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On solving the (3+1)-dimensional B-type Kadomtsev-Petviashvili equation by using two efficient method

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Abstract: This paper employs two distinct yet potent methodologies in order to tackle the intricate difficulties posed by nonlinear partial differential equations. Our primary focus is on deriving novel exact solutions for the (3+1)-dimensional B-type Kadomtsev-Petviashvili equation. The (3+1)-dimensional B-type Kadomtsev-Petviashvili equation serves as the focal point of this research. By employing the unified method and the generalized Kudryashov method, solitary wave solutions for this equation are obtained. These methods not only contribute to the theoretical analysis of nonlinear systems but also facilitate a deeper understanding of multidimensional wave phenomena. The newly derived exact solutions provide significant insights into the physical interpretations of these equations, paving the way for advanced applications in fields such as energy transmission, signal processing, and wave dynamics.

(3+1)-Boyutlu B-tipi Kadomtsev-Petviashvili Denkleminin Çözümü için İki Etkili Yöntemin Kullanılması

Anahtar Kelimeler
The generalized Kudryashov yöntemi,
B-type Kadomtsev-Petviashvili denklemi,
Unified yöntem

Öz: Bu çalışma, doğrusal olmayan kısmi diferansiyel denklemler tarafından ortaya konulan karmaşık zorlukların üstesinden gelmek amacıyla iki farklı ve güçlü yöntemi ele almaktadır. Çalışmanın temel amacı, (3+1)-boyutlu B-tipi Kadomtsev-Petviashvili denklemi için yeni ve tam çözümler türetmektir. Araştırmanın odak noktası olarak ele alınan bu denklem, birleşik yöntem ve genelleştirilmiş Kudryashov yöntem kullanılarak dalga çözümleri elde edilerek analiz edilmiştir. Bu yöntemler, doğrusal olmayan sistemlerin teorik analizine katkı sağlarken, çok boyutlu dalga fenomenlerinin daha derinlemesine anlaşılmasını da mümkün kılmaktadır. Türetilen yeni ve tam çözümler, bu denklemlerin fiziksel yorumlarına dair önemli içgörüler sunmakta ve enerji aktarımı, sinyal işleme ve dalga dinamikleri gibi alanlarda ileri düzey uygulamalara zemin hazırlamaktadır.

1. Introduction

In recent years, nonlinear partial differential equations (NPDEs) have become pivotal in advancing our understanding of complex phenomena that pervade various scientific and engineering domains. The study of NPDEs is motivated by their exceptional ability to capture the intricate interplay between spatial and temporal variables in nonlinear systems. These equations serve as a mathematical cornerstone for modeling diverse processes characterized by abrupt transitions and sensitivity to initial or boundary conditions, which often pose significant analytical and computational challenges.

NPDEs are integral to the mathematical representation of complex dynamical systems where linear approximations fail to provide accurate descriptions. For instance, in fluid dynamics, NPDEs govern the behavior of turbulent flows and nonlinear wave propagation, while in viscoelasticity, they elucidate stress-strain relationships in complex materials. Similarly, in control theory, NPDEs facilitate the design of systems capable of withstanding disturbances, and in electrochemistry, they describe reaction-diffusion processes critical to understanding electrochemical kinetics.

The utility of NPDEs extends far beyond physical sciences, finding applications in fields as varied as acoustics, finance, and biological systems, etc. Their ability to incorporate nonlinear interactions enables precise modeling of phenomena such as financial market dynamics, where small perturbations can lead

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to significant systemic changes, and biological pattern formation, which involves intricate chemical signaling pathways. Advanced mathematical techniques, including symmetry analysis, perturbation methods, and numerical simulations, have been employed to address the inherent challenges posed by NPDEs, leading to groundbreaking insights and innovative solution frameworks.

This emerging field continues to evolve, driven by the demand for accurate and efficient models that can predict and control nonlinear behaviours. The interdisciplinary nature of NPDEs not only bridges the theoretical and mathematical realms with practical applications, but also fosters collaboration across diverse scientific communities. Consequently, the study of NPDEs remains at the forefront of modern research, offering unparalleled opportunities to unravel the complexities of nonlinear systems and their implications in the real world [1].

To obtain a thorough understanding of the physical implications of NPDEs, researchers have developed a wide range of advanced techniques for deriving exact solutions. These methods, including the differential transformation method [2,3], the Adomian decomposition method [4,5], the fractional sub-equation method [6], the (G'/G)-expansion method [7-9], the φ^6-expansion method [10], the tanh-function expansion method [11], the sub-ODE method [12, 13], the unified method [14], the Sardar sub-equation method [15], the exponential function method [16, 17], the homogeneous balance method [18], the generalized Kudryashov method [19], etc. These approaches have significantly advanced the theoretical and practical understanding of nonlinear nonlinear differential equations, enabling precise descriptions of diverse nonlinear phenomena.

The Kadomtsev-Petviashvili (KP) equation is a fundamental model describing the propagation of weakly dispersive, small-amplitude waves in multidimensional media [20]. As a two-dimensional extension of the classical Korteweg-de Vries (KdV) equation, the KP equation has been extensively studied for its mathematical properties and physical applications, particularly in the context of integrable systems [21]. Various studies have demonstrated the rich mathematical structure of the KP equation, including the Lax pair formulation, infinite conservation laws, and the existence of multi-soliton solutions, which are fundamental in the study of nonlinear wave dynamics [22].

A significant extension of the KP equation is the B-Type Kadomtsev-Petviashvili (B-KP) equation, derived from the bilinear equations of the constrained B-KP hierarchy through pseudo-differential calculus [23]. The B-KP equation introduces additional complexity through its association with symmetric reductions and constrained flows, which distinguish it from the classical KP framework. These mathematical features provide new insights into multidimensional wave interactions and nonlinear dynamics.

One of the most remarkable aspects of the B-KP equation is its analytical solutions, often expressed through determinant-based representations such as Grammian solutions [22]. Within the Hirota bilinear formalism, these determinant solutions offer a powerful algebraic and geometric approach to understanding the dynamics of nonlinear wave interactions. Determinant structures have proven to be a crucial tool in analyzing solitonic interactions and providing exact solutions to nonlinear wave equations.

The B-KP hierarchy is further categorized into two distinct subhierarchies: the B-KP hierarchy and the C-Type Kadomtsev-Petviashvili (C-KP) hierarchy. Each of these subhierarchies presents unique mathematical structures and physical applications. The B-KP hierarchy is closely related to symmetric reductions and constrained flows, while the C-KP hierarchy explores alternative symmetry properties and additional integrable structures [24]. These classifications enable researchers to systematically study the interplay between integrability and symmetry in nonlinear wave systems [25].

Within this context, the B-KP equation and its hierarchy play a crucial role in the study of multidimensional integrable systems. The mathematical richness of this system, coupled with its determinant-based solutions and symmetry analysis, provides a valuable framework for understanding nonlinear wave dynamics in both theoretical and applied mathematical contexts.

Recent research in nonlinear science, particularly in fluid dynamics and plasma physics, has uncovered foundational phenomena that deepen our understanding of complex systems. Of particular note are solitary waves, which have been shown to possess the remarkable ability to maintain shape and coherence over long distances, thus challenging traditional wave behaviour paradigms. These insights have opened avenues for innovative applications in energy transmission and signal processing. The extended (3+1)-dimensional B-KP equation, introduced by [26], marks a significant generalization of the classical KP equation. This higher-dimensional extension incorporates an additional spatial dimension, enabling the exploration of more intricate wave interactions and dynamics in multidimensional media. The B-type structure modifies the nonlinear and dispersive terms, allowing for a broader spectrum of solutions, including lump solutions and higher-order solitons. These advancements make the B-KP equation a powerful tool for analyzing physical systems where higher-dimensional propagation and interactions are critical, significantly expanding the applicability of the Kadomtsev-Petviashvili framework in mathematical physics and beyond.

In [27], KP hierarchy of B-type [28, 29], introduced by Date, is discussed. This includes the (2+1)-dimensional equation proposed by Jimbo, Kashiwara, and Miwa, given as

$$9v_t + v_{5x} - 5\left(v_{xxy} - \partial_x^{-1} v_{yy}\right) + 15\left(v_x v_{xx} + v v_{xxx} - v v_y - v_x\, \partial_x^{-1} v_y\right) + 45 v^2 v_x = 0. \tag{1}$$

The study focuses on the generalized (2+1)-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation, which represents a significant extension of two well-known nonlinear partial differential equations: the Bogoyavlenskii-Schiff (BS) equation and the Kadomtsev-Petviashvili (KP) equation [27]. The BKP equation is rooted in B-type Lie algebras, setting it apart from the classical KP equation, which is derived from the mathematical framework of A-type Lie algebras. This distinction underscores the rich algebraic structure of the BKP equation and highlights its role in broadening the scope of integrable systems theory.

The Bogoyavlenskii-Schiff equation, from which the BKP equation partially extends, emerges in the context of constrained flows within B-type Lie algebras. These algebras are known for their unique symmetries and structural properties, which play a pivotal role in characterizing solutions to the BS equation. In contrast, the KP equation, derived from A-type Lie algebras, has served as a foundational model in nonlinear wave theory, particularly for describing weakly dispersive, small-amplitude waves in quasi-two-dimensional systems. The BKP equation seamlessly integrates features from both of these equations, offering a versatile framework for studying multidimensional nonlinear wave phenomena.

One of the notable characteristics of the BKP equation is its ability to encode higher-dimensional dynamics while maintaining integrability. Through its derivation and analysis, the BKP equation provides new avenues for exploring the interplay between geometry, symmetry, and nonlinear dynamics. Its formulation extends the scope of classical integrable systems by incorporating elements of pseudo-differential calculus, Grammian determinants, and bilinear transformations. These tools enable the construction of explicit solutions, such as solitons, dromions, and other localized wave structures, which are of significant interest in mathematical physics.

Furthermore, the BKP equation's connection to B-type Lie algebras introduces additional algebraic richness, allowing for the study of alternative symmetry reductions and hierarchical structures. This opens the door to a deeper understanding of constrained flows, Hamiltonian systems, and the interplay between different types of Lie algebras in the theory of integrable systems. As such, the generalized BKP equation not only serves as a bridge between the BS and KP equations but also acts as a powerful tool for investigating complex nonlinear wave phenomena in higher-dimensional settings.

Additionally, [27] explores a (3+1)-dimensional extension of the DJKM equation, formulated as:

$$9v_t + v_{5x} - 5\left(v_{xxy} - \partial_x^{-1} v_{yy}\right) + 15\left(v_x v_{xx} + v v_{xxx} - v v_y - v_x\, \partial_x^{-1} v_y\right) + 45 v^2 v_x + \alpha v_z = 0 \tag{2}$$

where the term v_z is incorporated into the (2+1)-dimensional form, and α is a arbitrary constant, as presented in [27]. The study considers an extended to Eq. (2), introducing additional terms to enhance its applicability and complexity. The generalized equation is given by:

$$9v_t + v_{5x} - 5\left(v_{xxy} - \partial_x^{-1} v_{yy}\right) + 15\left(v_x v_{xx} + v v_{xxx} - v v_y - v_x\, \partial_x^{-1} v_y\right) + 45 v^2 v_x + \alpha v_x + \beta v_y + \gamma v_z = 0, \tag{3}$$

where the last three terms αv_x , βv_y and γv_z are additional terms included for generalization. Here α, β and γ are real parameters representing external influences or modifications to the system's dynamics. Building on this, we focus on an extended (3+1)-dimensional B-KP equation derived under the substitution $u_x = v$, which transforms the equation into:

$$9u_{xt} + u_{6x} - 5\left(u_{xxxy} + u_{yy}\right) + 15\left(u_{xx} u_{3x} + u_x u_{4x} - u_x u_{xy} - u_{xx} u_y\right) + 45 (u_x)^2 u_{xx} + \alpha u_{xx} + \beta u_{xy} + \gamma u_{xz} = 0. \tag{4}$$

This formulation provides a higher-dimensional framework to explore the intricate interplay between spatial and temporal dynamics in nonlinear wave systems. The inclusion of the parameters α, β and γ broadens the scope of the analysis, enabling the modeling of more complex physical phenomena.

This study can be summarized as follows: In Section 2, we present a detailed description of the generalized Kudryashov method and the unified method, which serve as the primary analytical tools for addressing nonlinear partial differential equations in this work. Section 3 is dedicated to the application of these methods, wherein we derive abundant exact solutions for a selection of nonlinear partial differential equations, showcasing the efficacy and versatility of the approaches. Finally, in the concluding section, we discuss the significance of our findings, provide insights into their implications, and outline potential directions for future research in this domain.

2. Material and Method

The generalized Kudryashov method:

In this section, we give the generalized Kudryashov method as an effective approach for solving NPDEs. Consider a general nonlinear evolution equation expressed as:

$$P(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xt}, u_{yt}, u_{zt}, u_{xx}, \dots) = 0 \quad (5)$$

where P is a polynomial involving the unknown function $u(x, y, z, t)$ and its various partial derivatives. The generalized Kudryashov method has been developed for the purpose of constructing characteristic and broad-spectrum soliton solutions to nonlinear partial differential equations (NPDEs). This method is designed to address both temporal and spatial dependencies [30]. The method involves a systematic sequence of steps, which can be outlined as follows:

Step 1: In order to facilitate the analysis, a new variable ξ is given and the following transformation is applied:

$$u(x, y, z, t) = q(\xi)\ , \qquad \xi = x + y + z - kt \quad (6)$$

where k is parameter. The transformation specified in Eq. (6) leads to the reduction of Eq. (5) to the following nonlinear ordinary differential equation (NODE);

$$G\left(q, -kq', q', q', q', k^2q'', -k(q')^2, -k(q')^2, -k(q')^2, q'\right) = 0 \quad (7)$$

where G is a polynomial that involves the function q and its derivatives with respect to ξ. The resulting Eq. (7) is then integrated one or more times, with the constants of integration set to zero.

Step 2: Assume that the solution of Eq. (7) takes the following form:

$$q(\xi) = \frac{a_0 + \sum_{i=1}^{m} a_i U^i(\xi)}{b_0 + \sum_{j=1}^{n} b_j U^j(\xi)} \quad (8)$$

where $a_i (i = 0,1,2,3, \cdots, m)$ and $b_j (j = 0,1,2,3, \cdots, n)$ are constants to be determined, with the conditions $a_m \neq 0\ , b_n \neq 0$. Here, $U(\xi)$ is defined as:

$$U(\xi) = \frac{1}{1 + \lambda \exp(\xi)}$$

which is the general solution of the Riccati equation:

$$U'(\xi) = U^2(\xi) - U(\xi) \quad (9)$$

where λ is constant, and the prime $(\,'\,)$ denotes the ordinary derivative with respect to ξ.

Step 3: The values of m and n are determined through the method of homogeneous balancing, a technique that involves equating the highest-order derivative terms with the highest-order nonlinear terms in Eq. (7). This process ensures the consistency of the solution by balancing the contributions of the various terms in the equation. To apply this method, we substitute the expression for $q(\xi)$ from Eq. (8) into Eq. (7), along with the general solution for $U(\xi)$ provided in Eq. (9). Next, we equate the coefficients of like powers of $U(\xi)$ on both sides of the equation, setting them to zero. This results in a system of algebraic equations, which can be solved to determine the unknown constants a_i and b_j, as well as the values of m and n.

Step 4: The system of algebraic equations obtained from the balancing process is solved using advanced Maple software programme. These tools facilitate the efficient computation of the unknown constants $a_i (i = 0,1, \dots, n)$, $b_j (j = 0,1, \dots, m)$, as well as the parameters k and λ. Once these constants and parameters are determined, they are substituted back into the expression for $q(\xi)$ given in Eq. (8). This step results in the complete solution to the nonlinear evolution equation described by Eq. (7), providing explicit forms for the solution that capture the essential dynamics of the system under consideration.

The unified method:

The fundamental phases of the unified method are outlined as follows:

Step 1: The wave variable assigned in Eq. (6) transforms Eq. (5), we obtain Eq. (7).

Step 2: We express the exact solution of Eq. (7) in the following form:

$$q(\xi) = a_0 + \sum_{i=1}^{M} [a_i\, \varphi^i + b_i\, \varphi^{-i}]\ , \quad (10)$$

where M is positive integers, a_0, a_i, b_i ($i = 1, 2, 3, \dots, M$) are constants to be determined and $\varphi = \varphi(\xi)$ satisfies following the Riccati differential Eq. (8).

$$\varphi'(\xi) = \varphi^2(\xi) + \lambda\ , \quad (11)$$

where $\varphi' = \frac{d\varphi}{d\xi}$ and λ is a constant. The general solution of Eq. (7) as follows:

Set 1: When $\lambda < 0$, the solutions to Eq. (11) are as follows:

$$\varphi(\xi)=\begin{cases}\dfrac{\sqrt{-(A^2+B^2)\lambda}-A\sqrt{-\lambda}\cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)}{A\sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+B},\\ \dfrac{-\sqrt{-(A^2+B^2)\lambda}-A\sqrt{-\lambda}\cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)}{A\sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+B},\\ \sqrt{-\lambda}-\dfrac{2A\sqrt{-\lambda}}{A+\cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)-\sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)},\\ -\sqrt{-\lambda}+\dfrac{2A\sqrt{-\lambda}}{A+\cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+\sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)},\end{cases}$$

where A, B and ξ_0 are arbitrary constants.

Set 2: When $\lambda > 0$, the solutions to Eq. (11) are as follows:

$$\varphi(\xi)=\begin{cases}\dfrac{\sqrt{(A^2-B^2)\lambda}-A\sqrt{\lambda}\cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)}{A\sin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+B},\\ \dfrac{-\sqrt{(A^2-B^2)\lambda}-A\sqrt{\lambda}\cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)}{A\sin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+B},\\ i\sqrt{\lambda}-\dfrac{2Ai\sqrt{\lambda}}{A+\cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)-i\sin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)},\\ -i\sqrt{\lambda}+\dfrac{2A\sqrt{\lambda}}{A+\cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+i\sin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)},\end{cases}$$

where A, B and ξ_0 are arbitrary constants.

Set 3: When $\lambda = 0$, the solutions to Eq. (11) are as follows:

$$\varphi(\xi)=-\frac{1}{\xi+\xi_0},$$

where ξ_0 arbitrary constant [31].

Step 3: Employing the homogeneous balance method outlined in Eq. (7) enables us to determine the positive integer values of M corresponding to the solution described in Eq. (10). By substituting the solution from Eq. (10) into Eq. (7) and incorporating the Riccati equation depicted in Eq. (11), we obtain a polynomial expression in terms of $U(\xi)$. This polynomial, upon equating coefficients of similar powers of $U(\xi)$ to zero, yields specific sets of algebraic equations.

Step 4: Upon substituting Eq. (10) into Eq. (7) alongside Eq. (11), a polynomial expression in terms of $U(\xi)$ is derived. Equating all coefficients of $U(\xi)$ to zero leads to a system of algebraic equations. By employing the Maple program, we can effectively solve this system to determine the values of parameters such as a_0, a_i, b_i ($i = 1, 2, 3, \dots, M$), and λ. Subsequently, upon substituting these values and Eq. (11) into Eq. (10), exact solutions for the reduced Eq. (5) can be obtained.

3. Results

We employ the generalized Kudryashov method to solve the extended (3+1)-dimensional B-KP equation. By utilizing both the generalized Kudryashov method and the unified method, we investigate the exact traveling wave solutions of the (3+1)-dimensional B-KP equation. The B-KP equation is

$$9u_{xt}+u_{6x}-5\left(u_{xxxy}+u_{yy}\right)+15\left(u_{xx}u_{3x}+u_xu_{4x}-u_xu_{xy}-u_{xx}u_y\right)+45(u_x)^2u_{xx}+\alpha u_{xx}+\beta u_{xy}+\gamma u_{xz}=0. \tag{12}$$

where $u(x,y,z,t)$ is a differentiable function.

The generalized Kudryashov method:

We apply the generalized Kudryashov method to Eq.(12). Let us assume that

$$u(x,y,z,t)=q(\xi), \quad \xi=x+y+z-kt \tag{13}$$

where k denotes the wave velocity. By substituting Eq.(13) into Eq.(12), we derive the corresponding nonlinear ordinary differential equation.

$$(\alpha+\beta+\gamma-5-9k)q'+q^{(5)}+15q'q'''-5q'''-15\left(q'\right)^2+15\left(q'\right)^3=0 \tag{14}$$

where $q'=\frac{dq}{d\xi}$. By applying the method of homogeneous balancing, specifically by balancing the $q^{(5)}$ term and the $\left(q'\right)^3$ term in Eq. (14), we determine that $m=2$, $n=1$. Consequently, from Eq. (8) we obtain the following expression

$$q(\xi)=\frac{a_0+a_1U(\xi)+a_2U^2(\xi)}{b_0+b_1U(\xi)}. \tag{15}$$

Next, we substitute Eq. (15) into Eq. (14) and rearrange all terms such that the coefficients of $U^i(\xi)$ $(i=0,1,\dots,12)$ are set to zero. This results in a system of algebraic equations. By solving these equations using mathematical software, a set of solutions for $k, b_0, b_1, a_i (i=0,1,2)$ is obtained.

Case 1:

$$a_0=\frac{b_0}{b_1}(a_1+4b_0), a_2=-4b_1, k=\frac{1}{9}(\alpha+\beta+\gamma-9).$$

Plugging these values into Eq.(15), hence the solution for the Eq.(12) is

$$q(\xi)=\frac{a_1+4b_0}{b_1}-\frac{4}{\lambda[\sinh(\xi)+\cosh(\xi)]+1},$$

where $\xi=x+y+z-\frac{1}{9}(\alpha+\beta+\gamma-9)t$.

Case 2:

$$a_0=\frac{b_0}{b_1}(a_1+2b_0), a_2=-2b_1, k=\frac{1}{9}(\alpha+\beta+\gamma-9).$$

Plugging these values into Eq.(15), hence the solution for the Eq.(12) is

$$q(\xi)=\frac{a_1+2b_0}{b_1}-\frac{2}{\lambda[\sinh(\xi)+\cosh(\xi)]+1},$$

where $\xi=x+y+z-\frac{1}{9}(\alpha+\beta+\gamma-9)t$.

Case 3:

$$a_0=\frac{b_0}{b_1}(a_1+2b_1+4b_0), a_2=-2b_1, k=\frac{1}{9}(\alpha+\beta+\gamma-9).$$

Plugging these values into Eq.(15), hence the solution for the Eq.(12) is

$$q(\xi)=\frac{a_1+2b_1+4b_0}{b_1}-\frac{2}{\lambda[\sinh(\xi)+\cosh(\xi)]+1},$$

where $\xi=x+y+z-\frac{1}{9}(\alpha+\beta+\gamma-9)t$.

The unified method:

Now we utilize the unified method to the Eq.(12). Suppose that

$$u(x,y,z,t)=q(\xi),\quad \xi=x+y+z-\frac{kt^{\alpha}}{\Gamma(1+\alpha)} \tag{16}$$

where k is wave velocity. Substituting Eq.(13) into Eq.(12) reduces to the nonlinear ODE

$$(\alpha+\beta+\gamma-5-9k)q'+q^{(5)}+15q'q'''-5q'''-15(q')^2+15(q')^3=0 \tag{17}$$

where $q'=\frac{dq}{d\xi}$. By applying the method of homogeneous balancing, specifically by balancing the $q^{(5)}$ term and the $(q')^3$ term in Eq. (14), we determine that $M=1$. Consequently, from Eq. (8), we obtain the following expression

$$q(\xi)=a_0+a_1U(\xi)+\frac{b_1}{U(\xi)}. \tag{18}$$

Next, we substitute Eq. (18) into Eq. (17) and rearrange all terms such that the coefficients of $U^i(\xi)$ $(i=0,1,\ldots,12)$ are set to zero. This results in a system of algebraic equations. By solving these equations using mathematical software, a set of solutions for λ,k,b_1,a_0,a_1 is obtained.

Case 1:

$$a_1=-2, b_1=2\lambda, k=\frac{1}{9}(\alpha+\beta+\gamma+80\lambda+256\lambda^2-5).$$

Substituting these results into Eq.(18), we reach following the results:

(a) When $\lambda<0$:

$$q_{11}(\xi)=a_0-2U(\xi)+\frac{2\lambda}{U(\xi)}$$

$$q_{11}(\xi)=a_0+\frac{-2\sqrt{-(A^2+B^2)\lambda}+2A\sqrt{-\lambda}cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)}{Asinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+B}+\frac{2\lambda Asinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+2\lambda B}{\sqrt{-(A^2+B^2)\lambda}-A\sqrt{-\lambda}cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)},$$

$$q_{12}(\xi)=a_0-\frac{2\sqrt{(A^2-B^2)\lambda}+2A\sqrt{\lambda}cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)}{Asin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+B}+\frac{2\lambda Asin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+2\lambda B}{2\sqrt{(A^2-B^2)\lambda}+2A\sqrt{\lambda}cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)},$$

$$q_{13}(\xi)=a_0-2\sqrt{-\lambda}+\frac{4A\sqrt{-\lambda}}{A+cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)-sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)}+\frac{2\lambda\left(A+cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)-sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)\right)}{-A\sqrt{-\lambda}+\sqrt{-\lambda}cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)-\sqrt{-\lambda}sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)}$$

$$q_{14}(\xi)=a_0+2\sqrt{-\lambda}-\frac{4A\sqrt{-\lambda}}{A+cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)}+\frac{2\lambda\left(A+cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)+sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)\right)}{A\sqrt{-\lambda}-\sqrt{-\lambda}cosh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)-\sqrt{-\lambda}sinh\left(2\sqrt{-\lambda}(\xi+\xi_0)\right)},$$

(b) When $\lambda>0$:

$$q_{15}(\xi)=a_0+\frac{-2\sqrt{(A^2-B^2)\lambda}+2A\sqrt{\lambda}cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)}{Asin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+B}+\frac{2\lambda Asin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+2\lambda B}{-2\sqrt{(A^2-B^2)\lambda}+2A\sqrt{\lambda}cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)},$$

$$q_{16}(\xi)=a_0+\frac{2\sqrt{(A^2-B^2)\lambda}+2A\sqrt{\lambda}cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)}{Asin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+B}-\frac{2\lambda Asin\left(2\sqrt{\lambda}(\xi+\xi_0)\right)+2\lambda B}{\sqrt{(A^2-B^2)\lambda}+A\sqrt{\lambda}cos\left(2\sqrt{\lambda}(\xi+\xi_0)\right)},$$

,

$$q_{17}(\xi) = a_0 - 2i\sqrt{\lambda} + \frac{4Ai\sqrt{\lambda}}{A + cos\left(2\sqrt{\lambda}(\xi + \xi_0)\right) - i\,sin\left(2\sqrt{\lambda}(\xi + \xi_0)\right)} + \frac{2\lambda A + 2\lambda cos\left(2\sqrt{\lambda}(\xi + \xi_0)\right) - i\,2\lambda\,sin\left(2\sqrt{\lambda}(\xi + \xi_0)\right)}{i\sqrt{\lambda}\,cos\left(2\sqrt{\lambda}(\xi + \xi_0)\right) + \sqrt{\lambda}\,sin\left(2\sqrt{\lambda}(\xi + \xi_0)\right) - Ai\sqrt{\lambda}},$$

$$q_{18}(\xi) = a_0 + 2i\sqrt{\lambda} - \frac{4A\sqrt{\lambda}}{A + cos\left(2\sqrt{\lambda}(\xi + \xi_0)\right) + i\,sin\left(2\sqrt{\lambda}(\xi + \xi_0)\right)} + \frac{2\lambda A + 2\lambda cos\left(2\sqrt{\lambda}(\xi + \xi_0)\right) + i\,2\lambda\,sin\left(2\sqrt{\lambda}(\xi + \xi_0)\right)}{Ai\sqrt{\lambda} - i\sqrt{\lambda}\,cos\left(2\sqrt{\lambda}(\xi + \xi_0)\right) + \sqrt{\lambda}\,sin\left(2\sqrt{\lambda}(\xi + \xi_0)\right)},$$

(b) When $\lambda = 0$;

(c)

$$q_{19}(\xi) = a_0 + \frac{2}{\xi + \xi_0} - \lambda(\xi + \xi_0)\,,$$

where $\xi = x + y + z - \frac{1}{9}(\alpha + \beta + \gamma + 80\lambda + 256\lambda^2 - 5)t$.

Other cases of solutions can be obtained in a similar manner to the above case; however, these are omitted here for simplicity.

Case 2:

$$a_1 = -2, b_1 = 0, k = \frac{1}{9}(\alpha + \beta + \gamma + 20\lambda + 16\lambda^2 - 5).$$

Case 3:

$$a_1 = 0, b_1 = 2\lambda, k = \frac{1}{9}(\alpha + \beta + \gamma + 20\lambda + 16\lambda^2 - 5).$$

Case 4:

$$a_1 = -2, b_1 = -\frac{1}{2}, k = \frac{1}{9}\left(\alpha + \beta + \gamma - \frac{35}{4}\right), \lambda = 0.$$

Case 5:

$$a_1 = -4, b_1 = 0, k = \frac{1}{9}(\alpha + \beta + \gamma - 9), \lambda = -\frac{1}{4}.$$

Case 6:

$$a_1 = -4, b_1 = -\frac{1}{4}, k = \frac{1}{9}(\alpha + \beta + \gamma - 9), \lambda = -\frac{1}{16}.$$

Case 7:

$$a_1 = 0, b_1 = -1, k = \frac{1}{9}(\alpha + \beta + \gamma - 9), \lambda = -\frac{1}{4}.$$

4. Conclusion

In this study, we have explored the extended B-type Kadomtsev-Petviashvili (B-KP) equation, a crucial nonlinear partial differential equation that models wave interactions in multidimensional media. By introducing the real parameters α, β and γ, we extended the classical B-KP equation, offering a more general framework for understanding complex wave phenomena.

Through the application of a wave transformation, we reduced the high-dimensional PDE to an ordinary differential equation, which allowed us to use two advanced analytical methods-namely, the generalized Kudryashov method and the unified method. These methods facilitated the derivation of exact traveling wave solutions, providing valuable insight into the dynamic behavior of the system under different conditions.

The results obtained not only contribute to the theoretical understanding of wave dynamics in nonlinear systems but also pave the way for further research into more complex and generalized models. The analytical solutions obtained can be applied to a range of physical contexts, such as fluid dynamics, plasma physics, and other areas where wave phenomena play a crucial role. Future work can extend this approach by considering additional physical effects and further generalizations of the B-KP equation.

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