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Economic Lot-Sizing of Three Warehouses EOQ Type Model: AM-GM Inequality-based Optimization Technique

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Abstract. This paper focuses on the optimal policy of a classical three warehouses inventory problem, which is modelled under some assumptions like the Harris-Wilson model. In this model, among three warehouses, one is own warehouse and remaining two are rented. Now, the main purpose is to minimize the system's average cost and find out economic order quantity and cycle length using arithmetic mean-geometric mean (AM-GM) inequality. Finally, all the findings illustrating the system's optimality conditions are validated by numerical examples.

Keywords: Three Warehouse · Optimality · Arithmetic Mean · Geometric Mean.

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1 Introduction

A two or multi-warehouse inventory model is a strategic approach in inventory/supply chain management to optimize the distribution and storage of goods across multiple locations. In a two-warehouse model, a primary, centralized warehouse with large capacity named as own warehouse works in tandem with a smaller local secondary warehouse named as rented warehouse to reduce transportation and other costs, improve service levels, and balance inventory. The multiwarehouse model involves several distributed warehouses to serve different regions, enhancing responsiveness and reducing risk by diversifying storage locations. Both models aim to meet regional demand efficiently, but they introduce complexity in coordination and cost management. These models are essential in industries like retail, manufacturing, and e-commerce, where efficient inventory management is key to maintaining a competitive edge. In the area of inventory/supply chain management, several researchers [1,2,3,4,5,6] accomplished their works on two/multi-warehouse-based inventory/supply chain problems under different realistic scenarios.

On the other hand, numerous researchers have employed the arithmetic mean-geometric mean (AM-GM) inequality to determine the optimal policy for their inventory and supply chain models. Grubbstrom [7] was the first to use the AM-GM inequality to derive the optimal policy for the classical Harris-Wilson EOQ model. Subsequently, Grubbstrom and Erdem [8] applied the AM-GM inequality to find the optimality conditions for an extended EOQ model that allowed shortages. Cardenas-Barron [9] also utilized this approach to determine the optimal policy for the classical economic production quantity (EPQ) model. To address the EOQ model under uncertainty, Gani et al. [10] treated alluncertain inventory parameters as fuzzy numbers and used the AM-GM inequality to establish the optimality conditions. More recently, Rahman et al. generalized the AM-GM inequality in an interval environment to study the optimal policy of the classical EOQ model under interval uncertainty. However, till date, no one has used the AM-GM inequality to derive the optimality conditions for three or multi-warehouse-based inventory models.

In this work, a three warehouses-based inventory model has been formulated under some fundamental assumptions like Harris-Wilson's EOQ model. Here, one warehouse is own and others two are rented. The total order quantity has been shared in the three warehouses in a given ratio. All the inventory costs related to the model are constant and customers demand rate is uniform. Then the corresponding average cost of the system is minimized using the AM-GM inequality and the formulae for optimal order quantity, cycle length and other components has been obtained. Also, the optimality formulae for several particular cases are also obtained. Finally, all the optimality conditions of the system are illustrated with the help of a set of numerical examples.

2 Notation and Assumptions

2.1 Notation

The following notations are used to formulate the model mathematically:

Table 1: Notations used in the model formulation

Notation	Description
Q	Order quantity
A	Ordering cost per order
h_o	Per unit holding cost in OW
h_1	Per unit holding cost in RW1
h_2	Per unit holding cost in RW2
T_1	Time at which stock of RW2 becomes zero
T_2	Time at which stock of RW1 becomes zero
T	Cycle length

2.2 Assumptions

The present work deals with multi-warehouses based EOQ type model which has been formulated under the following assumptions:

1. In the present model, three warehouses are considered for stocking of goods among which, one is own warehouse (OW) and others two are rented warehouses named as RW1 and RW2.
2. The capacity of OW is the λ_1 fraction of the total order quantity Q, i.e., $\lambda_1 Q$. The amount of items occupied in RW1 and RW2 are $\lambda_2 Q$ and $\lambda_3 Q$, respectively, where $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $(\lambda_1, \lambda_2, \lambda_3 \geq 0)$.
3. The demand of the customer (D) is uniform throughout the business cycle.
4. Inventory planning horizon is infinite.
5. Shortages are not permissible.
6. The per unit holding cost per unit time in RW2 (h_2) is higher than that of the RW1 (h_1), and the holding cost in RW1 is greater than that of in OW (h_o), due to different preservation facilities in RWs than OW.

3 Model formulation

The total order quantity Q of the given system is occupied in the three warehouses OW, RW1 and RW2. Initially, the stock level in RW2 declines due to the customers' demand rate D and reaches to zero at $t = T_1$. Then, the stock level in RW1 decreases to zero at $t = T_2$ and finally, the stock of OW declines to zero at the time $t = T$ due to the customers' demand rate D. The variation of the inventory of this system is depicted in the Figure 1.

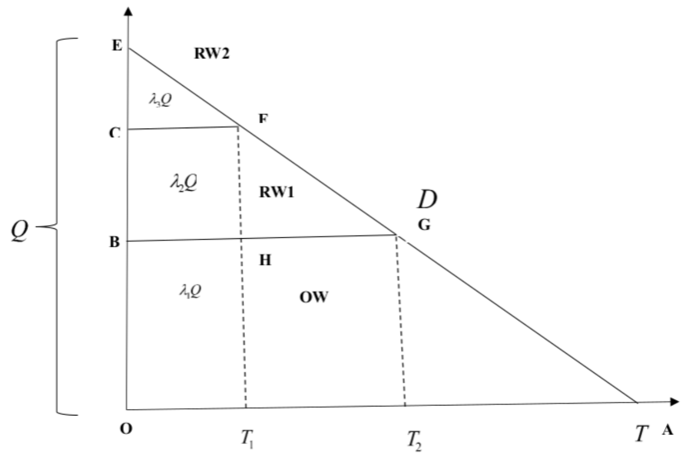

Fig. 1: Variation of inventory of the present system.

Since, total order quantity Q is consumed by the customer's demand (D) during the time $[0, T]$, one can observe the following relation:

$$Q = DT \tag{1}$$

Since, the order quantities in RW2 $\lambda_2 Q$ and in RW1 $\lambda_3 Q$ are consumed by the customer's demand (D) during the time T_1 and $(T_2 - T_1)$, one can see the relations

$$\lambda_2 Q = DT_1 \tag{2}$$

$$\lambda_2 Q = D(T_2 - T_1) \tag{3}$$

$$\lambda_1 Q = D(T - T_2) \tag{4}$$

$$(\lambda_3 + \lambda_2)Q = DT_2 \tag{5}$$

3.1 Cost components of the system

The cost components related to the present classical model are calculated as follows:

Ordering cost: A.

Total holding cost in RW2 is

$$\begin{aligned} HC_2 &= h_2 \times \text{area of } \triangle CEF \\ &= h_2 \times \frac{1}{2}\lambda_3 QT_1 \\ &= \frac{1}{2}h_2\lambda_3 QT_1. \end{aligned}$$

Total holding cost in RW1 is

$$\begin{aligned} HC_1 &= h_1 \times \text{area of } (\square BCFH + \triangle FHG) \\ &= h_1 \times \left(\lambda_2 QT_1 + \frac{1}{2}\lambda_2 Q(T_2 - T_1)\right). \end{aligned}$$

Total holding cost in OW is

$$\begin{aligned} HC_O &= h_0 \times \text{area of } (\square OBGT_2 + \triangle T_2GT) \\ &= h_0 \times \left(\lambda_1 QT_2 + \frac{1}{2}\lambda_1 Q(T - T_2)\right). \end{aligned}$$

Hence, total cost of the system is

$$\begin{aligned} TC(Q) &= A + HC_2 + HC_1 + HC_O \\ &= A + \frac{1}{2}h_2\lambda_3 QT_1 + h_1 \times \left(\lambda_2 QT_1 + \frac{1}{2}\lambda_2 Q(T_2 - T_1)\right) + h_o \times \left(\lambda_1 QT_2 + \frac{1}{2}\lambda_1 Q(T - T_2)\right) \\ &= A + \frac{h_2}{2D}\lambda_3^2 Q^2 + \frac{h_1}{2D}\left(2\lambda_3\lambda_2 + \lambda_2^2\right)Q^2 + \frac{h_o}{2D}\left(2(\lambda_3 + \lambda_2)\lambda_1 + \lambda_1^2\right)Q^2. \end{aligned}$$

Therefore, the average cost is

$$\begin{aligned} AC(Q) &= \frac{A}{T} + \frac{h_2}{2DT}\lambda_3^2 Q^2 + \frac{h_1}{2DT}\left(2\lambda_3\lambda_2 + \lambda_2^2\right)Q^2 + \frac{h_0}{2DT}\left(2(\lambda_3 + \lambda_2)\lambda_1 + \lambda_1^2\right)Q^2 \\ &= \frac{AD}{Q} + \frac{h_2}{2}\lambda_3^2 Q + \frac{h_1}{2}\left(2\lambda_3\lambda_2 + \lambda_2^2\right)Q + \frac{h_0}{2}\left(2(\lambda_3 + \lambda_2)\lambda_1 + \lambda_1^2\right)Q \\ &= \frac{AD}{Q} + \frac{1}{2}\left(2h_0(\lambda_3 + \lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)Q. \end{aligned}$$

Hence the corresponding minimization problem is

$$AC(Q) = \frac{AD}{Q} + \frac{1}{2}\left(2h_0(\lambda_3 + \lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)Q. \quad (6)$$

4 Optimality conditions

In this section, we have optimized the corresponding average cost (6) using the Arithmetic Mean-Geometric Mean (AM-GM) inequality-based optimization technique.

Applying AM-GM inequality one can obtain the following:

$$AC(Q) = \frac{AD}{Q} + \frac{1}{2}\Big(\Big(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_n\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\Big)Q\Big)$$

$$= \frac{\Big\{\frac{2AD}{Q} + \Big(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\Big)Q\Big\}}{2}$$

$$\geq \sqrt{\frac{2AD}{Q} \times \Big(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\Big)Q}$$

$$= \sqrt{2AD\Big(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\Big)}$$

And using the equality condition of AM-GM inequality one can obtain the following:

$$\frac{2AD}{Q} = \Big(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\Big)Q$$

$$\Rightarrow Q^2 = \frac{2AD}{\Big(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\Big)}$$

$$\Rightarrow Q = \sqrt{\frac{2AD}{\left(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_2\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)}}$$

Hence the optimal lot size and minimum average cost of the given system has been calculated by the following formulae:

$$Q^* = \sqrt{\frac{2AD}{\left(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)}} \tag{7}$$

$$AC(Q)_{min} = \sqrt{2AD(2h_0(\lambda_3+\lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2)} \tag{8}$$

Using the formula (7) in relation (1), one can obtain the formula for optimal cycle length as follows:

$$T^* = \sqrt{\frac{2A}{D\left(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)}} \tag{9}$$

Similarly, using the formula (7) in relations (2) and (5), one can calculate the optimal values of stock-in periods in RW2 and RWI as follows:

$$T_1^* = \lambda_3 \sqrt{\frac{2A}{D\left(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)}} \tag{10}$$

$$T_2^* = \left(\lambda_3+\lambda_2\right) \sqrt{\frac{2A}{D\left(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)}} \tag{11}$$

5 Particular cases of optimal policy

In this section, the optimality conditions of the system under some particular situations regarding the holding costs and availability of the warehouses have been discussed.

Case-1: If the per unit holding costs in all warehouses are the same, i.e., $h_0 = h_1 = h_2 = h$ (say), then from the optimality formulae (7)-(11), one can get the followings:

$$\begin{aligned} Q^* &= \frac{2AD}{\sqrt{\left(2h_0\left(\lambda_3+\lambda_2\right)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_o\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2\right)}} \\ &= \sqrt{\frac{2AD}{h\left(2\left(\lambda_3+\lambda_2\right)\lambda_1 + 2\lambda_3\lambda_2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)}} \\ &= \sqrt{\frac{2AD}{h\left(\lambda_1+\lambda_2+\lambda_3\right)^2}} \\ &= \sqrt{\frac{2AD}{h}} \quad \left(\because \lambda_1+\lambda_2+\lambda_3 = 1\right) \end{aligned}$$

And

$$\begin{aligned} AC(Q)_{\min} &= \sqrt{2AD(2h_0(\lambda_3+\lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2)} \\ &= \sqrt{2ADh(2(\lambda_3+\lambda_2)\lambda_1 + 2\lambda_3\lambda_2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2)} \\ &= \sqrt{2ADh(\lambda_1+\lambda_2+\lambda_3)^2} \\ &= \sqrt{2ADh} \quad (\because \lambda_1+\lambda_2+\lambda_3 = 1). \end{aligned}$$

Similarly, the formula for cycle length, stack-in period in RW2 and RW1 can be obtained. Therefore, the optimal values of all the unknown components for this case are presented in the Table 2.

Table 2: Optimal values of different known variables of the model for Case-1

Unknown Component	**Optimal Value**
Q^*	$\sqrt{\frac{2AD}{h}}$
T^*	$\sqrt{\frac{2A}{Dh}}$
T_1^*	$\lambda_3\sqrt{\frac{2A}{Dh}}$
T_2^*	$(\lambda_2+\lambda_3)\sqrt{\frac{2A}{Dh}}$
$AC(Q)_{\min}$	$\sqrt{2ADh}$

Note 1. From the dimension in Case-1, one can say that the optimal lot-sizing of this case is similar to that of the Harris-Wilson's model.

Case-2: If the present system consists only one rented warehouse, i.e., if $\lambda_3 = 0$, then from the optimality formulae (7)-(11) followings can be obtained:

$$\begin{aligned} Q^* &= \sqrt{\frac{2AD}{(2h_0(\lambda_3+\lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2)}} \\ &= \sqrt{\frac{2AD}{(2\lambda_1\lambda_2 h_0 + h_0\lambda_1^2 + h_1\lambda_2^2)}} \end{aligned}$$

And

$$\begin{aligned} AC(Q)_{\min} &= \sqrt{2AD\,(2h_0(\lambda_3+\lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2)} \\ &= \sqrt{2AD\,(2\lambda_1\lambda_2 h_0 + h_0\lambda_1^2 + h_1\lambda_2^2)} \end{aligned}$$

Similarly, the formulae for cycle length, stock-in periods in RW1 can be obtained. Therefore, the optimal values of all the unknown components for this case are presented in the Table3.

Note 2. From the discussions in Case-2, one can see that the optimal policy for this case is actually for the two-warehouse inventory system.

Case-3: If the present system consists one own warehouse and no rented warehouse, i.e., if $\lambda_2 = \lambda_3 = 0$, then from the optimality formulae (7)-(11) followings can be obtained:

Table 3: Optimal values of different known variables of the model for Case-2

Unknown Component	**Optimal Value**
Q^*	$\frac{2AD}{\sqrt{2\lambda_1\lambda_2 h_0 + h_0\lambda_1^2 + h_1\lambda_2^2}}$
T^*	$\frac{2A}{\sqrt{D(2\lambda_1\lambda_2 h_0 + h_0\lambda_1^2 + h_1\lambda_2^2)}}$
T_1^*	0
T_2^*	$\frac{2A}{\sqrt{D(2\lambda_1\lambda_2 h_0 + h_0\lambda_1^2 + h_1\lambda_2^2)}}$
$AC(Q)_{\min}$	$\sqrt{2AD(2\lambda_1\lambda_2 h_0 + h_0\lambda_1^2 + h_1\lambda_2^2)}$

$$\begin{aligned} Q^* &= \sqrt{\frac{2AD}{(2h_0(\lambda_3+\lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2)}} \\ &= \sqrt{\frac{2AD}{h_0\lambda_1^2}} \\ &= \sqrt{\frac{2AD}{h_0}} \quad (\because \lambda_1 = 1) \end{aligned}$$

And

$$\begin{aligned} AC(Q)_{\min} &= \sqrt{2AD\,(2h_0(\lambda_3+\lambda_2)\lambda_1 + 2h_1\lambda_3\lambda_2 + h_0\lambda_1^2 + h_1\lambda_2^2 + h_2\lambda_3^2)} \\ &= \sqrt{2ADh_0\lambda_1^2} \\ &= \sqrt{2ADh_0} \quad (\because \lambda_1 = 1). \end{aligned}$$

Similarly, the formula for cycle length can be obtained as,

$$T^* = \sqrt{\frac{2A}{Dh_0}}.$$

Note 3. From the discussions in Case-3, one can see that the optimal lot-sizing of this case is nothing different from that of the Harris-Wilson's model.

6 Numerical illustrations

In this section, we have illustrated all the optimality conditions with the help of following examples:

Example-1. Suppose, a retailer of garments has rented two warehouses viz. RW-1, RW-2 apart form his own warehouse viz. OW for his/her business purpose. The demand of the of the customers in his/her shop is 1,000 clothes per

year. It is observed from the previous cycle that about 40% of the total ordered quantity can be stored in OW, and remaining 60% have to store in the RW-1 and RW-2 in the ratio 2:3.
The costs for holdings in OW, RW-1, and RW-2 are \$5, \$6, and \$7 per unit per year, and the fixed cost to place an order is \$90.Find the optimal policy of the retailer.

Solution. The values of system parameters for Example-1 are as follows:

$$A = \$90, \quad h_0 = \$5, \quad h_1 = \$6, \quad h_2 = \$7, \quad D = 1000, \quad \lambda_0 = 0.4, \quad \lambda_1 = 0.24, \quad \lambda_0 = 0.36$$

The optimal values of order quantity, cycle length, average cost and others components are calculated using the formulae (7)-(11) and obtained results are presented in Table 4.

Table 4: Optimal values of different known variables of the model for Example-1

Unknown Component	Optimal Value
Q^*	181.08
T^*	0.18
T_1^*	0.065
T_2^*	0.11
$AC(Q)_{\min}$	994.05

7 Conclusion

As a summary, in this study, the optimality conditions of an inventory model based on a three-warehouse system, incorporating key assumptions from Harris-Wilson's EOQ model has been derived using the AM-GM inequality. The optimality conditions have been properly illustrated by the numerical example. From the findings of the case study, one can observe that this study is a more generalised study of inventory control management than the single warehouse or two warehouse cases. From the optimal policy of this study, we can conclude that it is optimality of Harris-Wilson's EOQ model for three warehouses. For future research, one can generalise the model features for n- warehouse or one can extend the same model in uncertain environment like, fuzzy, interval/Type-2 interval, etc. environments.

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