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Conformable Fractional Milne-Type Inequalities Through Twice-Differentiable Convex Functions

İzzettin Demir [1,†,*] **and Esra Üneş** [1,‡]

[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye*
[†] *izzettindemir@duzce.edu.tr*, [‡] *esrauness.25@gmail.com*
[*] *Corresponding Author*

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Abstract

In this article, we present a new integral identity based on conformable fractional integral operators with the help of twice-differentiable functions. Then, using this newly derived identity, we propose several Milne-type inequalities for twice-differentiable convex functions by means of conformable fractional integral operators and offer an example with an associated graph. Also, we note that the obtained results improve and expand some of the previous discoveries in the field of integral inequalities. Moreover, along with expanding on previous results, our results suggest effective approaches and methods for dealing with a variety of mathematical and scientific issues.

1. Introduction

In mathematics, the concept of convexity emerges as a fundamental idea, supported by extensive research and numerous practical applications, with a significant impact in various disciplines. Besides, convexity, by providing an essential framework for analyzing the geometric properties of sets and functions, forms the basis of various theories such as optimization theory, measure theory, approximation theory, and information theory, as well as their applications in science and engineering [1, 2, 3]. The formal definition of a convex function is given by the following:

Definition 1.1. *A function $\Lambda : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and $\lambda \in (0,1)$, we have*

$$\Lambda(\lambda x + (1-\lambda)y) \leq \lambda\Lambda(x) + (1-\lambda)\Lambda(y), \tag{1.1}$$

where I is an interval of the real numbers. In case the inequality (1.1) is reversed, Λ is known as concave.

Integral inequalities, used to determine the error bounds of numerical integration formulas, are an indispensable tool. For this reason, their applications have increased and impacted many contemporary areas of mathematics. The Hermite-Hadamard inequality [4], when expressed as below, is a fundamental inequality related to the concept of convexity:

$$\Lambda\left(\frac{\theta+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\theta}\int_{\theta}^{\upsilon}\Lambda(x)dx \leq \frac{\Lambda(\theta)+\Lambda(\upsilon)}{2} \tag{1.2}$$

where $\Lambda : I \to \mathbb{R}$ is a convex function on I and $\theta, \upsilon \in I$ with $\theta < \upsilon$. When Λ is concave, both inequalities in the statement hold in the opposite direction. For a deeper exploration of the historical context of inequality (1.2), we suggest [5, 6, 7], and the sources they reference.

The Milne inequality, among integral inequalities, is the most prominent and widely cited inequality and it is formulated as follows:

$$\left|\frac{1}{3}\left[2\Lambda(\theta) - \Lambda\left(\frac{\theta+\upsilon}{2}\right) + 2\Lambda(\upsilon)\right] - \frac{1}{\upsilon-\theta}\int_{\theta}^{\upsilon}\Lambda(x)dx\right| \leq \frac{7(\upsilon-\theta)^4}{23040}\left\|\Lambda^{(4)}\right\|_{\infty},$$

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where $\Lambda : [\theta,\upsilon] \to \mathbb{R}$ is a four times differentiable mapping on (θ,υ) and $\left\|\Lambda^{(4)}\right\|_\infty = \sup_{x\in(\theta,\upsilon)} \left|\Lambda^{(4)}(x)\right| < \infty$.

This inequality, which determines the error bound of the integral value using the Milne rule, is extremely important. Therefore, it ensures the accuracy and reliability of numerical integration in various applications. For this reason, there has been a notable increase in research focusing on Milne inequality. In 2013, Alomari and Liu [8] conducted a study to predict the bounds of Milne's quadrature rule using reduced derivatives and convex functions. In [9], Román-Flores et al. derived several Milne-type inequalities for interval-valued functions and explained their connections with other classical inequalities. In 2022, Djenaoui and Meftah [10] developed some new methods for Milne's quadrature rule applying the functions whose first derivative is π-convex. Focusing on strong multiplicative convex functions, Umar et al.[11] proved various Milne-type and Hermite-Hadamard-type integral inequalities.

On the other hand, fractional calculus, which extends the concepts of derivatives and integrals to non-integer orders, has increasingly become an effective tool in scientific fields such as physics, engineering, and chemistry [12, 13]. Since the beginning of fractional calculus, various fractional derivative and integral operators have been developed. Some notable examples include the Riemann-Liouville, conformable, Caputo, and Hadamard fractional integral operators, each of which plays a critical role in solving problems in applied mathematics and analysis.

Kilbas et al. [14] introduced the Riemann-Liouville fractional integral operators using the following approach:

Definition 1.2 ([14])**.** *The Riemann-Liouville integrals $I^{\varepsilon}_{\theta+}\Lambda(x)$ and $I^{\varepsilon}_{\upsilon-}\Lambda(x)$ of order $\varepsilon > 0$ are given by*

$$I^{\varepsilon}_{\theta+}\Lambda(x) = \frac{1}{\Gamma(\varepsilon)} \int_{\theta}^{x} (x-\mu)^{\varepsilon-1}\Lambda(\mu)d\mu, \quad x > \theta, \tag{1.3}$$

and

$$I^{\varepsilon}_{\upsilon-}\Lambda(x) = \frac{1}{\Gamma(\varepsilon)} \int_{x}^{\upsilon} (\mu-x)^{\varepsilon-1}\Lambda(\mu)d\mu, \quad x < \upsilon, \tag{1.4}$$

respectively, where $\Lambda \in L_1[\theta,\upsilon]$. Here, Γ is the Gamma function defined by

$$\Gamma(\varepsilon) := \int_0^{\infty} \mu^{\varepsilon-1}e^{-\mu}d\mu.$$

Riemann-Liouville integrals are equal to the classical integrals for the case of $\varepsilon = 1$.

Through these integral operators, several scientists conducted the studies to develop various integral inequalities. With the use of these operators, the studies focusing on the Milne inequality has gradually gained more importance through the years. Significant contributions can be found in [15, 16, 17, 18, 19] and further references therein.

New operators have been proposed to better define certain situations that classical fractional integral operators struggle to model effectively [20, 21]. Especially, conformable fractional integral operators, specified by Jarad et al. [22] as presented follows, not only come closer to the classical integral and differentiation principles but also generalize a range of fractional integral operators like Riemann-Liouville and Hadamard.

Definition 1.3. *The fractional conformable integral operator ${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta+}\Lambda(x)$ and ${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon-}\Lambda(x)$ of order $\varepsilon \in \mathbb{R}^+$ and $\sigma \in (0,1]$ are presented by*

$${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta+}\Lambda(x) = \frac{1}{\Gamma(\varepsilon)} \int_{\theta}^{x} \left(\frac{(x-\theta)^{\sigma}-(\mu-\theta)^{\sigma}}{\sigma}\right)^{\varepsilon-1} \frac{\Lambda(\mu)}{(\mu-\theta)^{1-\sigma}}d\mu, \quad \mu > \theta, \tag{1.5}$$

and

$${}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon-}\Lambda(x) = \frac{1}{\Gamma(\varepsilon)} \int_{x}^{\upsilon} \left(\frac{(\upsilon-x)^{\sigma}-(\upsilon-\mu)^{\sigma}}{\sigma}\right)^{\varepsilon-1} \frac{\Lambda(\mu)}{(\upsilon-\mu)^{1-\sigma}}d\mu, \quad \mu < \upsilon, \tag{1.6}$$

respectively, where $\Lambda \in L_1[\theta,\upsilon]$.

Take notice that the fractional integral in (1.5) reduces to the Riemann-Liouville fractional integral in (1.3) if $\sigma = 1$. Additionally, the fractional integral in (1.6) simplifies to the Riemann-Liouville fractional integral in (1.4) if $\sigma = 1$.

Following the discovery of these innovative operators, remarkable research has been carried out to formulate inequalities based on such integral operators. For example, the aim of Set et al. [23] was to prove an identity for convex functions using fractional conformable integral operators and two types of Hermite-Hadamard inequalities. By utilizing conformable fractional integrals, in 2023, Hezenci et al. [24] developed new inequalities for the left and right sides of the Hermite-Hadamard inequality for twice-differentiable mappings. In 2024, Ying et al. [25], who investigated conformable fractional Milne-type inequalities, presented a comprehensive example with graphical representations that provide numerical support and visual confirmation of the established inequalities. Hezenci and Budak [26], with the aid of conformable fractional integrals, proved various trapezoid-type inequalities with π-convex functions. Moreover, they [27] developed many Bullen-type inequalities for twice-differentiable functions. In [28], Çelik et al. created new Milne type inequalities with the help of these operators for bounded functions, Lipschitzian functions and functions of bounded variation. For more information on the inequalities derived through this fractional integral operators with various functions, readers are referred to [29, 30, 31] and the references mentioned there.

In the sequel, the following definition will be utilized.

Definition 1.4. *Let $\sigma,\varepsilon>0$. Then, the beta function is defined by*

$$\mathcal{B}(\sigma,\varepsilon):=\int_0^1 \mu^{\sigma-1}(1-\mu)^{\varepsilon-1}\,d\mu.$$

Also, let $0\le x\le 1$. The incomplete beta function, a generalization of the beta function, is defined as

$$\mathcal{B}_x(\sigma,\varepsilon):=\int_0^x \mu^{\sigma-1}(1-\mu)^{\varepsilon-1}\,d\mu.$$

Meanwhile, the development of integral inequalities, has often been dependent on classical techniques like Hölder inequality and its alternative form, the power mean inequality.

Theorem 1.5 (Hölder inequality)**.** *Let $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\Lambda,g:[\theta,\upsilon]\to\mathbb{R}$. If $|\Lambda|^p$ and $|g|^q$ are integrable functions on $[\theta,\upsilon]$, then*

$$\int_\theta^\upsilon |\Lambda(\mu)g(\mu)|d\mu\le\left(\int_\theta^\upsilon |\Lambda(\mu)|^p d\mu\right)^{\frac{1}{p}}\left(\int_\theta^\upsilon |g(\mu)|^q d\mu\right)^{\frac{1}{q}}.$$

Theorem 1.6 (Power mean inequality)**.** *Let $q\ge 1$ and $\Lambda,g:[\theta,\upsilon]\to\mathbb{R}$. If $|\Lambda|$ and $|\Lambda||g|^q$ are integrable functions on $[\theta,\upsilon]$, then*

$$\int_\theta^\upsilon |\Lambda(\mu)g(\mu)|d\mu\le\left(\int_\theta^\upsilon |\Lambda(\mu)|d\mu\right)^{1-\frac{1}{q}}\left(\int_\theta^\upsilon |\Lambda(\mu)||g(\mu)|^q d\mu\right)^{\frac{1}{q}}.$$

In line with ongoing research and the articles mentioned above, this article aims to present similar versions of Milne-type inequalities in the context of Riemann integrals through the use of conformable fractional integral operators. To achieve this goal, we will first present an identity for twice-differentiable functions using conformable fractional integral operators. Then, we derive some important Milne-type inequalities by utilizing convexity, the Hölder inequality, and the power mean inequality. Given the proper assumptions on σ and ε, these results advance and generalize the inequalities obtained in prior studies.

2. Main results

This section focuses on deriving Milne-type inequalities for twice-differentiable convex functions within the framework of conformable fractional integrals. To achieve this, we begin by establishing the following identity, which serves as a foundation for obtaining conformable fractional forms of Milne-type inequalities.

Lemma 2.1. *If $\Lambda:[\theta,\upsilon]\to\mathbb{R}$ is a twice-differentiable function on (θ,υ) and $\Lambda''\in L_1[\theta,\upsilon]$, then the following equality holds:*

$$\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^\varepsilon\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]=\frac{(\upsilon-\theta)^2\sigma^\varepsilon}{8}\left[I_1+I_2\right], \tag{2.1}$$

where

$$\begin{cases} I_1=\int\limits_0^1\left(\int\limits_\mu^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)d\mu,\\ I_2=\int\limits_0^1\left(\int\limits_\mu^1\left[\left(\frac{1-(1-\pi)^\sigma}{\sigma}\right)^\varepsilon+\frac{1}{3\sigma^\varepsilon}\right]d\pi\right)\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)d\mu.\end{cases}$$

Proof. From the application of integration by parts, it follows that

$$\begin{aligned}
I_1 &= \int_0^1 \left(\int_\mu^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda'' \left(\frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon \right) d\mu \\
&= \frac{2}{\upsilon-\theta} \left(\int_\mu^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda' \left(\frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon \right) \Bigg|_0^1 \\
&\quad + \frac{2}{\upsilon-\theta} \int_0^1 \left[\left(\frac{1-(1-\mu)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] \Lambda' \left(\frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon \right) d\mu \\
&= -\frac{2}{\upsilon-\theta} \left(\int_0^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda' \left(\frac{\theta+\upsilon}{2} \right) \\
&\quad + \frac{2}{\upsilon-\theta} \left\{ \frac{2}{\upsilon-\theta} \left[\left(\frac{1-(1-\mu)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] \Lambda \left(\frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon \right) \Bigg|_0^1 \right. \\
&\quad \left. - \frac{2\varepsilon}{\upsilon-\theta} \int_0^1 \left(\frac{1-(1-\mu)^\sigma}{\sigma} \right)^{\varepsilon-1} (1-\mu)^{\sigma-1} \Lambda \left(\frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon \right) d\mu \right\}.
\end{aligned}$$

If we utilize the change of variables $x = \frac{1-\mu}{2}\theta + \frac{1+\mu}{2}\upsilon$, we obtain

$$\begin{aligned}
I_1 &= -\frac{2}{\upsilon-\theta} \left(\int_0^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda' \left(\frac{\theta+\upsilon}{2} \right) \qquad (2.2) \\
&\quad + \frac{4}{(\upsilon-\theta)^2 \sigma^\varepsilon} \left[\frac{4}{3}\Lambda(\upsilon) - \frac{1}{3}\Lambda\left(\frac{\theta+\upsilon}{2} \right) \right] - \left(\frac{2}{\upsilon-\theta} \right)^{\sigma\varepsilon+2} \varepsilon \int_{\frac{\theta+\upsilon}{2}}^{\upsilon} \left(\frac{\left(\frac{\upsilon-\theta}{2}\right)^\sigma - (\upsilon-x)^\sigma}{\sigma} \right)^{\varepsilon-1} \frac{\Lambda(x)}{(\upsilon-x)^{1-\sigma}} dx \\
&= -\frac{2}{\upsilon-\theta} \left(\int_0^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda' \left(\frac{\theta+\upsilon}{2} \right) + \frac{4}{(\upsilon-\theta)^2 \sigma^\varepsilon} \left[\frac{4}{3}\Lambda(\upsilon) - \frac{1}{3}\Lambda\left(\frac{\theta+\upsilon}{2} \right) \right] \\
&\quad - \left(\frac{2}{\upsilon-\theta} \right)^{\sigma\varepsilon+2} \Gamma(\varepsilon+1)\, {}^{\varepsilon}\mathfrak{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2} \right).
\end{aligned}$$

By following a similar approach, the following result is achieved:

$$\begin{aligned}
I_2 &= \int_0^1 \left(\int_\mu^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda'' \left(\frac{1+\mu}{2}\theta + \frac{1-\mu}{2}\upsilon \right) d\mu \qquad (2.3) \\
&= \frac{2}{\upsilon-\theta} \left(\int_0^1 \left[\left(\frac{1-(1-\pi)^\sigma}{\sigma} \right)^\varepsilon + \frac{1}{3\sigma^\varepsilon} \right] d\pi \right) \Lambda' \left(\frac{\theta+\upsilon}{2} \right) + \frac{4}{(\upsilon-\theta)^2 \sigma^\varepsilon} \left[\frac{4}{3}\Lambda(\theta) - \frac{1}{3}\Lambda\left(\frac{\theta+\upsilon}{2} \right) \right] \\
&\quad - \left(\frac{2}{\upsilon-\theta} \right)^{\sigma\varepsilon+2} \Gamma(\varepsilon+1)\, {}^{\varepsilon}\mathfrak{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2} \right).
\end{aligned}$$

As a result, by merging the findings in (2.2) and (2.3) and multiplying it with $\frac{(\upsilon-\theta)^2\sigma^\varepsilon}{8}$, the equality given in (2.1) is established. □

Remark 2.2. *Let us choose $\sigma = 1$ in Lemma 2.1. From this, we get the identity*

$$\begin{aligned}
&\frac{1}{3}\left[2\Lambda(\theta) - \Lambda\left(\frac{\theta+\upsilon}{2} \right) + 2\Lambda(\upsilon) \right] - \frac{2^{\varepsilon-1}}{(\upsilon-\theta)^\varepsilon}\Gamma(\varepsilon+1)\left[I^{\varepsilon}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2} \right) + I^{\varepsilon}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2} \right) \right] \\
&= \frac{(\upsilon-\theta)^2}{24(\varepsilon+1)} \left[\int_0^1 \left(\varepsilon+4-\mu(\varepsilon+1)-3\mu^{\varepsilon+1} \right) \left[\Lambda''\left(\frac{1-\mu}{\mu}\theta + \frac{1+\mu}{2}\upsilon \right) + \Lambda''\left(\frac{1+\mu}{2}\theta + \frac{1-\mu}{2}\upsilon \right) \right] d\mu \right],
\end{aligned}$$

which was presented by Budak et al. in [32].

Corollary 2.3. *By taking $\sigma=\varepsilon=1$ in Lemma 2.1, we derive*

$$\begin{aligned}&\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_{\theta}^{\upsilon}\Lambda(\mu)d\mu\\ =&\frac{(\upsilon-\theta)^2}{48}\left[\int_0^1(3\mu+5)(1-\mu)\left[\Lambda''\left(\frac{1-\mu}{\mu}\theta+\frac{1+\mu}{2}\upsilon\right)+\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right]d\mu\right].\end{aligned}$$

Theorem 2.4. *Let $\Lambda:[\theta,\upsilon]\to\mathbb{R}$ be a twice-differentiable function on (θ,υ) such that $\Lambda''\in L_1[\theta,\upsilon]$. If $|\Lambda''|$ is a convex function on $[\theta,\upsilon]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right| \\ \le&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\psi_1(\sigma,\varepsilon)\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right],\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}\psi_1(\sigma,\varepsilon)&=\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\\&=\frac{1}{\sigma^{\varepsilon}}\int_0^1\left(\frac{1}{\sigma}\left(\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)\right)+\frac{1-\mu}{3}\right)d\mu.\end{aligned}$$

Proof. If we take the absolute value of the identity (2.1), we get

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|\\ \le&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\right|d\mu\right.\\ &\left.+\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right|d\mu\right].\end{aligned}$$

Considering that the function $|\Lambda''|$ is convex, we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|\\ \le&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left(\frac{1-\mu}{2}\left|\Lambda''(\theta)\right|+\frac{1+\mu}{2}\left|\Lambda''(\upsilon)\right|\right)d\mu\right.\\ &\left.+\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left(\frac{1+\mu}{2}\left|\Lambda''(\theta)\right|+\frac{1-\mu}{2}\left|\Lambda''(\upsilon)\right|\right)d\mu\right]\\ =&\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\int_0^1\left|\int_{\mu}^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right].\end{aligned}$$

Thus, we reach at the result (2.4). □

Remark 2.5. *Consider $\sigma = 1$ in Theorem 2.4. In this case, the inequality (2.4) is the Milne-type inequality for twice-differentiable convex functions, involving the Riemann-Liouville fractional integral operators:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^{\varepsilon}}\Gamma(\varepsilon+1)\left[I_{\theta^+}^{\varepsilon}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I_{\upsilon^-}^{\varepsilon}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\psi_1(1,\varepsilon)\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right],$$

satisfying

$$\psi_1(1,\varepsilon)=\int_0^1\left|\int_\mu^1\left(\pi^{\varepsilon}+\frac{1}{3}\right)d\pi\right|d\mu=\int_0^1\left(\frac{1}{\varepsilon+1}+\frac{1}{3}-\frac{\mu^{\varepsilon+1}}{\varepsilon+1}-\frac{\mu}{3}\right)d\mu=\frac{\varepsilon+8}{6(\varepsilon+2)},$$

which was given by Budak et al. in [32].

Remark 2.6. *By taking $\sigma = 1$ and $\varepsilon = 1$, we get,*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_\theta^\upsilon\Lambda(\mu)d\mu\right|\leq\frac{(\upsilon-\theta)^2}{16}\left[\left|\Lambda''(\theta)\right|+\left|\Lambda''(\upsilon)\right|\right],$$

which was provided by Budak et al. in [32].

Example 2.7. *Considering the function $\Lambda(x)=x^4$ on the interval $[0,1]$, we proceed to calculate the right-hand side of inequality (2.4) as follows:*

$$\frac{3}{2}\int_0^1\left[\frac{1}{\sigma}\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)+\frac{1-\mu}{3}\right]d\mu:=\Psi_1.$$

In addition, it is apparent that

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}_{\theta^+}^{\sigma}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}_{\upsilon^-}^{\sigma}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$=\left|\frac{31}{48}-\frac{\varepsilon}{2}\left(\frac{1}{\varepsilon}-2\mathcal{B}\left(\frac{1}{\sigma}+1,\varepsilon\right)+\frac{3}{2}\mathcal{B}\left(\frac{2}{\sigma}+1,\varepsilon\right)-\frac{1}{2}\mathcal{B}\left(\frac{3}{\sigma}+1,\varepsilon\right)+\frac{1}{8}\mathcal{B}\left(\frac{4}{\sigma}+1,\varepsilon\right)\right)\right|:=\Psi_2.$$

Thus, as Figure 1 illustrates, the left side of the inequality (2.4) is always situated beneath the right side of this inequality for all $0<\sigma<1$ and $0<\varepsilon<10$.

Theorem 2.8. *Let $\Lambda:[\theta,\upsilon]\to\mathbb{R}$ be a twice-differentiable function on (θ,υ) such that $\Lambda''\in L_1[\theta,\upsilon]$. If $|\Lambda''|^q$ is a convex function on $[\theta,\upsilon]$ with $q>1$, then the following inequalities hold:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}_{\theta^+}^{\sigma}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}_{\upsilon^-}^{\sigma}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\varphi(\sigma,\varepsilon,p)\left[\left(\frac{|\Lambda''(\theta)|^q+3|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda''(\theta)|^q+|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}4^{\frac{1}{p}}\varphi(\sigma,\varepsilon,p)\left[|\Lambda''(\theta)|+|\Lambda''(\upsilon)|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\begin{aligned}\varphi(\sigma,\varepsilon,p)&=\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^p d\mu\right)^{\frac{1}{p}}\\&=\frac{1}{\sigma^{\varepsilon}}\left(\int_0^1\left[\frac{1}{\sigma}\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)+\frac{1-\mu}{3}\right]^p d\mu\right)^{\frac{1}{p}}.\end{aligned}$$

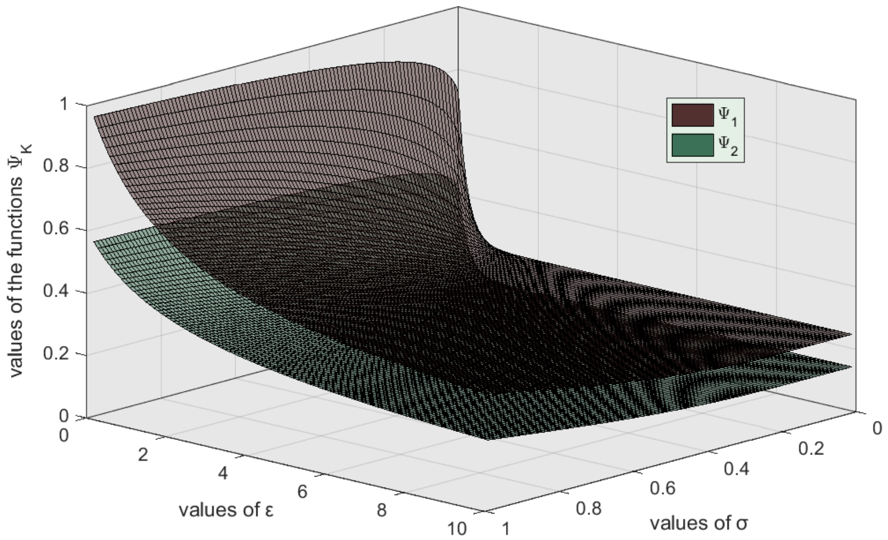

Figure 1: The graph of both sides of inequality (2.4) according to Example 1, which is computed and drawn by MATLAB program, depending on $\sigma \in (0,1)$ and $\varepsilon \in (0,10)$.

Proof. By utilizing the well-known Hölder's inequality, according to Lemma 2.1, we establish

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\right|^q d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].$$

Since $|\Lambda''|^q$ is a convex function, we have

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^p d\mu\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left(\frac{1-\mu}{2}\left|\Lambda''(\theta)\right|^q+\frac{1+\mu}{2}\left|\Lambda''(\upsilon)\right|^q\right)d\mu\right)^{\frac{1}{q}}+\left(\int_0^1\left(\frac{1+\mu}{2}\left|\Lambda''(\theta)\right|^q+\frac{1-\mu}{2}\left|\Lambda''(\upsilon)\right|^q\right)d\mu\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|^p d\mu\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{\left|\Lambda''(\theta)\right|^q+3\left|\Lambda''(\upsilon)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|\Lambda''(\theta)\right|^q+\left|\Lambda''(\upsilon)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

Moreover, for $0 \leq \pi < 1$ and $\eta_k, \rho_k \geq 0$ with $k \in \{1,2,...,n\}$, the inequality

$$\sum_{k=1}^{n}(\eta_k+\rho_k)^{\pi} \leq \sum_{k=1}^{n}\eta_k^{\pi}+\sum_{k=1}^{n}\rho_k^{\pi}$$

is a widely acknowledged property. Therefore, the proof of the second inequality follows easily by choosing $\eta_1 = 3|\Lambda''(\theta)|^q$, $\rho_1 = |\Lambda''(\upsilon)|^q$, $\eta_2 = |\Lambda''(\theta)|^q$ and $\rho_2 = 3|\Lambda''(\upsilon)|^q$, under the assumption that $1+3^{\frac{1}{q}} \leq 4$. □

Remark 2.9. *By specifying $\sigma = 1$ in Theorem 2.8, we get a Milne-type inequality for twice-differentiable convex functions, incorporating the Riemann-Liouville fractional integral operators:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^{\varepsilon}}\Gamma(\varepsilon+1)\left[I^{\varepsilon}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I^{\varepsilon}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\varphi(1,\varepsilon,p)\left[\left(\frac{|\Lambda''(\theta)|^q+3|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda''(\theta)|^q+|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\upsilon-\theta)^2}{8}4^{\frac{1}{p}}\varphi(1,\varepsilon,p)\left[|\Lambda''(\theta)|+|\Lambda''(\upsilon)|\right],$$

where

$$\varphi(1,\varepsilon,p)=\left(\int_0^1\left|\int_\mu^1\left(\pi^{\varepsilon}+\frac{1}{3}\right)^p d\pi\right|d\mu\right)^{\frac{1}{p}}=\frac{1}{3(\varepsilon+1)}\left(\int_0^1\left(\varepsilon+4-\mu(\varepsilon+1)-3\mu^{\varepsilon+1}\right)^p d\mu\right)^{\frac{1}{p}}.$$

This finding was established by Budak et al. [32].

Remark 2.10. *Setting $\sigma = 1$ and $\varepsilon = 1$ in Theorem 2.8 yields a Milne-type inequality for twice-differentiable convex functions, based on classical Riemann integral operator:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_\theta^\upsilon\Lambda(x)\,dx\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\left(\int_0^1\left(\int_\mu^1\left(\pi+\frac{1}{3}\right)d\pi\right)^p d\mu\right)^{\frac{1}{p}}\left[\left(\frac{|\Lambda''(\theta)|^q+3|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda''(\theta)|^q+|\Lambda''(\upsilon)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\upsilon-\theta)^2}{48}\left(\int_0^1[(5+3\mu)(1-\mu)]^p d\mu\right)^{\frac{1}{p}}\left[|\Lambda''(\theta)|+|\Lambda''(\upsilon)|\right].$$

The validity of this result was confirmed by Budak et al. [32].

Theorem 2.11. *Let $\Lambda : [\theta,\upsilon] \to \mathbb{R}$ be a twice-differentiable function on (θ,υ) such that $\Lambda'' \in L_1([\theta,\upsilon])$. If $|\Lambda''|^q$ is a convex function on $[\theta,\upsilon]$ with $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}(\psi_1(\sigma,\varepsilon))^{1-\frac{1}{q}}\left[\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\theta)|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\upsilon)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\theta)|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)|\Lambda''(\upsilon)|^q\right)^{\frac{1}{q}}\right],$$

where $\psi_1(\sigma,\varepsilon)$ is expressed as in Theorem 2.4 and

$$\begin{aligned}\psi_2(\sigma,\varepsilon) &= \int_0^1\frac{\mu}{2}\left|\int_0^\mu\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\\ &= \frac{1}{2\sigma^{\varepsilon}}\int_0^1\mu\left(\frac{1}{\sigma}\mathcal{B}_{(1-\mu)^{\sigma}}\left(\frac{1}{\sigma},\varepsilon+1\right)+\frac{1-\mu}{3}\right)d\mu.\end{aligned}$$

Proof. By applying the absolute value to the identity (2.1), by the power mean inequality, we have

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1-\mu}{2}\theta+\frac{1+\mu}{2}\upsilon\right)\right|^q d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left|\Lambda''\left(\frac{1+\mu}{2}\theta+\frac{1-\mu}{2}\upsilon\right)\right|^q d\mu\right)^{\frac{1}{q}}\right].$$

Taking into account the convexity of the $|\Lambda''|^q$, we obtain

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left[\frac{1-\mu}{2}\left|\Lambda''(\theta)\right|^q+\frac{1+\mu}{2}\left|\Lambda''(\upsilon)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\int_\mu^1\left[\left(\frac{1-(1-\pi)^{\sigma}}{\sigma}\right)^{\varepsilon}+\frac{1}{3\sigma^{\varepsilon}}\right]d\pi\right|\left[\frac{1+\mu}{2}\left|\Lambda''(\theta)\right|^q+\frac{1-\mu}{2}\left|\Lambda''(\upsilon)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].$$

Therefore, it is inferred that

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\sigma\varepsilon-1}\sigma^{\varepsilon}\Gamma(\varepsilon+1)}{(\upsilon-\theta)^{\sigma\varepsilon}}\left[{}^{\varepsilon}\mathcal{J}^{\sigma}_{\theta^+}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+{}^{\varepsilon}\mathcal{J}^{\sigma}_{\upsilon^-}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2\sigma^{\varepsilon}}{8}\left(\psi_1(\sigma,\varepsilon)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{\psi_1(\sigma,\varepsilon)}{2}+\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(\sigma,\varepsilon)}{2}-\psi_2(\sigma,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].$$

□

Remark 2.12. *Under the assumption $\sigma = 1$ in Theorem 2.11, we arrive at the following inequality:*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{2^{\varepsilon-1}}{(\upsilon-\theta)^{\varepsilon}}\Gamma(\varepsilon+1)\left[I_{\theta^+}^{\varepsilon}\Lambda\left(\frac{\theta+\upsilon}{2}\right)+I_{\upsilon^-}^{\varepsilon}\Lambda\left(\frac{\theta+\upsilon}{2}\right)\right]\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{8}\left(\psi_1(1,\varepsilon)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{\psi_1(1,\varepsilon)}{2}-\psi_2(1,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(1,\varepsilon)}{2}+\psi_2(1,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{\psi_1(1,\varepsilon)}{2}+\psi_2(1,\varepsilon)\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{\psi_1(1,\varepsilon)}{2}-\psi_2(1,\varepsilon)\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right]$$

$$= \frac{(\upsilon-\theta)^2}{48}\left(\frac{\varepsilon+8}{\varepsilon+2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{2\varepsilon^2+19\varepsilon+48}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{4\varepsilon^2+47\varepsilon+96}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left(\frac{4\varepsilon^2+47\varepsilon+96}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\theta)\right|^q+\left(\frac{2\varepsilon^2+19\varepsilon+48}{6(\varepsilon+2)(\varepsilon+3)}\right)\left|\Lambda''(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].$$

Here, $\psi_1(1,\varepsilon)$ is presented in Remark 2.5 and also

$$\psi_2(1,\varepsilon)=\frac{1}{2}\int_0^1\left[\int_\mu^1\mu\left(\pi^{\varepsilon}+\frac{1}{3}\right)d\pi\right]d\mu=\frac{1}{2}\int_0^1\left[\int_0^\pi\mu\left(\pi^{\varepsilon}+\frac{1}{3}\right)d\mu\right]d\pi=\frac{\varepsilon+12}{36(\varepsilon+3)}.$$

As shown by Budak et al. [32], this result holds true.

Remark 2.13. *By taking $\sigma = 1$ and $\varepsilon = 1$ in Theorem 2.11, we get*

$$\left|\frac{1}{3}\left[2\Lambda(\theta)-\Lambda\left(\frac{\theta+\upsilon}{2}\right)+2\Lambda(\upsilon)\right]-\frac{1}{\upsilon-\theta}\int_\theta^\upsilon\Lambda(x)\,dx\right|$$

$$\leq \frac{(\upsilon-\theta)^2}{16}\left[\left(\frac{23\left|\Lambda''(\theta)\right|^q+49\left|\Lambda''(\upsilon)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{49\left|\Lambda''(\theta)\right|^q+23\left|\Lambda''(\upsilon)\right|^q}{72}\right)^{\frac{1}{q}}\right].$$

This outcome was established by Budak et al. [32].

Declarations

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ORCID
İzzettin Demir https://orcid.org/0000-0003-0298-2872
Esra Üneş https://orcid.org/0009-0005-8670-6075

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