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Research Article

Numerical Solution of First and Higher Order IVPs Via a Single Continuous Block Method

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ABSTRACT

This article focuses on the development and implementation of a single continuous collocation numerical scheme for solving first and higher-order ordinary differential equations (ODEs). By employing the interpolation and collocation technique on power series as basis function, we were able to come up with a continuous scheme from which block methods for effectively solving first and higher order ODEs were derived. This is better and faster than the traditional way of developing a continuous scheme for a specific order of ODE. The method's accuracy is determined to be of order seven, establishing its consistency. Results from the implementation of our method show its applicability on nonlinear equations and application problems from first, second, third, and fourth order ODEs that are of significant implications on various fields in physics, engineering, biology and mathematics. Furthermore, the numerical results generated by the method reveal its effectiveness and accuracy, and also its superiority over some methods that exist in literature.

Keywords: Accuracy, Collocation, Continuous scheme, First order ODEs, Higher order ODEs, Interpolation

1. INTRODUCTION

Mathematical modeling entails the ability to transform issues from a particular field of application into workable mathematical expressions, whose theoretical and numerical evaluations provide understanding, solutions, and recommendations that are advantageous for the initial applications [35]. Numerous fundamental laws of nature in Physics, Chemistry, Biology, and astronomy are most effectively articulated through differential equations [34]. Their applications are widespread in mathematics, particularly in geometry, as well as in engineering, behavioural sciences, industrial mathematics, artificial intelligence. This kind of problem can be represented using either first-order or higher-order ODEs. First-order ODEs are commonly used in studying problems for example, determining the movement of an object that is ascending or descending while experiencing air resistance, and calculating the current in an electrical circuit; population expansion; radioactive decay; mixture problems; and so on. Second-order ODEs are also commonly used while analyzing vibrating systems, electromagnetism, and electrical circuits with capacitors, resistors, and inductors. Third order differential equations can also be used to solve physical problems like thin film flow, electromagnetic waves and gravity-driven flows. In general, solutions to differential equations are used to forecast the behavior of a system at a later time or in an unknown location. But there aren't many analytical ways to solve ODEs in a continuous or closed form, and nonlinear ODEs can be hard to solve or may not have a closed form of solution at all.

Given the challenges associated with solving most of these problems, numerous researchers have focused extensively on applying numerical methods to provide approximate solutions for differential equations. Numerical methods are particularly effective for addressing mathematical problems, leveraging the speed and efficiency of modern digital computers in performing arithmetic operations [42]. The process of solving these problems with high-precision digital computers typically begins with initial data, followed by the execution of suitable algorithms to produce the desired outcomes [43]. Common approaches for solving IVPs are classified into single-step methods, such as the Runge-Kutta methods [21, 43], and multistep methods [1 – 10].

Over the years, various techniques for deriving continuous linear multistep methods (LMMs) aimed at directly solving IVPs have been extensively discussed in the literature. Key techniques include collocation, interpolation, integration,

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and the use of interpolating polynomials. Basis functions like power series, Chebyshev polynomials, Legendre polynomials, and trigonometric functions have been used for this purpose. (see [7, 10,17, 18, 22, 44, 45, 46]). Researchers have found linear multistep method very useful in solving ODEs. Some authors [7, 10, 11, 12] have develop effective methods for first order ODEs while [6, 13, 27, 35, 42, 44] have implemented linear multistep methods on second order ODEs with comparative accuracy. Other higher ODEs have been solved numerically by [2, 14, 15 31, 37, 39, 43, 47, 48, 49]. All of the linear multistep methods developed by these researchers were for a specific order of ODEs. Recently Kuboye and Adeyefa [22] developed a linear multistep method for first, second and third order ODEs.

In this research, we derive a single continuous technique from a hybrid block approach that effectively handles first, second, third, and fourth-order initial value problems. The subsequent portion of this article describes the method's derivation, while the third sets out the analysis that ensures the method's validity. In part four, a variety of numerical problems are addressed, results are displayed graphically, and the method's efficacy is demonstrated by comparing absolute errors to those of recent existing methods in literature. In section five, we offer broad conclusions on our findings.

The general form of the ODE is expressed as follows:

$$\begin{aligned} &z^{(m)} = \left(z(t), z'(t), \ldots, z^{(m-1)}(t)\right) \\ &z(t_0) = z_0, z'(t_0) = z'_0, \ldots, z^{(m-1)}(t_0) = z_0^{(m-1)} \end{aligned} \tag{1}$$

2. DERIVATION OF THE METHOD

As stated in the introduction, the continuous representation of our method will be generated from one of the orthogonal basis functions, preferably the power series, due to its ease of usage. Therefore, the approximate solution to (1) is as follows:

$$Z(t) = \sum_{j=0}^{m+n-1} c_j t^j \tag{2}$$

where *m* and *n* are respectively the number of interpolation and collocation points. We derive a two-step linear multistep method using (2) with degree *m + n – 1* = 10. (2) is interpolated at $t = t_n$ and the first, second and third derivatives are collocated at $t = t_{n+2}$. Furthermore, we collocate the fourth derivative at distinct points $t = t_{n+j}; \left(j = 0, \frac{1}{4}, \frac{3}{4}, 1, \frac{5}{4}, \frac{7}{4}, 2 \right)$. All these processes lead to a system of nonlinear equations in t_{n+j} with c_j's as the unknowns.

$$\left. \begin{aligned} &Z(t_n) = z_n \\ &Z'(t_{n+2}) = f_{n+2} \\ &Z''(t_{n+2}) = g_{n+2} \\ &Z'''(t_{n+2}) = p_{n+2} \\ &Z^{iv}(t_{n+j}) = g_{n+j} \end{aligned} \right\} \tag{3}$$

We solved for the unknowns using the matrix inversion method via a mathematical software – Maple 2015; and then substitute c_j's values back into (2) to obtain the continuous scheme in the form:

$$Z(t)=z_n+h\alpha'(t)f_{n+2}+h^2\alpha''(t)g_{n+2}+h^3\alpha'''(t)p_{n+2} +h^4\left(\beta_0(t)q_n+\beta_{\frac{1}{4}}(t)q_{n+\frac{1}{4}}+\beta_{\frac{3}{4}}(t)q_{n+\frac{3}{4}}+\beta_1(t)q_{n+1}+\beta_{\frac{5}{4}}(t)q_{n+\frac{5}{4}}+\beta_{\frac{7}{4}}(t)q_{n+\frac{7}{4}}+\beta_2(t)q_{n+2}\right) \tag{4}$$

Evaluating (4) at the non-interpolating points gives 6 discrete schemes that form the block method as follows:

$$z_{n+\frac{1}{4}}=z_n+\frac{1}{4}hf_{n+2}-\frac{15}{32}h^2g_{n+2}+\frac{169}{384}h^3p_{n+2} +\frac{1}{13005619200}h^4\begin{pmatrix}1837375q_n-21942710q_{n+\frac{1}{4}}-353393782q_{n+\frac{3}{4}}+131048330q_{n+1}\\-1219981490q_{n+\frac{5}{4}}-1715628930q_{n+\frac{7}{4}}-409914793q_{n+2}\end{pmatrix} \tag{5}$$

$$z_{n+\frac{3}{4}}=z_n+\frac{3}{4}hf_{n+2}-\frac{39}{32}h^2g_{n+2}+\frac{129}{128}h^3p_{n+2} +\frac{1}{4335206400}h^4\begin{pmatrix}1070517q_n-8616430q_{n+\frac{1}{4}}-163366014q_{n+\frac{3}{4}}+67852190q_{n+1}\\-776333082q_{n+\frac{5}{4}}-1257048730q_{n+\frac{7}{4}}-312696051q_{n+2}\end{pmatrix} \tag{6}$$

$$z_{n+1}=z_n+hf_{n+2}-\frac{3}{2}h^2g_{n+2}+\frac{7}{6}h^3p_{n+2}+\frac{1}{793800}h^4\begin{pmatrix}283q_n-1880q_{n+\frac{1}{4}}-28840q_{n+\frac{3}{4}}+11585q_{n+1}\\-150248q_{n+\frac{5}{4}}-260760q_{n+\frac{7}{4}}-66265q_{n+2}\end{pmatrix} \tag{7}$$

$$z_{n+\frac{5}{4}}=z_n+\frac{5}{4}hf_{n+2}-\frac{55}{32}h^2g_{n+2}+\frac{485}{384}h^3p_{n+2} +\frac{1}{2601123840}h^4\begin{pmatrix}1221691q_n-7249910q_{n+\frac{1}{4}}-88889206q_{n+\frac{3}{4}}+28468370q_{n+1}\\-491995826q_{n+\frac{5}{4}}-906420930q_{n+\frac{7}{4}}-234924589q_{n+2}\end{pmatrix} \tag{8}$$

$$z_{n+\frac{7}{4}}=z_n+\frac{7}{4}hf_{n+2}-\frac{63}{32}h^2g_{n+2}+\frac{511}{384}h^3p_{n+2} +\frac{1}{1857945600}h^4\begin{pmatrix}1082161q_n-5977070q_{n+\frac{1}{4}}-58808638q_{n+\frac{3}{4}}+10672550q_{n+1}\\-34408610q_{n+\frac{5}{4}}-664954650q_{n+\frac{7}{4}}-176256247q_{n+2}\end{pmatrix} \tag{9}$$

$$z_{n+2}=z_n+2hf_{n+2}-2h^2g_{n+2}+\frac{4}{3}h^3p_{n+2}+\frac{1}{99225}h^4\begin{pmatrix}29q_n-160q_{n+\frac{1}{4}}-1568q_{n+\frac{3}{4}}+280q_{n+1}\\-1568q_{n+\frac{5}{4}}-17760q_{n+\frac{7}{4}}-4712q_{n+2}\end{pmatrix} \tag{10}$$

3. ANALYSIS OF THE BLOCK METHOD

At this point, following the approaches adopted in [18, 27, 28], the order of accuracy of our method is:

$(7,7,7,7,7,7)^T$ and error constants given as

$$\begin{pmatrix}\frac{221815417}{3515887072051200},\frac{66674131}{390654119116800},\frac{593}{2682408960},\frac{184617553}{703177414410240},\\\frac{150066817}{502269581721600},\frac{251}{838252800}\end{pmatrix}^T.$$

We show the working of equation (10) as follows:

$$\left.\begin{array}{l}
\varepsilon_0 = 1-1=0 \\
\varepsilon_1 = 2(1)-2=0 \\
\varepsilon_2 = \frac{1}{2!}(2^2)-(2\times 2)-(-2)=0 \\
\varepsilon_3 = \frac{1}{3!}(2^3)-\frac{1}{2!}(2^2\times 2)-(-2\times 2)-\left(\frac{4}{3}\right)=0 \\
\varepsilon_4 = \frac{1}{4!}(2^4)-\frac{1}{3!}(2^3\times 2)-\frac{1}{2!}(-2\times 2^2)-\left(2\times\frac{4}{3}\right)=0 \\
\varepsilon_5 = \frac{1}{5!}(2^5)-\frac{1}{4!}(2^4\times 2)-\frac{1}{3!}(-2\times 2^3)-\frac{1}{2!}\left(2^2\times\frac{4}{3}\right)- \\
\left(\frac{58}{99225}+\frac{64}{19845}+\frac{64}{2025}-\frac{16}{2835}+\frac{2624}{14175}+\frac{2368}{6615}+\frac{9424}{99225}\right)=0 \\
\vdots \\
\varepsilon_{11} = \frac{1}{11!}(2^{11})-\frac{1}{10!}(2^{10}\times 2)-\frac{1}{9!}(-2\times 2^9)-\frac{1}{8!}\left(2^8\times\frac{4}{3}\right)- \\
\frac{1}{7!}\left(\left(\frac{1}{4}\right)^7\frac{64}{19845}+\left(\frac{3}{4}\right)^7\frac{64}{2025}-(1)^7\frac{16}{2835}+\left(\frac{5}{4}\right)^7\frac{2624}{14175}+\left(\frac{7}{4}\right)^7\frac{2368}{6615}+(2)^7\frac{9424}{99225}\right) \\
=\frac{251}{838252800}
\end{array}\right\}$$

The techniques described in sections (5) through (10) can typically be represented using a matrix-based difference equation, which is detailed in the following steps.:

$$A^{(1)}Z_w = A^{(0)}Z_{w-1}+hBF_w+h^2CG_w+h^3DP_w+h^4\left[E^{(0)}Q_{w-1}+E^{(1)}Q_w\right] \tag{11}$$

where

$$\left.\begin{array}{l}
Z_w = \left(z_{n+\frac{1}{4}}, z_{n+\frac{3}{4}}, z_{n+1}, z_{n+\frac{5}{4}}, z_{n+\frac{7}{4}}, z_{n+2}\right)^T, \\
Z_{w-1} = \left(z_{n-\frac{1}{4}}, z_{n-\frac{3}{4}}, z_{n-1}, z_{n-\frac{5}{4}}, z_{n-\frac{7}{4}}, z_{n-2}\right)^T, \\
F_w = (f_{n+2})^T, G_w = (g_{n+2})^T, P_w = (p_{n+2})^T, \\
Q_w = \left(q_{n+\frac{1}{4}}, q_{n+\frac{3}{4}}, q_{n+1}, q_{n+\frac{5}{4}}, q_{n+\frac{7}{4}}, q_{n+2}\right)^T, \\
Q_{w-1} = \left(q_{n-\frac{1}{4}}, q_{n-\frac{3}{4}}, q_{n-1}, q_{n-\frac{5}{4}}, q_{n-\frac{7}{4}}, q_{n-2}\right)^T
\end{array}\right\}$$

and the matrices $A^{(1)}, A^{(0)}, B, C, D, E^{(1)}$ and $E^{(0)}$ are matrices whose elements are given by the coefficients of the block method.

Definition 3.1. The newly developed two-step hybrid block method (5 – 10) is considered zero-stable if and only if the first characteristic polynomial $\xi(t)$ has roots that satisfy $|t|\le 1$, and for the roots $|t|\le 1$, their multiplicity does not exceed one. The characteristic function of this newly derived method is presented below:

$$\xi(\lambda)=\lambda A^{(1)}-A^{(0)} \tag{12}$$

$$\xi(\lambda)=\lambda\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\\0&0&0&0&0&-1\end{pmatrix}=(\lambda+1)\lambda^5 \tag{13}$$

the solution of which is $\lambda=(0,0,0,0,0,1)$. Hence, our method is zero-stable.

Definition 3.2. If the order of the hybrid block approach is greater than or equal to one, it is consistent. Our method is consistent as a result of order *p*=7.

Theorem 3.1. Zero stability and consistency are adequate requirements for the convergence of a linear multistep method [26, 17, 27]. The new hybrid block technique is convergent due to its zero-stability and consistency.

To analyze the stability characteristics of the developed scheme, it is applied to a standard test problem.

$$z'=\lambda z, \qquad z''=\lambda^2 z, \qquad z'''=\lambda^3 z, \qquad z^{iv}=\lambda^4 z, \qquad \mathrm{Re}(\lambda)<0 \tag{14}$$

to yield

$$Z_w=\xi(w)Z_{w-1}, \quad w=z\lambda \tag{15}$$

where the matrix $\xi(w)$ is given as:

$$\zeta(w)=\left(Z^{(1)}-wF-w^2G-w^3P-w^4Q^{(1)}\right)^{-1}\left(Z^{(0)}+w^4Q^{(0)}\right) \tag{16}$$

The Matrix $\zeta(w)$ has eigenvalues $(0,0,0,0,0,\lambda_6)$, and the dominant eigenvalue $\lambda_6:\mathbb{C}\to\mathbb{C}$ is a rational function with real coefficients given by

$$R(w)=\frac{Y(w)}{X(w)}$$

The stability region is illustrated in Figure 1, indicating that the method is A-stable.

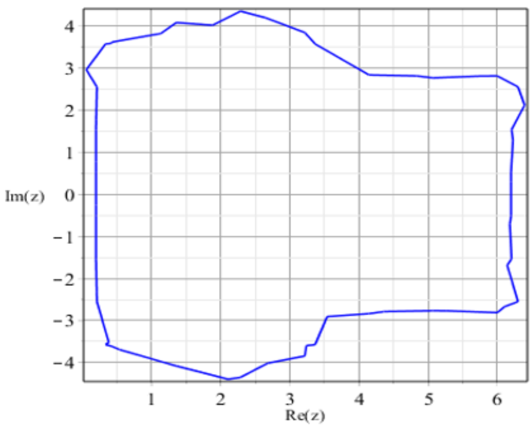

Fig 1. Region of absolute stability of the method

4. NUMERICAL EXPERIMENT

The recently developed hybrid block method is applied to both first and higher-order ODEs. This continuous approach results in a primary discrete two-step multi-derivative block method (as outlined in equations 5 – 10), along with supplementary methods that are integrated and utilized as a block method to generate approximations concurrently. $\{y_{n+1}, y_{n+2}\}$ at a block points $\{x_{n+1}, x_{n+2}\}$, $h = x_{n+1} - x_n$, $n = 0,..., N-2$, on a partition $[a,b]$, where $a,b \in \mathbb{R}$ is the interval of integration, h is the constant step-size, n is a grid index and $N > 0$ is the number of steps. We obtain initial conditions at $x_{n+2}, n = 0,1,..., N-2$, using the computed values y_{n+2} over smaller intervals $[x_0, x_2],...,[x_{N-2}, x_N]$. For example, when $n = 0, [y_1, y_2]$ are acquired at the same time across the smaller interval $[x_0, x_2]$, as y_0 is established from the initial value problem, for $n = 2, [y_3, y_4]$ are also acquired simultaneously over the smaller interval $[x_3, x_4]$, as is now understood from the earlier section, and so forth. Consequently, the smaller interval $[x_n, x_{n+2}]$ does not overlap, and the solutions derived in this manner are more precise than those obtained through traditional predictor-corrector methods. Similar approach is applied to higher order ODEs considered – the first, second and third derivatives of the continuous scheme are evaluated at all points to cater for the higher derivative terms in the general higher order ODEs considered. We present the graphical solution of the problems and compare absolute errors with some existing methods in the literature.

4.1 First-Order Problems

The block method (5) to (10) is used directly on some first order problems.

Problem 1: The SIR model is an epidemiological framework that tracks the hypothetical number of individuals within a closed population who are affected by an infectious disease over time. This type of model derives its name from the interconnected equations that describe the populations of susceptible individuals *S(t)*, infected individuals *I(t)*, and recovered individuals *R(t)*. It serves as an effective and simple model for various infectious diseases, such as measles, mumps, and rubella. The model is represented by the three interconnected equations provided below.:

$$\frac{dS}{dt} = \mu(1-S) - \beta IS$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS$$

$$\frac{dR}{dt} = \mu R + \gamma I$$

where μ, β and γ are positive parameters. Define *z* to be:

$$z = S + I + R$$

and adding (13), (14) and (15), the following evolution equation for *z* is obtained.

$$z' = \mu(1-z)$$

Kuboye and Adeyefa [22] solved this problem with the following parameters:

$$\mu = \frac{1}{2}, \quad z(0) = \frac{1}{2}, \quad h = 0.1$$

Exact solution: $z(t) = 1 - \mu e^{-\mu t}$

Problem 2: We consider the Riccati differential equation solved in Khalsaraei *et. al.* [20]

$$z' = 1 + 2z - z^2; \quad z(0) = 0, \quad 0 \le t \le 10$$

Exact solution: $z(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$

Problem 3: We consider the nonlinear system of stiff chemical problem solved in Akinfenwa *et. al.* [11]

$$\left.\begin{array}{l} z_1' = \lambda z_1 + z_2^2, \quad z_1(0) = -\dfrac{1}{\lambda+2} \\ z_2' = -z_2, \quad z_2(0) = 1 \end{array}\right\}$$

The exact solution is given as $z_1(t) = -\dfrac{\exp(-2t)}{\lambda+2}, \quad z_2(t) = \exp(-t)$ where $\lambda = 10000$.

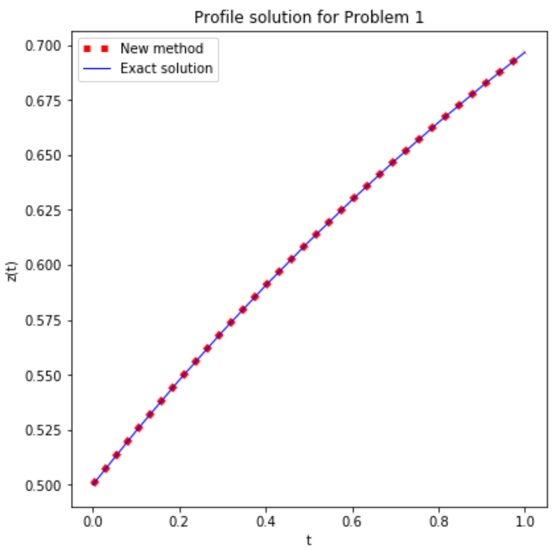

Fig 2. The profile solution for problem 1

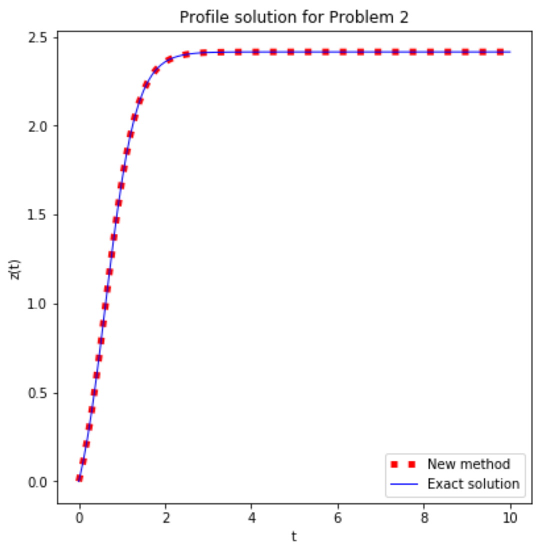

Fig 3. The profile solution for problem 2

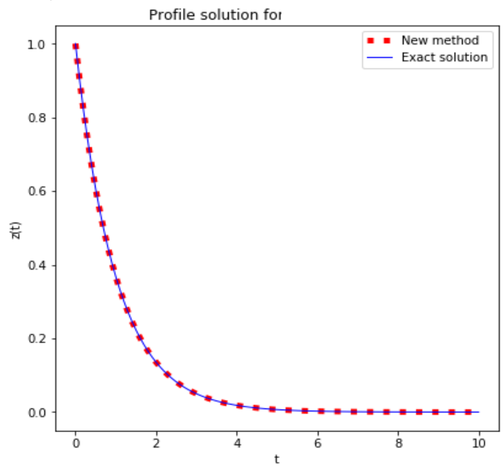

Fig 4. The profile solution for problem 3

Table 1. Error comparison for Problem 1

t	Error in [22] $(h=0.1)$	Error in [19] $(h=0.1)$	Error in [4] $(h=0.1)$	Error in New method $(h=0.1)$
0.1	3.846×10^{-13}	1.998×10^{-15}	9.104×10^{-15}	4.736×10^{-22}
0.2	7.319×10^{-13}	3.886×10^{-15}	7.105×10^{-15}	6.225×10^{-22}
0.3	1.044×10^{-12}	5.440×10^{-15}	8.882×10^{-15}	1.021×10^{-21}
0.4	1.324×10^{-12}	6.994×10^{-15}	2.121×10^{-14}	1.127×10^{-21}
0.5	1.575×10^{-12}	8.216×10^{-15}	1.368×10^{-13}	1.459×10^{-21}
0.6	1.797×10^{-12}	9.548×10^{-15}	7.983×10^{-13}	1.529×10^{-21}
0.7	1.995×10^{-12}	1.055×10^{-14}	3.699×10^{-12}	1.805×10^{-21}
0.8	2.168×10^{-12}	1.132×10^{-14}	-	1.845×10^{-21}
0.9	2.320×10^{-12}	1.221×10^{-14}	-	1.845×10^{-21}
1.0	2.452×10^{-12}	1.288×10^{-14}	-	2.086×10^{-21}

Table 2. Error comparison for Problem 2

t	Error in [19] $(h=0.05)$	Error in [4] $(h=0.1)$	Error in New method $(h=0.1)$
1	1.418×10^{-11}	9.104×10^{-15}	2.168×10^{-11}
2	7.234×10^{-13}	7.105×10^{-15}	1.052×10^{-12}
3	1.163×10^{-13}	8.882×10^{-15}	5.216×10^{-14}
4	2.132×10^{-14}	2.121×10^{-14}	3.144×10^{-15}
5	2.664×10^{-15}	1.368×10^{-13}	1.970×10^{-16}
6	4.441×10^{-16}	7.983×10^{-13}	1.233×10^{-17}
7	4.441×10^{-16}	3.699×10^{-12}	7.213×10^{-19}
8	4.441×10^{-16}	-	4.702×10^{-20}
9	4.441×10^{-16}	-	2.977×10^{-21}
10	4.441×10^{-16}	-	1.850×10^{-22}

Table 3. Error comparison for Problem 3

t	z_i	Error in [20] $(h=0.0001)$	Error in [11] $(h=0.01)$	Error in [26] $(h=0.1)$	Error in New method $(h=0.1)$
3	(z_1)	1.779×10^{-20}	2.030×10^{-19}	3.790×10^{-22}	1.972×10^{-23}
	(z_2)	2.079×10^{-20}	1.440×10^{-14}	2.998×10^{-18}	1.981×10^{-18}
5	(z_1)	2.493×10^{-19}	1.200×10^{-20}	1.60×10^{-21}	6.019×10^{-25}
	(z_2)	4.664×10^{-13}	3.210×10^{-15}	6.740×10^{-19}	4.468×10^{-19}
10	(z_1)	5.743×10^{-20}	1.110×10^{-20}	7.120×10^{-20}	5.466×10^{-29}
	(z_2)	6.346×10^{-12}	4.380×10^{-17}	9.080×10^{-21}	6.021×10^{-21}

4.2 Second-Order Problems

In order to implement our method on second order IVPs, we take the first derivative of the continuous scheme (4) and evaluated at points $t_{n+j},\left(j=0,\frac{1}{4},\frac{3}{4},1,\frac{5}{4},\frac{7}{4},2\right)$ with the block schemes (5 – 10).

Problem 4: Consider the nonlinear second order IVP from Abdelrahim and Omar [1].

$$z''=t\left(z'\right)^2,\ \ z(0)=1,\ z'(0)=0.5$$

The exact solution is: $z(t)=\arctan h\left(\frac{1}{2}t\right)+1$

Problem 5: *Cooling of a body.* Source: Kwanamu *et. al.* [24]

The temperature *z* degree of a body *t* minutes after being placed in a certain room, satisfies the differential equation

$$3\frac{d^2z}{dt}+\frac{dz}{dt}=0,\quad z(0)=60,\ \left.\frac{dz}{dt}\right|_{t=0}=-\frac{80}{9}$$

With the analytic solutions: $z(t)=\frac{80}{3}e^{-\frac{t}{3}}+\frac{100}{3}$

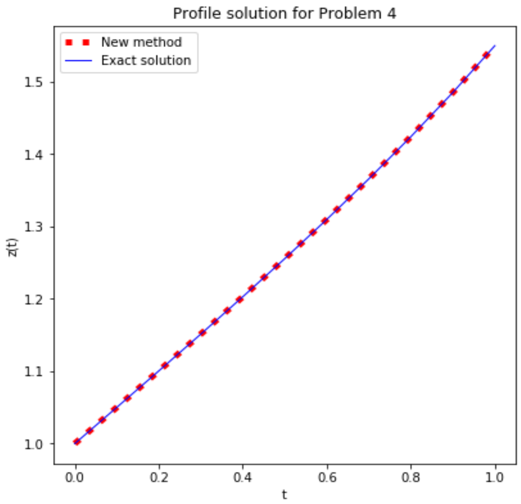

Fig 5. The profile solution for problem 4

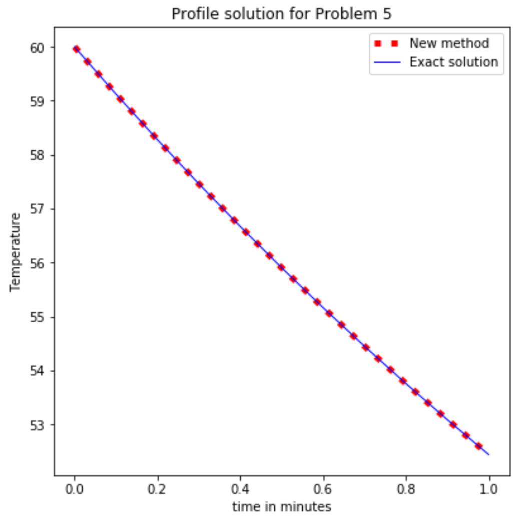

Fig 6. The profile solution for problem 5

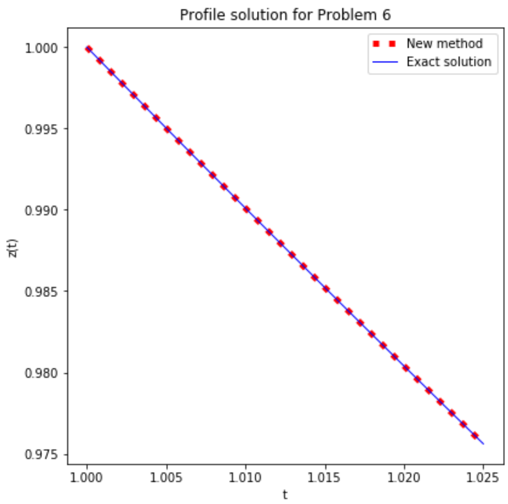

Fig 7. The profile solution for problem 6

Problem 6: Consider the nonlinear problem:

$z''-2z^3=0$ (Source: Ogunlaran & Kehinde [30])

with the following initial conditions

$z(1)=1, z'(1)=-1$ and $h=\frac{0.1}{40}$

Exact solution: $Z(t)=\frac{1}{t}$

Table 4. Error comparison for Problem 4

t	Error in [1] $\left(h=\frac{1}{30}\right)$	Error in [7] $(h=0.1)$	Error in new method $(h=0.1)$	Error in New method $\left(h=\frac{1}{30}\right)$
0.1	1.310×10^{-16}	1.957×10^{-13}	1.539×10^{-15}	6.552×10^{-20}
0.2	3.975×10^{-14}	6.040×10^{-13}	6.459×10^{-15}	2.934×10^{-19}
0.3	1.021×10^{-14}	1.262×10^{-12}	1.791×10^{-14}	7.771×10^{-19}
0.4	3.304×10^{-13}	3.715×10^{-12}	3.972×10^{-14}	1.726×10^{-18}
0.5	-	7.919×10^{-12}	8.698×10^{-14}	3.588×10^{-18}
0.6	-	1.416×10^{-11}	1.781×10^{-13}	7.353×10^{-18}
0.7	-	3.616×10^{-11}	3.987×10^{-13}	1.531×10^{-17}
0.8	-	7.473×10^{-11}	8.522×10^{-13}	3.306×10^{-17}
0.9	-	1.335×10^{-10}	2.182×10^{-12}	7.538×10^{-17}
1.0	1.293×10^{-12}	4.317×10^{-10}	5.160×10^{-12}	1.844×10^{-16}

Table 5. Error comparison for Problem 5, h = 0.1

t	Error in [38]	Error in [41]	Error in [33]	Error in [24]	Error in New method
0.1	3.550×10^{-11}	2.300×10^{-17}	7.476×10^{-06}	-	9.000×10^{-23}
0.2	4.580×10^{-11}	1.710×10^{-16}	2.939×10^{-05}	4.000×10^{-18}	3.600×10^{-22}
0.3	7.000×10^{-11}	4.370×10^{-16}	6.480×10^{-05}	9.000×10^{-18}	8.300×10^{-22}
0.4	6.500×10^{-11}	8.130×10^{-16}	1.128×10^{-05}	1.700×10^{-17}	1.450×10^{-21}
0.5	3.330×10^{-11}	1.290×10^{-15}	1.725×10^{-04}	2.600×10^{-17}	2.230×10^{-21}
0.6	4.200×10^{-11}	1.864×10^{-15}	2.431×10^{-04}	3.800×10^{-17}	3.170×10^{-21}
0.7	4.380×10^{-11}	2.525×10^{-15}	3.238×10^{-04}	5.100×10^{-17}	4.240×10^{-21}
0.8	1.070×10^{-10}	3.269×10^{-15}	4.139×10^{-04}	6.500×10^{-17}	5.410×10^{-21}
0.9	6.580×10^{-11}	4.089×10^{-15}	5.127×10^{-04}	8.100×10^{-17}	6.720×10^{-21}
1.0	1.6900×10^{-10}	4.980×10^{-15}	6.195×10^{-04}	9.700×10^{-17}	8.140×10^{-21}

Table 6. Error comparison for Problem 6

t	Exact Solution	Error in [30]	Error in New method
1.0025	0.997506234413965	3.160×10^{-17}	5.400×10^{-29}
1.0050	0.995024875621891	7.550×10^{-17}	2.480×10^{-28}
1.0075	0.992555831265509	1.190×10^{-16}	5.650×10^{-28}
1.0100	0.990099009900990	1.510×10^{-16}	1.005×10^{-27}
1.0125	0.987654320987654	1.810×10^{-16}	1.566×10^{-27}
1.0150	0.985221674876847	2.220×10^{-16}	2.241×10^{-27}
1.0175	0.982800982800983	2.630×10^{-16}	3.031×10^{-27}
1.0200	0.980392156862745	2.930×10^{-16}	3.929×10^{-27}
1.0225	0.977995110024450	3.210×10^{-16}	4.936×10^{-27}
1.0250	0.975609756097561	3.590×10^{-16}	6.044×10^{-27}

4.3 Third-Order Problems

In order to implement our method on third order IVPs, we take the first and second derivatives of the continuous scheme (4) and evaluated at points $t_{n+j},\left(j=0,\frac{1}{4},\frac{3}{4},1,\frac{5}{4},\frac{7}{4},2\right)$ with the block schemes (5 – 10).

Problem 7: Consider the nonlinear IVP:

$$z'''=\left(2tz''+1\right)z';\quad z\left(0\right)=1,\ z'\left(0\right)=0.5,\quad z''\left(0\right)=0$$

The exact solution is: $z\left(t\right)=\tan^{-1}\left(\frac{1}{2}t\right)+1$

Source: Yakusak &. Owolanke [47]

Problem 8: Consider the nonlinear ODE

$$z'''=\frac{3}{8z^5};\quad z\left(0\right)=1,\ z'\left(0\right)=\frac{1}{2},\quad z''\left(0\right)=-\frac{1}{4}$$

Exact solution: $z\left(t\right)=\frac{1}{\sqrt{1+t}}$

Source: Adeyeye & Omar [8]

Problem 9:

$$z'''=tz''-\left(tz\right)^2+t\sin\left(t\right)-\cos\left(t\right)+t^2\sin^2\left(t\right);\quad z\left(0\right)=0,\ z'\left(0\right)=1,\quad z''\left(0\right)=0$$

Exact solution: $z\left(t\right)=\sin\left(t\right)$

Source: Source: Adeyeye & Omar [8]

Table 7. Error comparison for Problem 7

t	Error in [47] $(h=0.01)$	Error in [9] $(h=0.01)$	Error in [31] $(h=0.01)$	Error in New method $(h=0.01)$
0.1	9.609×10^{-12}	1.931×10^{-08}	2.043×10^{-14}	4.401×10^{-24}
0.2	7.072×10^{-10}	5.608×10^{-07}	8.371×10^{-14}	3.661×10^{-23}
0.3	6.693×10^{-09}	3.755×10^{-06}	2.813×10^{-13}	1.353×10^{-22}
0.4	3.142×10^{-08}	1.340×10^{-05}	7.667×10^{-13}	3.665×10^{-22}
0.5	1.051×10^{-07}	3.259×10^{-05}	1.853×10^{-12}	8.554×10^{-22}
0.6	2.852×10^{-07}	5.816×10^{-05}	4.163×10^{-12}	1.855×10^{-21}
0.7	6.777×10^{-07}	7.152×10^{-05}	8.965×10^{-12}	3.905×10^{-21}
0.8	1.467×10^{-06}	2.564×10^{-05}	-	8.224×10^{-21}
0.9	2.983×10^{-06}	1.709×10^{-04}	-	1.774×10^{-20}
1.0	6.189×10^{-06}	6.706×10^{-04}	-	4.003×10^{-20}

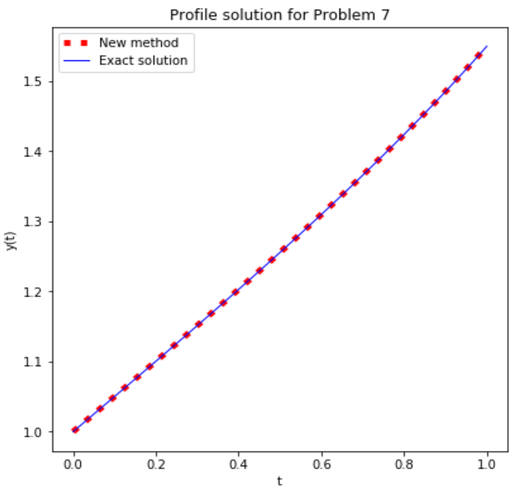

Fig 8. The profile solution for problem 7

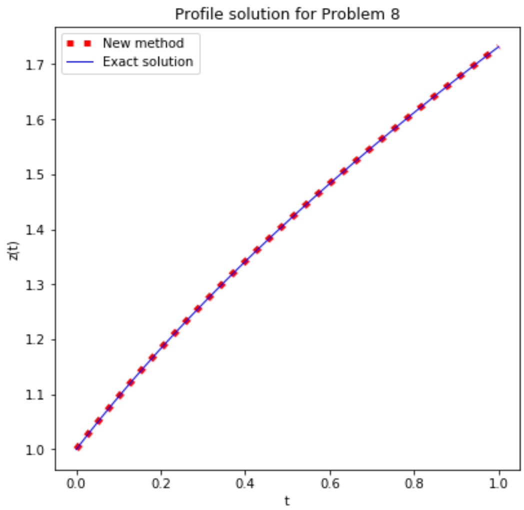

Fig 9. The profile solution for problem 8

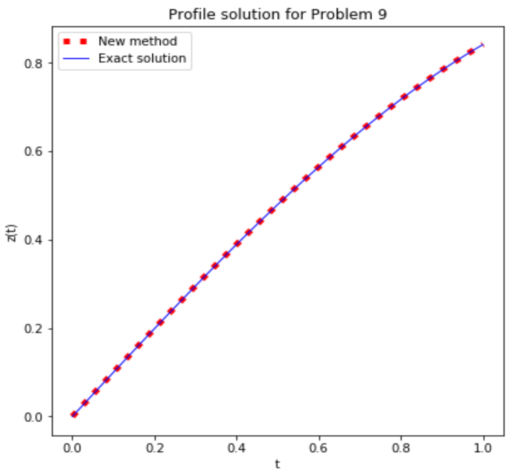

Fig 10. The profile solution for problem 9

Table 8. Comparison of errors for Problem 8

t	Exact Solution	Error in [8] $(h=0.1)$	Error in New method $(h=0.1)$	Error in New method $(h=0.01)$
0.2	1.09544511501033222691394 0	2.181×10^{-11}	3.890×10^{-13}	3.870×10^{-21}
0.4	1.183215956619923208513466	7.070×10^{-11}	1.812×10^{-12}	2.242×10^{-20}
0.6	1.264911064067351732799557	1.348×10^{-10}	4.673×10^{-12}	5.891×10^{-20}
0.8	1.341640786499873817845504	2.106×10^{-10}	9.038×10^{-12}	1.138×10^{-19}
1.0	1.414213562373095048801689	2.964×10^{-10}	1.490×10^{-11}	1.871×10^{-19}
1.2	1.483239697419132589742279	3.914×10^{-10}	2.225×10^{-11}	2.785×10^{-19}
1.4	1.549193338482966754071706	4.947×10^{-10}	3.106×10^{-11}	3.877×10^{-19}
1.6	1.612451549659709930473323	6.059×10^{-10}	4.128×10^{-11}	5.142×10^{-19}
1.8	1.673320053068151095956344	7.242×10^{-10}	5.290×10^{-11}	6.576×10^{-19}
2.0	1.732050807568877293527446	8.492×10^{-10}	6.586×10^{-11}	8.174×10^{-19}

Table 9. Error comparison for Problem 9

t	Exact Solution	Error in [8] $(h=0.1)$	Error in New method $(h=0.1)$	Error in New method $(h=0.01)$
0.1	0.099833416646828152307	6.661×10^{-16}	7.851×10^{-19}	3.435×10^{-27}
0.2	0.19866933079506121546	3.914×10^{-15}	3.028×10^{-18}	1.919×10^{-26}
0.3	0.29552020666133957511	1.243×10^{-14}	8.180×10^{-18}	6.472×10^{-26}
0.4	0.38941834230865049167	2.887×10^{-14}	1.779×10^{-17}	1.538×10^{-25}
0.5	0.47942553860420300027	5.601×10^{-14}	3.339×10^{-17}	3.017×10^{-25}
0.6	0.56464247339503535720	9.692×10^{-14}	5.667×10^{-17}	5.243×10^{-25}
0.7	0.64421768723769105367	1.547×10^{-13}	8.933×10^{-17}	8.383×10^{-25}
0.8	0.71735609089952276163	2.326×10^{-13}	1.333×10^{-16}	1.261×10^{-24}
0.9	0.78332690962748338846	3.346×10^{-13}	1.904×10^{-16}	1.812×10^{-24}
1.0	0.84147098480789650665	4.644×10^{-13}	2.627×10^{-16}	2.509×10^{-24}

4.4 Fourth-Order Problems

Problem 10: Consider the oscillatory problem arising from ship dynamics:

$$z^{iv}+3z''+z(2+\varepsilon\cos\lambda t)=0;\quad z(0)=1,\ z'(0)=0,\quad z''(0)=0, z'''(0)=0$$

Where $\lambda=0$ for existence of the exact solution: $z(t)=2\cos t-\cos\left(\sqrt{2}t\right)$

Source: Familua & Omole [16]

Problem 11: Consider the nonlinear problem:

$$z^{iv}=(z')^2-zz'''-4t^2+e^t\left(1-4t+t^2\right);\quad z(0)=1,\ z'(0)=1,\quad z''(0)=3, z'''(0)=1$$

Exact solution: $z(t)=t^2+e^t$. (Source: Familua & Omole [16])

Problem 12: consider the nonlinear sinusoidal problem:

$$z^{iv}=z^2+\sin^2(t)-\cos(t)-1;\quad z(0)=-1,\ z'(0)=0,\quad z''(0)=1, z'''(0)=0$$

Exact solution: $z(t)=-\cos(t)$. Source: Tiamiyu *et al.* [43]

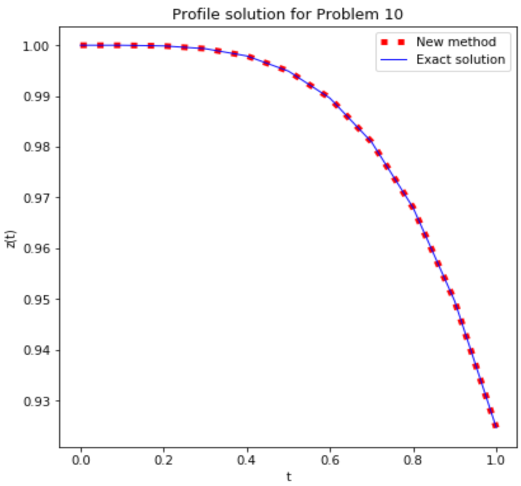

Fig 11. The profile solution for problem 10

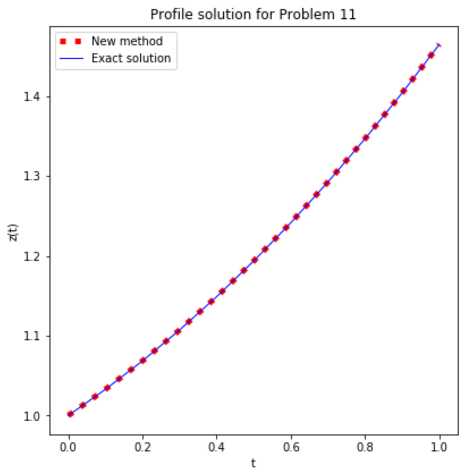

Fig 12. The profile solution for problem 11

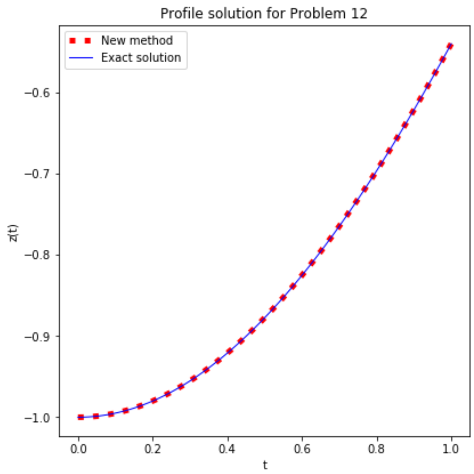

Fig 13. The profile solution for problem 12

Table 10. Error comparison for Problem 10 $(h = 0.003125)$

t	Error in [16] (Block mode)	Error in [16] (P-C mode)	Error in [45]	Error in New Method
0.003125	6.686×10^{-13}	5.686×10^{-10}	1.900×10^{-19}	3.657×10^{-36}
0.006250	1.458×10^{-11}	1.768×10^{-10}	2.300×10^{-19}	1.262×10^{-35}
0.009375	1.083×10^{-10}	5.910×10^{-09}	8.600×10^{-19}	4.233×10^{-35}
0.001250	3.918×10^{-10}	5.768×10^{-09}	1.380×10^{-18}	1.042×10^{-34}
0.015625	1.025×10^{-09}	1.100×10^{-08}	3.530×10^{-18}	2.271×10^{-34}
0.018750	2.217×10^{-09}	6.899×10^{-08}	5.310×10^{-18}	4.359×10^{-34}
0.021875	4.226×10^{-09}	4.636×10^{-08}	8.880×10^{-18}	7.728×10^{-34}
0.025000	7.358×10^{-09}	5.788×10^{-07}	3.922×10^{-17}	1.276×10^{-33}
0.028125	1.197×10^{-08}	2.246×10^{-07}	5.846×10^{-17}	2.001×10^{-33}
0.031250	1.846×10^{-08}	2.846×10^{-07}	8.477×10^{-17}	3.001×10^{-33}

Table 11. Error comparison for Problem 10 using different values of h

t	Exact solution	Error in New method $(h = 0.01)$	Error in New method $(h = 0.1)$
0.1	0.99999167499652860438	3.477×10^{-27}	4.196×10^{-18}
0.2	0.99986719911195714198	5.225×10^{-26}	1.437×10^{-17}
0.3	0.99933105226749824584	2.522×10^{-25}	4.756×10^{-17}
0.4	0.99790057330915505191	8.075×10^{-25}	1.157×10^{-16}
0.5	0.99492052670511528095	1.938×10^{-24}	2.4783×10^{-16}
0.6	0.98958301770794685383	3.941×10^{-24}	4.670×10^{-16}
0.7	0.98095229007588226219	7.137×10^{-24}	8.088×10^{-16}
0.8	0.96799382462962246366	1.186×10^{-23}	1.302×10^{-15}
0.9	0.94960705358355858551	1.844×10^{-23}	1.981×10^{-15}
1.0	0.92466091697090496135	2.718×10^{-23}	2.874×10^{-15}

Table 12. Error comparison for Problem 11

t	Error in [16]	Error in [23]	Error in New method
0.031250	1.149×10^{-12}	1.788×10^{-10}	2.000×10^{-24}
0.062500	1.885×10^{-11}	1.134×10^{-08}	9.000×10^{-24}
0.093750	9.780×10^{-11}	1.196×10^{-07}	2.300×10^{-23}
0.125000	3.166×10^{-10}	6.401×10^{-07}	4.800×10^{-23}
0.156250	7.909×10^{-10}	2.349×10^{-06}	9.000×10^{-23}
0.187500	1.676×10^{-09}	6.573×10^{-06}	1.500×10^{-22}
0.218750	3.169×10^{-09}	1.610×10^{-05}	2.3400×10^{-22}
0.250000	5.512×10^{-09}	3.501×10^{-05}	3.450×10^{-22}
0.281250	8.995×10^{-09}	6.985×10^{-05}	4.870×10^{-22}
0.312500	1.396×10^{-08}	1.245×10^{-04}	6.620×10^{-22}

Table 13. Error comparison for Problem 11 using different values of h

t	Exact solution	Error in New method $(h=0.1)$	Error in New method $(h=0.01)$
0.1	1.11517091807564762481	8.200×10^{-19}	2.430×10^{-27}
0.2	1.26140275816016983392	2.960×10^{-18}	1.891×10^{-26}
0.3	1.43985880757600310398	8.000×10^{-18}	6.317×10^{-26}
0.4	1.65182469764127031782	1.706×10^{-17}	1.486×10^{-25}
0.5	1.89872127070012814685	3.173×10^{-17}	2.881×10^{-25}
0.6	2.18211880039050897488	5.298×10^{-17}	4.947×10^{-25}
0.7	2.50375270747047652162	8.246×10^{-17}	7.811×10^{-25}
0.8	2.86554092849246760458	1.211×10^{-16}	1.160×10^{-24}
0.9	2.86554092849246760458	1.707×10^{-16}	1.645×10^{-24}
1.0	3.71828182845904523536	2.232×10^{-16}	2.251×10^{-24}

Table 14. Error comparison for Problem 12

t	Exact solution	Error in [43] $(h=0.01)$	Error in New method $(h=0.01)$
0.1	– 0.995004165278025766095561987804	1.400×10^{-29}	5.580×10^{-29}
0.2	−0.980066577841241631124196516748	8.240×10^{-28}	8.465×10^{-28}
0.3	−0.955336489125606019642310227568	7.825×10^{-27}	4.236×10^{-27}
0.4	−0.921060994002885082798526732052	3.697×10^{-26}	1.331×10^{-26}
0.5	−0.877582561890372716116281582604	1.209×10^{-25}	3.234×10^{-26}
0.6	−0.825335614909678297240952498955	3.148×10^{-25}	6.675×10^{-26}
0.7	−0.764842187284488426255859990192	7.021×10^{-25}	1.230×10^{-25}
0.8	−0.696706709347165420920749981642	1.400×10^{-24}	2.088×10^{-25}
0.9	−0.621609968270664456484716151407	2.564×10^{-24}	3.323×10^{-25}
1.0	−0.540302305868139717400936607443	4.393×10^{-24}	5.031×10^{-25}

Problem 13. And lastly, we also consider the following nearly sinusoidal problem in first-order system of equations. Source: (Akinfenwa *et. al.,* [10])

$$z_1' = -21z_1 + z_2 + 2\sin t, \qquad z_1(0) = 2$$

$$z_2' = 998z_1 - 999z_2 + 999\cos t - 999\sin t, \qquad z_2(0) = 3$$

Exact solution: $z_1(t) = 2e^{-t} + \sin t, \quad z_2(t) = 2e^{-t} + \cos t$

Table 15: Error comparison for Problem 13

	HBSDBDF [10]		New Method		
h	Maximum error	Relative error	Maximum error	Relative error	*ROC*
0.4	8.9924×10^{-7}	3.6279×10^{-7}	1.2119×10^{-11}	1.8909×10^{-10}	-
0.2	5.9042×10^{-9}	2.6294×10^{-9}	1.6035×10^{-14}	2.2570×10^{-13}	9.56
0.1	4.5695×10^{-11}	1.8848×10^{-11}	1.7159×10^{-17}	3.0416×10^{-16}	9.87
0.05	2.9376×10^{-13}	1.2826×10^{-13}	1.7479×10^{-20}	3.1990×10^{-19}	9.94

5. DISCUSSION OF RESULTS

Figures 1–3 respectively display the graphical solution of the SIR model in problem 1, the nonlinear Riccati equation in problem 2 and the nonlinear system of stiff chemical equations in problem 3. In the graphs, we plotted the results generated from our method (red boxes) and the exact solution (blue line). It is easily seen that the new method agrees with the analytical solutions to the problems. Going further, Table 1 shows the comparison of absolute errors for Problem 1. It is shown that the newly derived method has a computational advantage over the methods in ([22], [19] and [4]). Also in Table 2, we show the absolute errors of some methods and ours, for different step sizes in solving problem 2. A comparison of our method with [4] using the same step size (*h* = 0.1) indicates that our method outperforms the method in [4], and further comparison with the method in [19] (whose step size is smaller) also indicates the superiority of our method over the one in [19]. Table 3 presents a comparative analysis of absolute errors for problem 3 using different step sizes. We solved the nonlinear chemical stiff problem using the step size (*h* = 0.1) and analysis shows that our method performs better than the methods in [26] (with the same step size), [11] (whose step size is *h* = 0.01) and [20] (with step size *h* = 0.0001).

Problems 4–6 are second-order problems considered using the same block method as the first-order problems. Problem 4 is a nonlinear IVP, while Problem 5 is an application problem in the cooling of a body and Problem 6 is also a nonlinear problem. Their behavioural solutions are displayed in figures 4–6 demonstrating agreement in the numerical method that we derived and the exact solutions. To further validate the effectiveness of our method, we compare the absolute errors against those produced by existing techniques. This comparative analysis provides additional evidence supporting the superior performance of our approach. Table 4 shows the comparison of errors for problem 4. It is shown that the new method with *h* = 0.1 outperforms the method in [7] with the same step size and that in [1] using the same step size of $\left(h = \frac{1}{30}\right)$. Also, Table 5 shows the comparison of errors for Problem 5, which indicates the superiority of our method over those in [24, 33, 38, 41] using the same step size. And in Table 6, we show the numerical solution made by the new method, the absolute error and that in [30], it is shown that the new method is better in terms of accuracy.

The nonlinear third-order initial value problems were the next class of ODEs we looked at in this paper. The graphical solutions of problems 7–9 are displayed in Figures 7– 9. Furthermore, comparative analysis for Problem 7 in Table 7 indicates that our method is found to give better accuracy than the methods in [31], [47], and [9]. Similarly, analyses in Tables 8 and 9 also show that the newly derived method in this paper gives better accuracy than the method derived in [8] for problems 8 and 9, respectively.

Finally, a class of fourth-order ODEs is also solved by our method. Figures 10–12 display the graphical solutions for problems 10–12, respectively. Problem 10 is solved by [16] using block mode predictor–corrector (P–C) of the linear multistep method. Also, [45] solved this particular problem in the interval [0.003125, 0.03125] with h = 0.003125. Table 10 shows the comparison of errors between our method, [16] and [33] as mentioned above. It has been demonstrated that the new block method provides a more accurate approximation for the application problem related to ship dynamics. We further compare our method for variable step sizes over [0, 1] using h = 0.1 and h = 0.01 and Table 11 shows that the method gives better approximation as h becomes smaller. Similarly, the nonlinear equation in problem 11 is also solved by [16] and [23] and the results in Table 12 assert that the newly derived method is superior in terms of accuracy to the methods in [16, 23]. Table 13 gives the comparison of errors in our method based on different step sizes. And in Table 14, the comparison of errors for problem 12 is made between our method and that in [43]. It is shown that as the numerical iteration progresses, our method gets closer to the exact solution than the method in [43].

Lastly, we incorporated the sinusoidal problem into a system of first-order equations, as addressed in [10]. We showcased the method's accuracy, rate of convergence *(ROC*), and strong stability characteristics. We calculated the maximum error, $\left(\max\left|z(t_i)-z_i\right|\right)$, relative error $\max_i \dfrac{\left|z(t_i)-z_i\right|}{\left|1+z(t_i)\right|}$ and compared with the method in [10], which indicates that the new method has higher accuracy than the method in [10]. Also, the *ROC* $=\log_2\left(\dfrac{e^{2h}}{e^{h}}\right)$, for different step sizes h, where e^h is the maximum absolute error for each h is calculated as shown in Table 15.

6. CONCLUSION

In this paper, a novel numerical method in the class of linear multistep method is developed for first order and higher order ODEs. The method is derived through the usual interpolation and collocation techniques with power series used as the basis function. Basic numerical properties as established in section 3 show that the method converges and A-stable. The method conveniently solves first, second, third and fourth order IVPs as shown in Tables 1–15. Thirteen numerical problems were considered in all, with nonlinear equation as majority and some application problems. The numerical results generated by our method show its superiority over some existing methods as compared therein. The innovative aspect of this method lies in its capability to effectively solve first, second, third, and fourth order ordinary differential equations (ODEs) without requiring distinct numerical schemes for each type. Therefore, this new approach is proposed as a practical numerical algorithm for tackling first and higher order ODEs. In our upcoming research, we plan to expand this method to address partial differential equations.

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