long-term investment horizon returns have a Hurst exponent greater than 0.50. This finding suggests that BTC returns and returns in long-term investment horizons violate the efficient market hypothesis, as the return series exhibit persistent, trend-reinforcing, and continuous behaviour. This implies the existence of a long-term memory. Therefore, the fractal market hypothesis may be relevant to the Bitcoin market. According to observations, information shocks impacting the BTC cryptocurrency market tend to dissipate gradually over long-term investment horizons. Therefore, we can forecast future returns based on past returns. In addition, the fact that the Hurst exponent (the exponent) stays the same across the whole sample over long-term investment horizons shows that the data is self-similar and has fractal properties. Conversely, the Hurst exponents for returns in short- and medium-term investment horizons fall below 0.50. This result indicates a deviation from the assumption of the efficient market hypothesis, which posits H = 0.50. These findings suggest that short- and medium-term investment horizons may entirely reject the EMH, and the returns do not display a random walk behaviour. According to the fractal market hypothesis, a Hurst exponent (H) lower than 0.50 signifies that the series lacks persistence, tends to deviate from the mean, and exhibits negative autocorrelation. This implies that the series displays non-persistent behaviour and undergoes a mean reversion process. It also suggests that decreases will likely follow increases and vice versa. As a result, BTC returns in short- and medium-term investment horizons display a chaotic pattern. This suggests that investors react more strongly to new information than they would in an efficient market, leading to sharp fluctuations and heightened volatility.

Our research shows that the HYGARCH (1, d, 1) model best fits Bitcoin's return series. The statistical significance of the long-memory parameter d at the 1% level supports strong long-memory dynamics. The predicted log(α) value is less than 1, indicating that short-term volatility responses are limited, with market shocks having a substantial short-term impact that diminishes over time. The AR(1) FIGARCH(1, d, 1) and AR(1) MA(1) FIGARCH(1, d, 1) models are best for long-term investments because they have strong long-memory properties, as shown by the statistical significance of dd. These findings challenge the efficient market hypothesis and suggest the applicability of the fractal market hypothesis. For short- and medium-term investment horizons, different models are suitable. The AR(1) MA(1) C-GARCH(1, 1) model is most appropriate for D1, while the C-GARCH(1, 1) hypothesis fits D2. These models highlight the persistence of volatility in the short term, though its impact on total volatility is moderate. The positive θ coefficient suggests that an increase in short-term volatility contributes to long-term volatility. The AR(1) GARCH-M(1,1) model for the D3 horizon reveals a positive relationship between daily returns and a negative relationship between high volatility and average returns. This is different from the usual high-risk, high-return relationship. In medium-term horizons (D4 and D5), the AR(1) MA(1) GARCH(1,1) model is the most suitable. Here, 24% to 59% of volatility shocks stem from previous periods, and the sum of the α and β coefficients, while less than 1, suggests that shocks to conditional variance are temporary yet resilient. This indicates that volatility is predictable, with the impact of information shocks fading more gradually over medium- and long-term horizons.

During the periods covered by the study, the Bitcoin cryptocurrency market exhibited a chaotic structure, showing that the EMH is not valid, as the returns on short and medium-term investment horizons had a high degree of information asymmetry and formed a new average without reverting to the old average. On the other hand, the returns on long-term investment horizons show that the FPH may be valid because they create repeating trends and exhibit similar structures. This situation highlights the increasing importance of these cryptocurrencies as "a store of value" and "a long-term" asset, similar to gold and other precious metals. These research findings highlight the importance of different investment horizons and strategic approaches for portfolio managers, analysts, and investors when investing in cryptocurrency. For portfolio managers and investors who prefer to take risks, it may be advisable to focus on short-term investment horizons of 2-4 days, 4-8 days, and 8-16 days to evaluate the market's short-term fluctuations. This approach offers the potential for high returns from short-term market movements while also providing the opportunity to reassess and adjust their portfolios frequently. On the other hand, for risk-averse or risk-neutral portfolio managers, researchers, and investors, focusing on long-term investment horizons of 64-128 days and 128-256 days may be more appropriate. This strategy can minimize the risk of being affected by cryptocurrency markets' high volatility while offering more stable growth potential in their portfolios. Long-term investment strategies can contribute to portfolio balance by protecting capital, especially during market corrections and fluctuations. However, investors should consider different cryptocurrencies besides Bitcoin when diversifying their portfolios. This diversification can reduce the portfolio's overall risk profile by balancing each cryptocurrency's response to market conditions and news. Additionally, investors can adopt strategies suitable for their investment horizons to provide more effective protection against the volatility of cryptocurrency markets, thereby creating a portfolio better suited to market fluctuations. Diversification, a principle generally accepted in financial markets, applies to cryptocurrency markets. In this context, investors should diversify their portfolios with cryptocurrencies and traditional asset classes in a balanced manner. For example, traditional asset classes such as stocks, bonds, and commodities can balance the volatile nature of cryptocurrencies and serve as a stabilizer under general market conditions. Regarding risk management strategies, portfolio managers and investors may consider using tools such as stop-loss orders and risk-limiting techniques to counter sudden price movements in the cryptocurrency markets. Such measures help limit potential losses in portfolio value during unexpected market movements.

Additionally, continuously obtaining up-to-date information about cryptocurrency markets and closely monitoring market trends is critical for effective portfolio management and risk minimization. This can allow investors to make quick and informed decisions suitable for market conditions. These approaches allow for the development of a more suitable portfolio management strategy against the fluctuations in the cryptocurrency market. As a result, in the cryptocurrency markets, it is observed that information shocks occurring in the short and medium-term investment horizons quickly dissipate. In contrast, information shocks in the medium and long-term investment horizons are gradually eliminated. This situation indicates that past return data can be utilized to predict future returns. The analyzed findings reveal that the cryptocurrency market presents risks and profitable opportunities for investors. Investors can predict future price changes using technical analysis methods and thus minimize the risks in these markets. In this context, including cryptocurrencies in investors' portfolios is an approach suitable for long-term investment strategies and is thought to contribute to risk management and the diversification of portfolios. At the same time, considering the volatile nature of cryptocurrency markets, it may be beneficial for investors, portfolio managers, and analysts to focus on long-term investment horizons and the fundamental values of cryptocurrency projects rather than short-term market fluctuations to develop safer and more sustainable investment strategies.

The paper's findings have significant strategic and policy implications, especially for individual and institutional investors and regulatory authorities. Indeed, at long-term investment horizons, the long memory properties of investment returns and the hyperbolic decay of information shocks make it possible to shape investment decisions based on forecasts derived from historical data. In this context, individual and institutional investors are advised to be cautious about short-term market volatility and reduce risk through portfolio diversification. Moreover, regulatory authorities should develop transparency policies to reduce information asymmetry in cryptocurrency markets where market efficiency is weak, implement risk warning systems to protect investors, and establish differentiated capital adequacy criteria based on time horizon to ensure financial stability. Moreover, since traditional risk analysis methods may be insufficient in cryptocurrency markets where fractal structures and long memory effects are prominent, wavelet-based analyses and time-frequency decompositions should be used more effectively in macro-financial regulations. In this context, it is emphasized that the findings obtained for Bitcoin are not limited to this cryptocurrency but can be generalized to other cryptocurrency markets with similar structural characteristics; therefore, a new paradigm is needed in guiding investor behaviour and formulating regulatory policy sets.

Ethics Committee approval was not required for this study.

The authors declare that the study was conducted in accordance with research and publication ethics.

The authors confirm that Artificial Intelligence tools were employed solely to enhance the clarity and fluency of the language. Following the use of these tools, the authors thoroughly reviewed and edited the content as needed and assume full responsibility for the final version of the published article.

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The authors affirm that they contributed equally to all processes of the research.

REFERENCES

Açıkalın, S., & Sakınç, İ. (2022). Zayıf form etkinlik ve kripto para piyasası. *Maliye ve Finans Yazıları,* (117), 177–196. https://doi.org/10.33203/mfy.1084658

Al-Yahyaee, K. H., Mensi, W., & Yoon, S. M. (2018). Efficiency, multifractality, and the long-memory property of the Bitcoin market: A comparative analysis with stock, currency, and gold markets. *Finance Research Letters, 27,* 228–234. https://doi.org/10.1016/j.frl.2018.03.017

Assaf, A., Bhandari, A., Charif, H., & Demir, E. (2022). Multivariate long memory structure in the cryptocurrency market: The impact of COVID-19. *International Review of Financial Analysis, 82,* 102132. https://doi.org/10.1016/j.irfa.2022.102132

Baillie, R. T., Bollerslev, T., & Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics, 74*(1), 3–30. https://doi.org/10.1016/S0304-4076(95)01749-6

Bariviera, A. F. (2017). The inefficiency of Bitcoin revisited: A dynamic approach. *Economics Letters, 161,* 1–4. https://doi.org/10.1016/j.econlet.2017.09.013

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics, 31*(3), 307–327. https://doi.org/10.1016/0304-4076(86)90063-1

Brooks, C. (2008). *Introductory econometrics for finance*. Cambridge University Press.

Brooks, C. (2019). *Introductory econometrics for finance* (4th ed.). Cambridge University Press.

Cao, G., He, L. Y., & Cao, J. (2018). *Multifractal detrended analysis method and its application in financial markets.* Springer.

Caporale, G. M., Gil-Alana, L., & Plastun, A. (2018). Persistence in the cryptocurrency market. *Research in International Business and Finance, 46,* 141–148. https://doi.org/10.1016/j.ribaf.2018.01.002

Celeste, V., Corbet, S., & Gurdgiev, C. (2020). Fractal dynamics and wavelet analysis: Deep volatility and return properties of Bitcoin, Ethereum, and Ripple. *The Quarterly Review of Economics and Finance, 76,* 310–324. https://doi.org/10.1016/j.qref.2019.09.011

Çelik, İ. (2020). Can Bitcoin be a stable investment? *Financial Studies, 24*(2 (88)), 19–36. https://hdl.handle.net/10419/231697

CoinMarketCap. (2023, December 25). *Homepage*. CoinMarketCap. https://coinmarketcap.com/

Crowley, P. M. (2007). A guide to wavelets for economists. *Journal of Economic Surveys, 21*(2), 207-267.

Davidson, J. (2004). Moment and memory properties of linear conditional heteroscedasticity models, and a new model. *Journal of Business & Economic Statistics, 22*(1), 16–29. https://doi.org/10.1198/073500103288619359

Davidson, R., & MacKinnon, J. G. (2004). *Econometric theory and methods.* Oxford University Press.

Ding, Z., & Granger, C. W. (1996). Modeling volatility persistence of speculative returns: A new approach. *Journal of Econometrics, 73*(1), 185–215. https://doi.org/10.1016/0304-4076(95)01737-2

Dyhrberg, A. H. (2016). Bitcoin, gold and the dollar–A GARCH volatility analysis. *Finance Research Letters, 16,* 85–92. https://doi.org/10.1016/j.frl.2015.10.008

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of business & economic statistics, 20*(3), 339-350. https://doi.org/10.1198/073500102288618487

Eteman, V., & Işığıçok, E. (2022). Yüksek frekanslı kripto varlık oynaklığının uzun hafıza ve stokastik özelliklerinin FIGARCH modeli ile incelenmesi. *Dicle Üniversitesi İktisadi ve İdari Bilimler Fakültesi Dergisi, 12*(24), 284–310. https://doi.org/10.53092/duiibfd.1124966

Eyüboğlu, K., & Eyüboğlu, S. (2022). Bist ana sektör endekslerinde zayıf formda etkinliğin yapısal kırılmalı uzun hafiza modelleri ile analizi. *Abant Sosyal Bilimler Dergisi*, 22(2), 702-720. https://doi.org/10.11616/asbi.1097446

Fakhfekh, M., & Jeribi, A. (2020). Volatility dynamics of cryptocurrencies' returns: Evidence from asymmetric and long memory GARCH models. *Research in International Business and Finance, 51,* 101075. https://doi.org/10.1016/j.ribaf.2019.101075

Gençay, R., Selçuk, F., & Whitcher, B. (2002). *An introduction to wavelets and other filtering methods in finance and economics.* Academic Press.

Hosking, J. R. M. (1981). Fractional differencing. *Biometrika, 68(1)*, 165–176. https://doi.org/10.1093/biomet/68.1.165

Hurst, H. E. (1951). The long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers, 116*, 770–799. https://doi.org/10.1061/TACEAT.0006518.

In, F., & Kim, S. (2012). *An introduction to wavelet theory in finance: A wavelet multiscale approach.* World Scientific.

Kasman, A., & Torun, E. (2007). Long memory in the Turkish stock market return and volatility. *Central Bank Review, 7*(2), 13–27.

Lillo, F., & Farmer, J. D. (2004). The long memory of the efficient market. *Studies in Nonlinear Dynamics & Econometrics, 8*(3), 1–33. https://doi.org/10.2202/1558-3708.1226

Liu, G., Yu, C. P., Shiu, S. N., & Shih, I. T. (2022). The efficient market hypothesis and the fractal market hypothesis: Interfluves, fusions, and evolutions. *SAGE Open, 12*(1), 1–8. https://doi.org/10.1177/21582440221082137

Lo, A. W. (1991). Long-term memory in stock market prices. *Econometrica: Journal of the Econometric Society, 59*(5), 1279–1313. https://doi.org/10.2307/2938368

Mallat, S. (1989). *A wavelet tour of signal processing.* Academic Press.

Mandelbrot, B. (1972). Statistical methodology for nonperiodic cycles: From the covariance to R/S analysis. In *Annals of Economic and Social Measurement, Volume 1, Number 3* (pp. 259–290). NBER.

Mandelbrot, B. B. (1975). Limit theorems on the self-normalized range for weakly and strongly dependent processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 31,* 271–285. https://doi.org/10.1007/BF00532867

Mandelbrot, B. B., & Wallis, J. R. (1969). Computer experiments with fractional Gaussian noises: Averages and variances. *Water Resources Research, 5*(1), 228–241. https://doi.org/10.1029/WR005i001p00228

Mensi, W., Al-Yahyaee, K. H., & Kang, S. H. (2019). Structural breaks and double long memory of cryptocurrency prices: A comparative analysis from Bitcoin and Ethereum. *Finance Research Letters, 29,* 222–230. https://doi.org/10.1016/j.frl.2018.07.011

Mensi, W., Lee, Y. J., Al-Yahyaee, K. H., Sensoy, A., & Yoon, S. M. (2019). Intraday downward/upward multifractality and long memory in Bitcoin and Ethereum markets: An asymmetric multifractal detrended fluctuation analysis. *Finance Research Letters, 31,* 19–25. https://doi.org/10.1016/j.frl.2019.03.029

Mnif, E., & Jarboui, A. (2021). COVID-19, Bitcoin market efficiency, and herd behaviour. *Review of Behavioral Finance, 13*(1), 69–84. https://doi.org/10.1108/RBF-09-2020-0233

Moralı, T., & Uyar, U. (2018). Kıymetli metaller piyasasının fraktal analizi. *Hitit Üniversitesi Sosyal Bilimler Enstitüsü Dergisi, 11*(3), 2203–2218. https://doi.org/10.17218/hititsosbil.441151

Mulligan, R. F. (2004). Fractal analysis of highly volatile markets: An application to technology equities. *The Quarterly Review of Economics And Finance, 44*(1), 155-179, https://doi.org/10.1016/S1062-9769(03)00028-0

Nadarajah, S., & Chu, J. (2017). On the inefficiency of Bitcoin. *Economics Letters, 150*, 6–9. https://doi.org/10.1016/j.econlet.2016.10.033

Nakamoto, S. (2008). Bitcoin: A peer-to-peer electronic cash system. *Manubot.*

Özdemir Yazgan, S. D. (2022). Kripto para piyasasında uzun hafıza ve fraktal dinamikler: Hibrit model yaklaşımı. In T. Ünkaracalar (Ed.), *Finans, muhasebe ve iktisat alanlarında güncel araştırmalar-1* (pp. 1–19). Art Revisited.

Peters, E. E. (1994). *Fractal market analysis: Applying chaos theory to investment and economics.* Wiley.

Rachev, S. T., Mittnik, S., Fabozzi, F. J., Focardi, S. M., & Jasic, T. (2007). *Financial econometrics: From basics to advanced modeling techniques.* Wiley.

Sağlam Bezgin, M. (2023). Gelişmiş ve gelişmekte olan borsalar ile kripto varlık piyasasında Fraktal Piyasa Hipotezi'nin testi. *Doğuş Üniversitesi Dergisi*, 24(1), 81-91. https://doi.org/10.31671/doujournal.1101057

Tarı, R. (2014). *Ekonometri.* Umuttepe Yayınları. Kocaeli.

Tkacz, G. (2001). Estimating the fractional order of integration of interest rates using a wavelet OLS estimator. *Studies in Nonlinear Dynamics & Econometrics, 5(1),* 1–16. https://doi.org/10.2202/1558-3708.1068

Tsay, R. S. (2010). *Analysis of financial time series* (3rd ed.). John Wiley & Sons.

Urquhart, A. (2016). The inefficiency of Bitcoin. *Economics Letters, 148,* 80–82. https://doi.org/10.1016/j.econlet.2016.09.019

Uyar, U. (2019). Sistematik risk davranışında yatırım döngüsü: Wavelet analizi. *Hacettepe Üniversitesi İktisadi ve İdari Bilimler Fakültesi Dergisi, 37*(1), 135–168. https://doi.org/10.17065/huniibf.347775

Uyar, U., & Kangallı Uyar, S. (2021). Sermaye ve altın piyasaları arasındaki yayılım etkisi: Wavelet'e dayalı dinamik koşullu korelasyon yaklaşımı. In M. Ural & Ü. Aydın (Eds.), *Finansal ekonometri uygulamaları: Kavram-teori-uygulama* (pp. 309–335). Seçkin Yayıncılık. Zivot, E., & Wang, J. (2006). *Modeling financial time series with S-Plus* (2nd ed.). Springer.

Weron, R. (2002). Estimating long-range dependence: Finite sample properties and confidence intervals, *Physica A*, 312, 285-299.