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Deferred statistical convergence of sequences in octonion-valued metric spaces

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Abstract. By employing octonions, which offer a higher-dimensional and non-associative algebraic structure, octonion-valued metric spaces generalize conventional metric spaces. Every ring forms a module over itself, and every field forms a vector space over itself, as is commonly known. It should be noted, nevertheless, that octonions do not form a module over themselves and so cannot even be regarded as a ring because they lack the multiplicative union condition. The metric spaces we have defined and the findings produced in these spaces are very intriguing because of this aspect. Consequently, various conclusions pertaining to summability theory are examined utilizing some essential concepts associated with these mathematical structures. In particular, we present the concepts of deferred statistical convergence and deferred strong Cesàro summability in octonion-valued metric spaces and explore the connections between them. Additionally, we introduce and discuss the concepts of strong deferred invariant convergence, deferred invariant convergence in octonion-valued metric spaces, and deferred invariant statistical convergence.

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1. Introduction

In 1843, John T. Graves introduced octonions shortly after the groundbreaking discovery of quaternions by Hamilton. Arthur Cayley independently refined and extended this concept. The transition from real numbers to complex numbers, quaternions, and ultimately octonions exemplifies a systematic progression in hypercomplex number theory, governed by the Cayley-Dickson construction. This construction method successively increases the dimensionality, moving from the one-dimensional real numbers to the two-dimensional complex numbers, followed by the four-dimensional quaternions, and culminating in the eight-dimensional octonions. Each stage in this sequence reveals more sophisticated algebraic structures and properties, which contribute to new mathematical insights and potential applications.

Octonions are distinguished in this progression due to their unique mathematical properties. Unlike real and complex numbers, which are commutative, and quaternions, which are non-commutative but still associative, octonions exhibit neither commutativity nor associativity. Their non-associative nature means that the order of multiplication affects the result, such that $(ab)c \neq a(bc)$. This deviation prevents octonions from fitting into standard algebraic frameworks and instead classifies them within a broader category known as alternative algebras. These algebras relax the standard associativity, characterized by the Moufang identities, to accommodate the structure of octonions and similar algebraic systems.

The Cayley-Dickson construction, which extends quaternions to octonions, defines their distinctive multiplication rules. These rules are frequently illustrated using the Fano plane, a graphical representation that visually captures the relationships between the basis elements of the octonion space.

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This diagram serves as a useful tool in mathematical applications, offering a clear depiction of how the basis elements interact during multiplication. It highlights the intricate structure and properties of octonions, making it easier to understand their non-commutative and non-associative nature.

Beyond their theoretical significance, the unique non-associative structure of octonions proves to be valuable in applications that involve handling multidimensional data interactions. According to Kansu et al. [20], octonions are utilized in physics to develop duality-invariant field equations for dyons. These equations, similar to Maxwell's equations, efficiently represent electric-magnetic dualities. The multi-component nature of octonions, with their eight dimensions, enables the capture of complex relationships between electric and magnetic components in a unified framework.

In the field of machine learning, octonions have proven to be a valuable tool for representing and processing high-dimensional data. Wu et al. [41] introduced deep octonion networks (DONs), which leverage the compact structure of octonions to integrate multi-dimensional features across different layers of neural networks. Within this framework, octonions facilitate efficient data representation and processing, with tasks such as image classification showing improved performance and convergence.

Moreover, Takahashi et al. [40] extended the application of octonions to control systems, particularly for dynamic control of robotic manipulators. In this scenario, octonion-valued neural networks capture both spatial and temporal dynamics. Their non-associative nature enables the network to model complex multi-axis movements required for precise manipulator control.

Thus, although the non-associative and non-commutative properties of octonions initially posed challenges for traditional algebraic applications, they have enabled innovative uses in modern theoretical physics, artificial intelligence, and control systems, where multi-dimensionality and flexible data representation are essential.

For in-depth information on octonions, their subalgebraic structures, and interdisciplinary applications, one can refer to the works by [2, 4, 7, 9, 32].

Fast's groundbreaking research on statistical convergence, as detailed in [12], has profoundly impacted various scientific fields, shaping both methodological approaches and theoretical advancements. This seminal contribution, frequently cited in subsequent studies [5, 14–19, 24, 25, 30, 35, 42], has laid a solid groundwork for analyzing convergence properties in mathematical sequences and related areas. Connor [6] highlighted the relationship between statistical convergence and the strong p-Cesàro convergence of sequences. Additionally, several researchers have investigated the principles of invariant mean and invariant convergence [27–29, 34, 37–39]. Savaş and Nuray [36] introduced the concepts of σ-statistical convergence and its lacunary variation, establishing related correlation theorems.

Agnew [1] introduced the concept of the deferred Cesàro mean as an enhancement of the usual Cesàro mean, aiming to develop a summation technique with more robust characteristics. The deferred Cesàro mean, formulated through the sequences $(\mathfrak{n}_\mathfrak{u})$ and $(\mathfrak{m}_\mathfrak{u})$, serves as an advanced variation of the usual Cesàro mean, offering superior summation functionality. Notably, while the usual Cesàro mean incorporates results such as the Silverman-Toeplitz theorem, the deferred version—viewed as a generalization—exhibits an additional significant property: for every $k \in \mathbb{N}$, $a_{\mathfrak{u},k} = 0$ for almost all $\mathfrak{u} \in \mathbb{N}$. This feature renders the deferred Cesàro mean particularly effective in managing lower triangular matrix methods, transforming bounded sequences into convergent ones. Its broader applicability, when compared to other summation techniques, arises from these unique advantages. Küçükaslan and Yılmaztürk [22] developed the framework for deferred statistical convergence. Additional details can be found in [11, 21, 31].

In this study, we extend certain fundamental ideas, such statistical convergence and convergence, to octonion-valued metric spaces, which were initially created by Çetin et al. [8], Mursaleen et al. [26], and Quan et al. [33]. We present the essential ideas associated with this special mathematical structure, such as deferred strong Cesàro summability and deferred statistical convergence, by developing a partial order relation on octonions. We may investigate these ideas' characteristics and the relationships between them since they are generalized in the context of octonion-valued metric spaces. Furthermore, we investigate the effects of the non-associative structure of octonions on the behaviour of deferred strong Cesàro summability and deferred statistical convergence. By taking use of the rich and intricate algebraic features of octonions, octonion-valued metric spaces offer a higher-dimensional and non-associative framework that extends standard metric spaces. Octonions are of special relevance when applied to these specified metric spaces since they do not possess the multiplicative associativity feature as conventional vector spaces or rings do. This special characteristic not only sheds further light on convergence theory but also draws attention to the novel opportunities and difficulties presented by octonions in mathematical study.

2. Preliminaries

In the follow-up we will to examine on $\mathbb{O}$, Octonions, a non-associative generalization of the division algebra of quaternions.

In this section, we will begin by extending the basis elements of quaternions, represented as $\{1, i, j, k\}$, by incorporating an additional basis element ℓ. This extension enables us to construct the eight-dimensional octonion division algebra in detail, as described in [13], including its diagrammatic representation and algebraic operations.

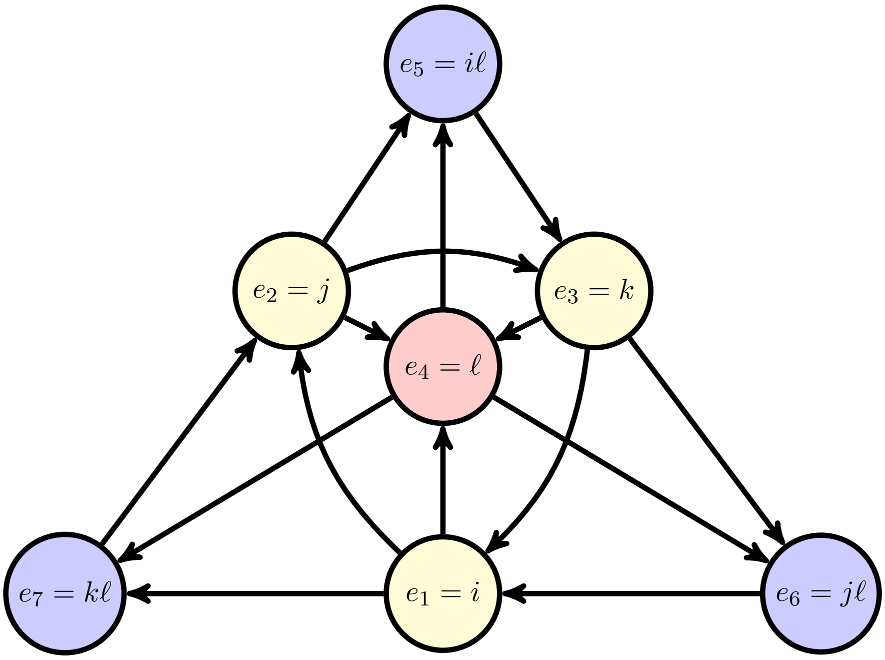

Figure 1. Diagram

Thus, each element $\mathbb{o} \in \mathbb{O}$ can be expressed in the form:

$$\mathbb{o} = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7, \quad o_n \in \mathbb{R}, \quad \text{where } n = 0, 1, 2, 3, 4, 5, 6, 7.$$

The basis elements of $\mathbb{O}$ are $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$. The detailed multiplication of these basis elements is shown in the table below.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The conjugate element $\bar{\mathbb{o}}$ is given by

$$\bar{\mathbb{o}} = o_0 - o_1e_1 - o_2e_2 - o_3e_3 - o_4e_4 - o_5e_5 - o_6e_6 - o_7e_7.$$

The norm of an arbitrary octonion is calculated as

$$\|\mathbb{o}\| = \sqrt{\mathbb{o} \cdot \bar{\mathbb{o}}} = \sqrt{o_0^2 + o_1^2 + o_2^2 + o_3^2 + o_4^2 + o_5^2 + o_6^2 + o_7^2}.$$

Additionally, the inverse of an arbitrary octonion $\mathbb{o}$ is given in the form

$$\mathbb{o}^{-1} = \frac{\bar{\mathbb{O}}}{\|\mathbb{o}\|^2}.$$

Any quaternion's imaginary part can be represented as a vector in three-dimensional Euclidean space, analogous to a movement vector, with its real part indicating the time of this movement. Similarly, octonions can be redefined in a seven-dimensional Euclidean space as a pair consisting of a scalar and a vector, allowing for a different perspective. While quaternions differ from real and complex numbers in their non-commutative multiplication, octonions, as a more complex structure, lose the associative property from the group axioms in multiplication. Consequently, division algebra over octonions becomes non-associative, adding to its intriguing properties.

We can represent octonions as an ordered set of eight real numbers $(o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ with coordinate-wise addition and multiplication defined by a specific table. Here, the first component, o_0, is called the real part, while the remaining seven-tuple $(o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ constitutes the imaginary part.

Thus, as noted above, any quaternion can be written in the form $\left(o_0, \overrightarrow{u}\right)$,where $\overrightarrow{u} = (o_1, o_2, o_3, o_4, o_5, o_6, o_7)$ and o_0 represents the real part. From here, the following properties can be easily observed:

$$\begin{aligned} \mathbb{o} : &= \left(o_0, \overrightarrow{u}\right), \quad \overrightarrow{u} \in \mathbb{R}^7; \quad o_0 \in \mathbb{R} \\ &= \left(o_0, (o_1, o_2, o_3, o_4, o_5, o_6, o_7)\right); \quad o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in \mathbb{R} \\ &= o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7. \end{aligned}$$

Now, let us define a partial ordering relation $\preceq$ on the non-associative and non-commutative octonion algebra $\mathbb{O}$ as follows.

$\mathbb{o} \preceq \mathbb{o}'$ if and only if $\mathrm{Re}(\mathbb{o}) \leq \mathrm{Re}(\mathbb{o}')$, $\mathrm{Im}_e(\mathbb{o}) \leq \mathrm{Im}_e(\mathbb{o}')$, $\mathbb{o}, \mathbb{o}' \in \mathbb{H}$; $e = e_1, e_2, e_3, e_4, e_5, e_6, e_7$, where $\mathrm{Im}_{e_n} = o_n; n = 1, 2, 3, 4, 5, 6, 7$. To confirm that it is $\mathbb{o} \preceq \mathbb{o}'$, satisfying any one of the 256 conditions derived from the sum of all possible combinations of 8, from 0 to 8 in respectively, will suffice.

Obtained from the 0-combinations of 8, meaning none of its components are equal; this 1 case constitute

(1) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

Obtained from the 1-combinations of 8, meaning only one component is equal; these 8 cases constitute

(2) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.
(3) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$.
(4) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$.
(5) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$.
(6) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.
(7) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(8) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(9) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.

Obtained from the 2-combinations of 8, meaning only two components are equal; these 27 cases constitute

(10) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$.
(11) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$.
(12) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$.
(13) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.
(14) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(15) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(16) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(17) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$.
(18) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$.
(19) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.
(20) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(21) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(22) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_1}(\mathbb{o}) = \mathrm{Im}_{e_1}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(23) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$.
(24) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 3, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.

(25) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 3, 4, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(26) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 3, 4, 5, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(27) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 3, 4, 5, 6$; $\mathrm{Im}_{e_2}(\mathbb{o}) = \mathrm{Im}_{e_2}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(28) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 5, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$.
(29) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 4, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(30) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 4, 5, 7$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(31) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 4, 5, 6$; $\mathrm{Im}_{e_3}(\mathbb{o}) = \mathrm{Im}_{e_3}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(32) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 3, 6, 7$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$.
(33) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 3, 5, 7$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(34) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 3, 5, 6$; $\mathrm{Im}_{e_4}(\mathbb{o}) = \mathrm{Im}_{e_4}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.
(35) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 3, 4, 7$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$; $\mathrm{Im}_{e_6}(\mathbb{o}) = \mathrm{Im}_{e_6}(\mathbb{o}')$.
(36) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) < \mathrm{Im}_{e_n}(\mathbb{o}')$, $n = 1, 2, 3, 4, 6$; $\mathrm{Im}_{e_5}(\mathbb{o}) = \mathrm{Im}_{e_5}(\mathbb{o}')$; $\mathrm{Im}_{e_7}(\mathbb{o}) = \mathrm{Im}_{e_7}(\mathbb{o}')$.

Following a similar approach, we can easily list the 56 cases where exactly 3 components are equal (derived from the 3-combinations of 8), 70 cases with 4 equal components, 56 cases with 5 equal components, and 27 cases with 6 equal components. However, to avoid making the article overly tedious, we will not elaborate in detail on the remaining 211 intermediate cases. For simplicity, let us focus only on the 8 cases with exactly 7 equal components, corresponding to the 7-combinations of 8 where just one component differs.

(248) $\mathrm{Re}(\mathbb{o}) < \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.
(249) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 2, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_1}(\mathbb{o}) < \mathrm{Im}_{e_1}(\mathbb{o}')$.
(250) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 3, 4, 5, 6, 7$; $\mathrm{Im}_{e_2}(\mathbb{o}) < \mathrm{Im}_{e_2}(\mathbb{o}')$.
(251) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 4, 5, 6, 7$; $\mathrm{Im}_{e_3}(\mathbb{o}) < \mathrm{Im}_{e_3}(\mathbb{o}')$.
(252) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 5, 6, 7$; $\mathrm{Im}_{e_4}(\mathbb{o}) < \mathrm{Im}_{e_4}(\mathbb{o}')$.
(253) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 6, 7$; $\mathrm{Im}_{e_5}(\mathbb{o}) < \mathrm{Im}_{e_5}(\mathbb{o}')$.
(254) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 7$; $\mathrm{Im}_{e_6}(\mathbb{o}) < \mathrm{Im}_{e_6}(\mathbb{o}')$.
(255) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6$; $\mathrm{Im}_{e_7}(\mathbb{o}) < \mathrm{Im}_{e_7}(\mathbb{o}')$.

Finally, let us consider the case derived from the 8-combinations of 8, where all corresponding components are equal, which indicates that the two octonions are identical.

(256) $\mathrm{Re}(\mathbb{o}) = \mathrm{Re}(\mathbb{o}')$; $\mathrm{Im}_{e_n}(\mathbb{o}) = \mathrm{Im}_{e_n}(\mathbb{o}')$, where $n = 1, 2, 3, 4, 5, 6, 7$.

Specifically, if $\|\mathbb{o}\| \neq \|\mathbb{o}'\|$ and any condition between (1) and (256) is satisfied,$\mathbb{o} \preceq \mathbb{o}'$ will be written. If only condition (256) is satisfied, we will denote this by $\mathbb{o} \prec \mathbb{o}'$. We will briefly denote this situation as

$$\mathbb{o} \preceq \mathbb{o}' \implies \|\mathbb{o}\| \leq \|\mathbb{o}'\|. \tag{1}$$

A careful examination of the 256 conditions above reveals that we can introduce octonion-valued metric spaces, which generalize the complex metric spaces defined by Azam and colleagues [3], by taking the codomain as the field of complex numbers.

Definition 1. *[3] Given a non-empty set S. If the transformation $\Omega_{\mathbb{C}} : S \times S \to \mathbb{C}$ on this set satisfies following conditions,*

(1) $0_{\mathbb{C}} \preceq \Omega_{\mathbb{C}}(s,t)$, *for all $s, t \in S$ and* $\Omega_{\mathbb{C}}(s,t) = 0_{\mathbb{C}} \iff s = t$.
(2) $\Omega_{\mathbb{C}}(s,t) = \Omega_{\mathbb{C}}(t,s)$ *for all $s, t \in S$.*
(3) $\Omega_{\mathbb{C}}(s,t) \preceq \Omega_{\mathbb{C}}(s,v) + \Omega_{\mathbb{C}}(v,t)$ *for all $s, t, v \in S$.*

Then the pair $(S, \Omega_{\mathbb{C}})$ is said to be a complex metric space.

These are then generalized to quaternion-valued metric spaces, as defined by Ahmed et al. [10], taking the codomain as the skew field of quaternions, which serve as a non-commutative extension of these metric spaces to Clifford algebra analysis.

Definition 2. *[10] Given a nonempty set S. If the transformation $\Omega_{\mathbb{H}} : S \times S \to \mathbb{H}$ on this set satisfies following conditions,*

(1) $0_{\mathbb{H}} \preceq \Omega_{\mathbb{H}}(s,t)$ *for all $s, t \in S$ and* $\Omega_{\mathbb{H}}(s,t) = 0_{\mathbb{H}} \iff s = t$,
(2) $\Omega_{\mathbb{H}}(s,t) = \Omega_{\mathbb{H}}(t,s)$ *for all $s, t \in S$,*
(3) $\Omega_{\mathbb{H}}(s,t) \preceq \Omega_{\mathbb{H}}(s,v) + \Omega_{\mathbb{H}}(v,t)$ *for all $s, t, v \in S$.*

Then $\Omega_{\mathbb{H}}$ is said to be a quaternion-valued metric on S, and the pair $(S, \Omega_{\mathbb{H}})$ is said to be a quaternion valued metric space.

Following, we will define octonion-valued metric spaces, an interesting generalization of metric spaces that are neither commutative nor associative.

The definitions, examples, theorems, and propositions in this section are taken from [8, 26, 33].

Definition 3. *Given a nonempty set S. If the transformation $\Omega_{\mathbb{O}} : S \times S \to \mathbb{O}$ on this set satisfies following conditions,*

(1) $0_{\mathbb{O}} \preceq \Omega_{\mathbb{O}}(s,t)$ *for all* $s,t \in S$ *and* $\Omega_{\mathbb{O}}(s,t) = 0_{\mathbb{O}}$ *if and only if* $s = t$,
(2) $\Omega_{\mathbb{O}}(s,t) = \Omega_{\mathbb{O}}(t,s)$ *for all* $s,t \in S$,
(3) $\Omega_{\mathbb{O}}(s,t) \preceq \Omega_{\mathbb{O}}(s,v) + \Omega_{\mathbb{O}}(v,t)$ *for all* $s,t,v \in S$.

Then $\Omega_{\mathbb{O}}$ is called be an octanion valued metric on S, and the pair $(S,\Omega_{\mathbb{O}})$ is called be an octanion valued metric space.

Example 1. *Let $\Omega_{\mathbb{O}} : \mathbb{O} \times \mathbb{O} \to \mathbb{O}$ be an octanion valued function defined as* $\Omega_{\mathbb{O}}(\mathbb{o},\mathbb{o}') = |o_0 - o_0'| + |o_1 - o_1'|e_1 + |o_2 - o_2'|e_2 + |o_2 - o_2'|e_2 + |o_3 - o_3'|e_3 + |o_4 - o_4'|e_4 + |o_5 - o_5'|e_5 + |o_6 - o_6'|e_6 + |o_7 - o_7'|e_7$, *where $\mathbb{o},\mathbb{o}' \in \mathbb{O}$ with*

$$\mathbb{o} = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7,$$
$$\mathbb{o}' = o_0' + o_1'e_1 + o_2'e_2 + o_3'e_3 + o_4'e_4 + o_5'e_5 + o_6'e_6 + o_7'e_7;$$
$$o_i, o_i' \in \mathbb{R};\ i = 0,1,2,3,4,5,6,7.$$

Then $(\mathbb{O},\Omega_{\mathbb{O}})$ defines an octanion valued metric space.

Below, we provide an example of an octonion-valued metric that does not have a known numerical set as its domain.

Example 2. *Let $X = \{a,b,c\}$ be an arbitrary set with three elements. Define the distances between the elements of the set by*

$$\Omega_{\mathbb{O}}(a,b) = \Omega_{\mathbb{O}}(b,a) = 3 + 4e_1 - 6e_2 + 4e_3 + 3e_4 + 3e_5 - 2e_6 + e_7$$
$$\Omega_{\mathbb{O}}(b,c) = \Omega_{\mathbb{O}}(c,b) = 1 + 2e_1 + 3e_3 - 5e_4 - 3e_6 + 4e_7$$
$$\Omega_{\mathbb{O}}(a,c) = \Omega_{\mathbb{O}}(c,a) = 2 + 3e_1 + e_2 + e_3 - 2e_4 + 2e_5 - e_6 + 5e_7$$
$$\Omega_{\mathbb{O}}(a,a) = \Omega_{\mathbb{O}}(b,b) = \Omega_{\mathbb{O}}(c,c) = 0 + 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6 + 0e_7.$$

Since they are $\|\Omega_{\mathbb{O}}(a,b)\| = 10$, $\|\Omega_{\mathbb{O}}(a,c)\| = 7$, $\|\Omega_{\mathbb{O}}(c,b)\| = 8$, $\|\Omega_{\mathbb{O}}(a,b) + \Omega_{\mathbb{O}}(a,c)\| = \sqrt{195}$, $\|\Omega_{\mathbb{O}}(a,b) + \Omega_{\mathbb{O}}(b,c)\| = \sqrt{200}$ *and* $\|\Omega_{\mathbb{O}}(c,b) + \Omega_{\mathbb{O}}(a,c)\| = \sqrt{169} = 13$, *it can be seen through straightforward calculations that the conditions given in Definition 3 above are satisfied.*

The definition we offered is a logical extension of the classical metric concept, as well as complex and quaternion-valued metrics, as the definitions and examples above demonstrate. Let's put out the following claims to illustrate the relationships between them.

Using Diagram in Figure 1, Definition 1, 2, and 3, we can give the following propositions.

Proposition 1. *Every quaternion-valued metric space can be embedded into an octonion-valued metric space.*

Proposition 2. *Every complex-valued metric space can be embedded into a quaternion-valued metric space and an octonion-valued metric space.*

Proposition 3. *Every metric space can be embedded into a complex-valued metric space, a quaternion-valued metric space and an octonion-valued metric space.*

Thus, we can now move on to define some basic concepts related to the above definition from the works [8], [26] and [33].

Definition 4. *Any point $s \in S$ is called be an interior point of set $A \subset S$ whenever there exists $0_{\mathbb{O}} \prec r \in \mathbb{O}$ such that*

$$B(s,r) = \{t \in S : \Omega_{\mathbb{O}}(s,t) \prec r\} \subset A.$$

Definition 5. *Any point $s \in S$ is called be a limit point of $A \subset S$ whenever for every $0_{\mathbb{O}} \prec r \in \mathbb{O}$*

$$B(s,r) \cap (A - \{s\}) \neq \emptyset.$$

Definition 6. *Set O is said to be an open set whenever each element of O is an interior point of O. Subset $C \subset S$ is called a closed set whenever each limit point of C belongs to C. The family*

$$F = \{B(s,r) : s \in S, 0_{\mathbb{O}} \prec r\}$$

is a subbase for Hausdorff topology τ on S.

Definition 7. *Let $s \in S$ and s_k be a sequence in the set S. If for each $\mathbb{o} \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \mathbb{o}$ there is $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_k, s) \prec \mathbb{o}$, then (s_k) is called convergence sequence. Then, in this case (s_k) sequence converges to the limit point s; as notation, $s_k \to s$ as $k \to \infty$ or $\lim_k s_k = s$.*

Definition 8. *If there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0, \Omega_{\mathbb{O}}(s_{k+m}, s_k) \prec \mathbb{o}$, then (s_k) is said to be a Cauchy sequence in the octanion-valued metric space $(S, \Omega_{\mathbb{O}})$. If every Cauchy sequence is convergent in $(S, \Omega_{\mathbb{O}})$, then $(S, \Omega_{\mathbb{O}})$ is said to be a complete octanion valued metric space.*

Definition 9. *A sequence (s_k) in an octonion-valued metric space $(S, \Omega_{\mathbb{O}})$ is said to converge statistically to a point $s \in S$ (denoted as $s_k \xrightarrow{stg} s$), if as for all $0_{\mathbb{O}} \prec \mathbb{o}$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{k \le N : \Omega_{\mathbb{O}}(s_k, s) \not\prec \mathbb{o}\right\}\right| = 0.$$

Above, we provide some definitions and statements on the concepts of convergence and statistical convergence in these special mathematical structures, as well as the previously established octonion-valued metric spaces. For more results, interested readers can see [8], [26], and [33].

Let σ be a mapping defined as $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$. A continuous linear functional Φ on l_∞, the space of real bounded sequences, is known as an invariant mean or a σ mean, if it meets the following criteria:
(1) $\Phi(\omega_\alpha) \ge 0$, when the sequence (ω_α) has $\omega_\alpha \ge 0$ for $\forall \alpha \in \mathbb{N}$;
(2) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$;
(3) $\Phi\left(\omega_{\sigma(\mathfrak{u})}\right) = \Phi(\omega_{\mathfrak{u}})$ for all $(\omega_{\mathfrak{u}}) \in l_\infty$.

The mappings Φ is assumed to be injective, ensuring $\sigma^{\mathfrak{m}}(\mathfrak{u}) \ne \mathfrak{u}$ for all $\mathfrak{u}, \mathfrak{m} \in \mathbb{Z}^+$, where $\sigma^{\mathfrak{m}}(\mathfrak{u})$ denotes the $\mathfrak{m}$th iterate of the mapping σ at $\mathfrak{u}$. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(\omega_{\mathfrak{u}}) = \lim_{\mathfrak{u}} \omega$, for all $(\omega_{\mathfrak{u}}) \in c$.

In the case where σ represents the translation mapping $\sigma(\mathfrak{u}) = \mathfrak{u}+1$, the σ-mean is known as a Banach limit.

The space V_σ, comprising bounded sequences whose invariant means coincide, can be shown as

$$V_\sigma = \left\{(\omega_\alpha) \in l_\infty : \lim_{\mathfrak{m}\to\infty} \frac{1}{\mathfrak{m}} \sum_{\alpha=1}^{\mathfrak{m}} \omega_{\sigma^\alpha(\mathfrak{t})} = L\right\}$$

uniformly in $\mathfrak{t}$.

In 1932, Agnew [1] introduced the concept of deferred Cesàro mean of real (or complex) valued sequences $\omega = (\omega_\alpha)$ defined by

$$\left(D_{\mathfrak{m},\mathfrak{n}}\omega\right)_{\mathfrak{u}} := \frac{1}{\mathfrak{n}(\mathfrak{u}) - \mathfrak{m}(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \omega_\alpha, \ \mathfrak{u} = 1, 2, 3...$$

where $\{\mathfrak{m}(\mathfrak{u})\}$ and $\{\mathfrak{n}(\mathfrak{u})\}$ are the sequences of non-negative integers satisfying

$$\mathfrak{m}(\mathfrak{u}) < \mathfrak{n}(\mathfrak{u}), \ \varpi(\mathfrak{u}) = \mathfrak{n}(\mathfrak{u}) - \mathfrak{m}(\mathfrak{u}) \text{ and } \lim_{n\to\infty} \mathfrak{n}(\mathfrak{u}) = \infty. \tag{2}$$

3. Deferred Statistical Convergence in Octonion-valued Metric Space

In this part, we expand the concepts of deferred statistical convergence and deferred strong Cesàro summability to octonion-valued metric space and analyze their connections.

Definition 10. *Let $(S, \Omega_{\mathbb{O}})$ represent an octonion-valued metric space, where $\omega \in S$ is a point, and $(\omega_\alpha) \subseteq S$ is a sequence. A sequence (ω_α) is said to be $DS_{\mathfrak{m},\mathfrak{n}}$-convergent (or deferred statistically convergent) ω if, for every $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$ such that*

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \left|\{\mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) \ \ (\mathfrak{u} \in \mathbb{N}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| = 0,$$

and denoted by $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ *or* $(\omega_\alpha)\overset{DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}}{\to}\omega$.

It is clear that;
(i) If $\mathfrak{n}(\mathfrak{u})=\mathfrak{u}$ and $\mathfrak{m}(\mathfrak{u})=0$, then deferred statistical convergence corresponds with statistical convergence in an octonion-valued metric space.
(ii) If we consider $\mathfrak{n}(\mathfrak{u})=k_{\mathfrak{u}}$ and $\mathfrak{m}(\mathfrak{u})=k_{\mathfrak{u}-1}$ (for any lacunary sequence of nonnegative integers with $k_{\mathfrak{u}}-k_{\mathfrak{u}-1}\to\infty$, as $\mathfrak{u}\to\infty$, then deferred statistical convergence corresponds with lacunary statistical convergence in an octonion-valued metric space.
(iii) If $\mathfrak{n}(\mathfrak{u})=\lambda_{\mathfrak{u}}$ and $\mathfrak{m}(\mathfrak{u})=0$ (where $\lambda_{\mathfrak{u}}$ is a strictly increasing sequence of natural numbers such that $\lim_{\mathfrak{u}\to\infty}\lambda_{\mathfrak{u}}=\infty$), then deferred λ-statistical convergence corresponds with λ-statistical convergence in an octonion-valued metric space.

Definition 11. *A sequence* (ω_α) *is said to be* $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}$*-summable (or deferred strongly-Cesàro summable) to* ω *such that*

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\varpi(\mathfrak{u})}\sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega)=0.$$

Here, we would write $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ *or* $(\omega_\alpha)\to\omega\left(Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}\right)$*. The set of all strongly Cesàro-summable sequences will be denoted by* $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}$.

If $\mathfrak{n}(\mathfrak{u})=\mathfrak{u}$ and $\mathfrak{m}(\mathfrak{u})=0$, for all $\mathfrak{u}\in\mathbb{N}$, then deferred strong Cesàro summability coincides strong Cesàro summability denoted by $w^{(S,\Omega_{\mathbb{O}})}$.
Also, if

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\varpi(\mathfrak{u})}\sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})}\left[\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\right]^{\mathfrak{s}}=0$$

holds, then (ω_α) is called to be deferred strongly $\mathfrak{s}$-Cesàro summable to ω. Here, we would write $Dw_{\mathfrak{m},\mathfrak{n}}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ or $(\omega_\alpha)\to\omega\left(Dw_{\mathfrak{m},\mathfrak{n}}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})}\right)$.
We can give the following theorem without proof.

Theorem 1. *Let* $\{\mathfrak{m}(\mathfrak{u})\}$ *and* $\{\mathfrak{n}(\mathfrak{u})\}$ *be sequences of non-negative integers satisfying the condition (2),* $(S,\Omega_{\mathbb{O}})$ *be an octonion-valued metric space, and* $(\omega_\alpha),(\gamma_\alpha)\subseteq S$ *be sequences, then*
(a) *If* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ *and* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\gamma_\alpha=\gamma$*, then* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}(\omega_\alpha+\gamma_\alpha)=\omega+\gamma$,
(b) *If* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ *and* $\beta\in\mathbb{C}$*, then* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\beta\omega_\alpha=\beta\omega$,
(c) *If* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$ *and* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\gamma_\alpha=\gamma$ *and* $(\omega_\alpha),(\gamma_\alpha)$ *are bounded in* $(S,\Omega_{\mathbb{O}})$ *(i.e.,* $(\omega_\alpha),(\gamma_\alpha)\in\ell_\infty^{(S,\Omega_{\mathbb{O}})}$*), then* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}(\omega_\alpha\gamma_\alpha)=\omega\gamma$.

Theorem 2. *If* $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$*, then* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$*. However, the opposite is untrue.*

Proof. Let $\varrho\in\mathbb{O}$ with $0_{\mathbb{O}}\prec\varrho$ and $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$. Then, we get

$$\begin{aligned}\frac{1}{\varpi(\mathfrak{u})}\sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega)&\geq\frac{1}{\varpi(\mathfrak{u})}\sum_{\substack{\alpha=\mathfrak{m}(\mathfrak{u})+1\\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega)+\frac{1}{\varpi(\mathfrak{u})}\sum_{\substack{\alpha=\mathfrak{m}(\mathfrak{u})+1\\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\\&\geq\varrho.\frac{1}{\varpi(\mathfrak{u})}\left|\{\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u})\ \ (\mathfrak{u}\in\mathbb{N}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|\end{aligned}$$

which gives the result. Hence, we have $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$.
For the converse, consider the sequence

$$\omega_\alpha:=\begin{cases}\alpha^2, & \left[\left|\sqrt{\mathfrak{n}(\mathfrak{u})}\right|\right]-\mathfrak{t}_0<\alpha\leq\left[\left|\sqrt{\mathfrak{n}(\mathfrak{u})}\right|\right],\ \mathfrak{u}=1,2,...\\ 0, & \text{if not.}\end{cases}$$

where $\mathfrak{n}(\mathfrak{u})$ is a monotone increasing sequence and $\mathfrak{t}_0 \neq 0$ is an arbitrary fixed natural number.
If we consider $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})}$ for the sequence $\mathfrak{m}(\mathfrak{u})$ satisfying $0 < \mathfrak{m}(\mathfrak{u}) \leq \left[\left|\sqrt{\mathfrak{n}(\mathfrak{u})}\right|\right] - \mathfrak{t}_0$, then for an arbitrary $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$, we have

$$\frac{1}{\varpi(\mathfrak{u})} \left|\{\mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) \ \ (\mathfrak{u} \in \mathbb{N}) : \Omega_{\mathbb{O}}(\omega_\alpha, 0) \not\prec \varrho\}\right| = \frac{\mathfrak{t}_0}{\varpi(\mathfrak{u})} \to 0, \text{ as } \mathfrak{u} \to \infty,$$

i.e., $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = 0$.

On the other hand,

$$\frac{1}{\varpi(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_\alpha, 0) \geq \frac{\mathfrak{t}_0 \left(\left[\left|\sqrt{\mathfrak{n}(\mathfrak{u})}\right|\right] - \mathfrak{t}_0\right)^2}{\varpi(\mathfrak{u})} \to \mathfrak{t}_0, \text{ as } \mathfrak{u} \to \infty,$$

i.e., $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha \neq 0$. It is also clear that the sequence does not convergent to zero in usual case. □

Theorem 3. *If* $(\omega_\alpha) \in \ell_\infty^{(S,\Omega_{\mathbb{O}})}$ *and* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$, *then* $Dw_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Let $(\omega_\alpha) \in \ell_\infty^{(S,\Omega_{\mathbb{O}})}$ and $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. Under the assumption on (ω_α), there exists a $\mathfrak{D} > 0$ such that $\Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \mathfrak{D}$. So, the inequality

$$\begin{aligned}
\frac{1}{\varpi(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_\alpha, \omega) &= \frac{1}{\varpi(\mathfrak{u})} \left(\sum_{\substack{\alpha=\mathfrak{m}(\mathfrak{u})+1 \\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} + \sum_{\substack{\alpha=\mathfrak{m}(\mathfrak{u})+1 \\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} \right) \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \\
&\leq \frac{1}{\varpi(\mathfrak{u})} \left(\mathfrak{D} \sum_{\substack{\alpha=\mathfrak{m}(\mathfrak{u})+1 \\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} 1 + \varrho \sum_{\substack{\alpha=\mathfrak{m}(\mathfrak{u})+1 \\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} 1 \right) \\
&\leq \frac{\mathfrak{D}}{\varpi(\mathfrak{u})} \left|\{\mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) \ \ (\mathfrak{u} \in \mathbb{N}) : \Omega_{\mathbb{O}}(\omega_\alpha, 0) \not\prec \varrho\}\right| \\
&\quad + \varrho . \frac{1}{\varpi(\mathfrak{u})} \left|\{\mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) \ \ (\mathfrak{u} \in \mathbb{N}) : \Omega_{\mathbb{O}}(\omega_\alpha, 0) \prec \varrho\}\right|
\end{aligned}$$

holds. From the limit relation we have $\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_\alpha, \omega) = 0$. So, the proof is completed. □

Theorem 4. *If the sequence* $\left\{\frac{\mathfrak{m}(\mathfrak{u})}{\varpi(\mathfrak{u})}\right\}_{\mathfrak{u}\in\mathbb{N}}$ *is bounded, then* $\mathcal{S}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Let's give a note about the sequences of positive natural numbers $(x_{\mathfrak{u}})$ and $(y_{\mathfrak{u}})$ without proof: if $\lim_{\mathfrak{u}\to\infty} x_{\mathfrak{u}} = x$, $\mathfrak{u} \in \mathbb{N}$ and $\lim_{\mathfrak{u}\to\infty} y_{\mathfrak{u}} = \infty$, then $\lim_{\mathfrak{u}\to\infty} x_{y_{\mathfrak{u}}} = x$.
From the assumption on (ω_α), the limit relation

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\mathfrak{u}} \left|\{\alpha : \alpha \leq \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| = 0,$$

holds for every $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$. Since the sequence $\mathfrak{n}(\mathfrak{u})$ satisfies (2), then the sequence

$$\left\{ \frac{\left|\{\alpha : \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right|}{\mathfrak{n}(\mathfrak{u})} \right\}_{\mathfrak{u}\in\mathbb{N}}$$

is convergent to zero. Therefore, the inclusion

$$\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\} \subset \{\alpha : \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\},$$

and the inequality

$$\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \leq \left|\{\alpha : \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right|,$$

are hold. From the last inequality we have

$$\begin{aligned}
&\frac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad \leq \left(1 + \frac{\mathfrak{m}(\mathfrak{u})}{\varpi(\mathfrak{u})}\right) \frac{1}{\mathfrak{n}(\mathfrak{u})} \left|\{\alpha : \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right|,
\end{aligned}$$

and from the limit relation we get $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$. □

Corollary 1. *Let $\{\mathfrak{n}(\mathfrak{u})\}$ be an arbitrary sequence with $\mathfrak{n}(\mathfrak{u}) < \mathfrak{u}$ for all $\mathfrak{u} \in \mathbb{N}$ and $\left\{\frac{\mathfrak{u}}{\varpi(\mathfrak{u})}\right\}_{\mathfrak{u}\in\mathbb{N}}$ be a bounded sequence. Then, $S^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$ implies $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$.*

Remark 1. *The converse of Theorem 4 is not true even if $\left\{\frac{\mathfrak{m}(\mathfrak{u})}{\varpi(\mathfrak{u})}\right\}_{\mathfrak{u}\in\mathbb{N}}$ is bounded.*

Example 3. *Let us consider $\mathfrak{m}(\mathfrak{u}) = 2\mathfrak{u}$; $\mathfrak{n}(\mathfrak{u}) = 4\mathfrak{u}$ and a sequence (ω_α) as*

$$\omega_\alpha := \begin{cases} \frac{\alpha+1}{2}, & \alpha \text{ is odd,} \\ -\frac{\alpha}{2}, & \alpha \text{ is even.} \end{cases}$$

It is clear that the assumption of Theorem 4 is fulfilled and $D^{(S,\Omega_{\mathbb{O}})}_{2\mathfrak{u},4\mathfrak{u}} - \lim_{\alpha\to\infty}\omega_\alpha = 0$. From Theorem 4 we get $DS^{(S,\Omega_{\mathbb{O}})}_{2\mathfrak{u},4\mathfrak{u}} - \lim_{\alpha\to\infty}\omega_\alpha = 0$. But, for an arbitrary small $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$,

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\mathfrak{u}}\left|\{\alpha : \alpha \le \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha, 0) \not\prec \varrho\}\right| \neq 0.$$

Theorem 5. *Let $\mathfrak{n}(\mathfrak{u}) = \mathfrak{u}$ for all $\mathfrak{u} \in \mathbb{N}$. Then, $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$ if and only if $S^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$.*

Proof. (⇒) Let us assume that $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$. We shall apply the technique which was used by Agnew in [1]. Then, for any $\mathfrak{u} \in \mathbb{N}$,

$$\mathfrak{m}(\mathfrak{u}) = \mathfrak{u}^{(1)} > \mathfrak{m}\left(\mathfrak{u}^{(1)}\right) = \mathfrak{u}^{(2)} > \mathfrak{m}\left(\mathfrak{u}^{(2)}\right) = \mathfrak{u}^{(3)} > ...,$$

and we may write the set $\{\alpha : \alpha \le \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}$ as

$$\begin{aligned}\{\alpha : \alpha \le \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\} &= \left\{\alpha : \alpha \le \mathfrak{u}^{(1)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} \\ &\cup \left\{\alpha : \mathfrak{u}^{(1)} < \alpha \le \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\},\end{aligned}$$

and the set $\left\{\alpha : 1 < \alpha \le \mathfrak{u}^{(1)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\}$ as

$$\begin{aligned}\left\{\alpha : 1 < \alpha \le \mathfrak{u}^{(1)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} &= \left\{\alpha : \alpha \le \mathfrak{u}^{(2)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} \\ &\cup \left\{\alpha : \mathfrak{u}^{(2)} < \alpha \le \mathfrak{u}^{(1)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\},\end{aligned}$$

and the set $\left\{\alpha : \alpha \le \mathfrak{u}^{(2)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\}$ as

$$\begin{aligned}\left\{\alpha : \alpha \le \mathfrak{u}^{(2)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} &= \left\{\alpha : \alpha \le \mathfrak{u}^{(3)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} \\ &\cup \left\{\alpha : \mathfrak{u}^{(3)} < \alpha \le \mathfrak{u}^{(2)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\},\end{aligned}$$

and if this process is continued we obtain

$$\begin{aligned}\left\{\alpha : \alpha \le \mathfrak{u}^{(\mathfrak{q}-1)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} &= \left\{\alpha : \alpha \le \mathfrak{u}^{(\mathfrak{q})} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\} \\ &\cup \left\{\alpha : \mathfrak{u}^{(\mathfrak{q})} < \alpha \le \mathfrak{u}^{(\mathfrak{q}-1)} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\}\end{aligned}$$

for a certain positive integer $\mathfrak{q} > 0$ depending on $\mathfrak{u}$ such that $\mathfrak{u}^{(\mathfrak{q})} \ge 1$ and $\mathfrak{u}^{(\mathfrak{q}-1)} = 0$. From the above discussion, the relation

$$\begin{aligned}&\tfrac{1}{\mathfrak{u}}\left|\{\alpha : \alpha \le \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\ &= \sum_{\mathfrak{j}=0}^{\mathfrak{q}} \tfrac{\mathfrak{u}^{(\mathfrak{j})}-\mathfrak{u}^{(\mathfrak{j}+1)}}{\mathfrak{u}} \tfrac{1}{\mathfrak{u}^{(\mathfrak{j})}-\mathfrak{u}^{(\mathfrak{j}+1)}} \left|\left\{\mathfrak{u}^{(\mathfrak{j}+1)} < \alpha \le \mathfrak{u}^{(\mathfrak{j})} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\}\right|\end{aligned}$$

holds for every $\mathfrak{u}$. This relation gives that statistical convergence of the sequence (ω_α) to ω is a linear combination of following sequence:

$$\left\{\frac{1}{\mathfrak{u}^{(\mathfrak{j})} - \mathfrak{u}^{(\mathfrak{j}+1)}}\left|\left\{\mathfrak{u}^{(\mathfrak{j}+1)} < \alpha \le \mathfrak{u}^{(\mathfrak{j})} : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\right\}\right|\right\}_{\mathfrak{j}\in\mathbb{N}}.$$

Let us consider the matrix

$$b_{\mathfrak{u},\mathfrak{j}} = \begin{cases} \frac{\mathfrak{u}^{(\mathfrak{j})}-\mathfrak{u}^{(\mathfrak{j}+1)}}{\mathfrak{u}}, & \mathfrak{j} = 0,1,2,..,\mathfrak{q}, \\ 0, & \text{if not.} \end{cases}$$

where $\mathfrak{u}^{(0)} = 0$.
The matrix $(b_{\mathfrak{u},\mathfrak{j}})$ is satisfied the Silverman Toeplitz theorem (see in [23]). So, we have

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\mathfrak{u}} \left|\{\alpha : \alpha \leq \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_\alpha, 0) \not\prec \varrho\}\right| = 0,$$

since

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\mathfrak{u}^{(\mathfrak{j})} - \mathfrak{u}^{(\mathfrak{j}+1)}} \left|\left\{\mathfrak{u}^{(\mathfrak{j}+1)} < \alpha \leq \mathfrak{u}^{(\mathfrak{j})} : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\right\}\right| = 0.$$

($\Leftarrow$) Since $\mathfrak{n}(\mathfrak{u}) = \mathfrak{u}$ is satisfied (2), then the inverse of the theorem is a simple consequence of Theorem 4. □

Corollary 2. *Assume that* $\{\mathfrak{n}(\mathfrak{u})\}_{\mathfrak{u}\in\mathbb{N}}$ *contains almost all positive integers. Then,* $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $S^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Let $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ for an arbitrary $\{\mathfrak{m}(\mathfrak{u})\}_{\mathfrak{u}\in\mathbb{N}}$ and choose sufficiently large positive integer $\mathfrak{g}$ such that the set $\{\mathfrak{n}(\mathfrak{u})\}_{\mathfrak{u}\in\mathbb{N}}$ contains all positive integers which is greater than $\mathfrak{g}$. Then, it can be constructed a sequence $(\mathfrak{t}_\alpha)$ as follows:

$$\mathfrak{t}_1 = \mathfrak{t}_2 = ... = \mathfrak{t}_{\mathfrak{g}} = 1$$

and for each $\alpha > \mathfrak{g}$ an index $\mathfrak{t}_\alpha$ such that $\mathfrak{n}_{\mathfrak{t}_\alpha} = \alpha$. It is clear from the construction that $(\mathfrak{t}_\alpha)$ is a monotone increasing sequence. So, from the assumption $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}_{\mathfrak{t}_\alpha},\mathfrak{n}_{\mathfrak{t}_\alpha}} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. Hence, the proof of Corollary follows from Theorem 5. □

Theorem 6. *Four sequences of non-negative real numbers,* $\{\mathfrak{m}(\mathfrak{u})\}$, $\{\mathfrak{n}(\mathfrak{u})\}$, $\{\mathfrak{m}'(\mathfrak{u})\}$ *and* $\{\mathfrak{n}'(\mathfrak{u})\}$, *should be considered such that*

$$\mathfrak{m}(\mathfrak{u}) < \mathfrak{m}'(\mathfrak{u}) < \mathfrak{n}'(\mathfrak{u}) < \mathfrak{n}(\mathfrak{u}) \text{ for all } \mathfrak{u} \in \mathbb{N}.$$

Then, the sets

$$\{\mathfrak{u} : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u})\} \text{ and } \{\mathfrak{u} : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u})\}$$

are finite sets for all $\mathfrak{u} \in \mathbb{N}$. *Then* $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Let us consider the sequence (ω_α) such that $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. For an arbitrary $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$, the equality

$$\begin{aligned}
&\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\} \\
&\quad = \{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\} \\
&\quad \cup \{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\} \\
&\quad \cup \{\alpha : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad = \tfrac{1}{\varpi(\mathfrak{u})} \{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\} \\
&\quad + \tfrac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad + \tfrac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right|,
\end{aligned}$$

are hold. On taking limits when $\mathfrak{u} \to \infty$; we obtain

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| = 0.$$

This proves our assertion. □

Theorem 7. *Four sequences of non-negative real numbers,* $\{\mathfrak{m}(\mathfrak{u})\}$, $\{\mathfrak{n}(\mathfrak{u})\}$, $\{\mathfrak{m}'(\mathfrak{u})\}$ *and* $\{\mathfrak{n}'(\mathfrak{u})\}$, *should be considered such that*

$$\mathfrak{m}(\mathfrak{u}) < \mathfrak{m}'(\mathfrak{u}) < \mathfrak{n}'(\mathfrak{u}) < \mathfrak{n}(\mathfrak{u}) \text{ for all } \mathfrak{u} \in \mathbb{N}. \tag{3}$$

Then, the sets

$$\{\mathfrak{u} : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u})\} \text{ and } \{\mathfrak{u} : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u})\}$$

are finite sets for all $\mathfrak{u} \in \mathbb{N}$. *Then* $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Let us consider the sequence (ω_α) such that $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'}-\lim_{\alpha\to\infty}\omega_\alpha=\omega$. For an arbitrary $\varrho\in\mathbb{O}$ with $0_{\mathbb{O}}\prec\varrho$, the equality

$$\begin{aligned}&\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\\&\quad=\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{m}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\\&\quad\cup\{\alpha:\mathfrak{m}'(\mathfrak{u})<\alpha\leq\mathfrak{n}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\\&\quad\cup\{\alpha:\mathfrak{n}'(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\end{aligned}$$

and

$$\begin{aligned}&\tfrac{1}{\varpi(\mathfrak{u})}\left|\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|\\&\quad=\tfrac{1}{\varpi(\mathfrak{u})}\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{m}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\\&\quad+\tfrac{1}{\varpi(\mathfrak{u})}\left|\{\alpha:\mathfrak{m}'(\mathfrak{u})<\alpha\leq\mathfrak{n}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|\\&\quad+\tfrac{1}{\varpi(\mathfrak{u})}\left|\{\alpha:\mathfrak{n}'(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|,\end{aligned}$$

are hold. On taking limits when $\mathfrak{u}\to\infty$; we obtain

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\varpi(\mathfrak{u})}\left|\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|=0.$$

This proves our assertion. □

Theorem 8. *Assume that (3) supplies. If*

$$\lim_{\mathfrak{u}\to\infty}\frac{\varpi(\mathfrak{u})}{\varpi'(\mathfrak{u})}=m>0,$$

then $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}}\subseteq DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'}$.

Proof. Assume that (3) supplies. It is evident that the inclusion

$$\begin{aligned}&\{\alpha:\mathfrak{m}'(\mathfrak{u})<\alpha\leq\mathfrak{n}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\\&\quad\subseteq\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\end{aligned}$$

and the inequality

$$\begin{aligned}&\left|\{\alpha:\mathfrak{m}'(\mathfrak{u})<\alpha\leq\mathfrak{n}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|\\&\quad\leq\left|\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|,\end{aligned}$$

hold true for given $\varrho\in\mathbb{O}$ with $0_{\mathbb{O}}\prec\varrho$. So, we obtain

$$\begin{aligned}&\tfrac{1}{\varpi'(\mathfrak{u})}\left|\{\alpha:\mathfrak{m}'(\mathfrak{u})<\alpha\leq\mathfrak{n}'(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|\\&\quad\leq\tfrac{\varpi(\mathfrak{u})}{\varpi'(\mathfrak{u})}\tfrac{1}{\varpi(\mathfrak{u})}\left|\{\alpha:\mathfrak{m}(\mathfrak{u})<\alpha\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho\}\right|.\end{aligned}$$

Therefore, we have $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}}\subseteq DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'}$. □

Theorem 9. *Four sequences of non-negative real numbers,* $\{\mathfrak{m}(\mathfrak{u})\}$, $\{\mathfrak{n}(\mathfrak{u})\}$, $\{\mathfrak{m}'(\mathfrak{u})\}$ *and* $\{\mathfrak{n}'(\mathfrak{u})\}$, *should be considered such that*

$$\mathfrak{m}'(\mathfrak{u})<\mathfrak{m}(\mathfrak{u})<\mathfrak{n}(\mathfrak{u})<\mathfrak{n}'(\mathfrak{u})\ \text{for all}\ \mathfrak{u}\in\mathbb{N}.\tag{4}$$

Let ρ,σ *be fixed real numbers such that* $0<\rho\leq\sigma\leq1$, *then*
(I) If

$$\lim_{\mathfrak{u}\to\infty}\frac{[\varpi(\mathfrak{u})]^{\rho}}{[\varpi'(\mathfrak{u})]^{\sigma}}=m>0,\tag{5}$$

then $DS^{\sigma,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'}\subseteq DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}}$.
(II) If

$$\lim_{\mathfrak{u}\to\infty}\frac{\varpi'(\mathfrak{u})}{[\varpi(\mathfrak{u})]^{\sigma}}=1,\tag{6}$$

then $DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}}\subseteq DS^{\sigma,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'}$.

Proof. (I) Omitted.
(II) Let Equation (6) hold and $DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$. Then, for given $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$, we have

$$\begin{aligned}
&\tfrac{1}{[\varpi(\mathfrak{u})]^\sigma}\left|\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \le \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&\le \tfrac{1}{[\varpi(\mathfrak{u})]^\sigma}\left|\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \le \mathfrak{m}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&+ \tfrac{1}{[\varpi(\mathfrak{u})]^\sigma}\left|\{\alpha : \mathfrak{n}(\mathfrak{u}) < \alpha \le \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&+ \tfrac{1}{[\varpi(\mathfrak{u})]^\sigma}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&\le \tfrac{\mathfrak{m}(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})+\mathfrak{n}'(\mathfrak{u})-\mathfrak{n}(\mathfrak{u})}{[\varpi(\mathfrak{u})]^\sigma} + \tfrac{1}{[\varpi(\mathfrak{u})]^\sigma}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&\le \tfrac{\left(\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})\right)-\left(\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})\right)}{[\varpi(\mathfrak{u})]^\sigma} + \tfrac{1}{[\varpi(\mathfrak{u})]^\sigma}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&= \tfrac{\varpi'(\mathfrak{u})-\varpi(\mathfrak{u})}{\varpi(\mathfrak{u})} + \tfrac{1}{[\varpi(\mathfrak{u})]^\rho}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&\le \left(\tfrac{\varpi'(\mathfrak{u})}{\varpi(\mathfrak{u})} - 1\right) + \tfrac{1}{[\varpi(\mathfrak{u})]^\rho}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right|.
\end{aligned}$$

On taking limit when $\mathfrak{u}\to\infty$, we have $DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} \subseteq DS^{\sigma,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'}$. □

Theorem 10. *Assume that (4) be satisfied. Let ρ,σ be fixed real numbers such that $0<\rho\le\sigma\le 1$, then*
(I) Let (5) holds. If $Dw^{\sigma,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$, then $DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty} S_\alpha = \omega$.
(II) Let (6) holds and assume that $(\omega_\alpha)\in\ell_\infty^{(S,\Omega_{\mathbb{O}})}$. If $DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$, then $Dw^{\sigma,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$.

Proof. (I) Omitted.
(II) Assume that $DS^{\rho,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$ and $(\omega_\alpha)\in\ell_\infty^{(S,\Omega_{\mathbb{O}})}$. Under the assumption on (ω_α), there exists a $\mathfrak{D}>0$ such that $\Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\mathfrak{D}$. We can write

$$\begin{aligned}
&\tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\alpha\in\mathfrak{m}'(\mathfrak{u})+1}^{\mathfrak{n}'(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) = \tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\alpha=\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) + \tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) \\
&\le \tfrac{\left(\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})\right)-\left(\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})\right)}{[\varpi'(\mathfrak{u})]^\sigma}\mathfrak{D} + \tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) \\
&\le \tfrac{\left(\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})\right)-\left(\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})\right)^\sigma}{[\varpi'(\mathfrak{u})]^\sigma}\mathfrak{D} + \tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) \\
&\le \left(\tfrac{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})}{[\varpi'(\mathfrak{u})]^\sigma} - 1\right)\mathfrak{D} + \tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\substack{\alpha\in\mathfrak{m}(\mathfrak{u})+1 \\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\not\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) + \tfrac{1}{[\varpi'(\mathfrak{u})]^\sigma}\sum_{\substack{\alpha\in\mathfrak{m}(\mathfrak{u})+1 \\ \Omega_{\mathbb{O}}(\omega_\alpha,\omega)\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})}\Omega_{\mathbb{O}}(\omega_\alpha,\omega) \\
&\le \left(\tfrac{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})}{[\varpi'(\mathfrak{u})]^\sigma} - 1\right)\mathfrak{D} + \tfrac{1}{[\varpi'(\mathfrak{u})]^\rho}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| + \varrho.\tfrac{\varpi'(\mathfrak{u})}{[\varpi'(\mathfrak{u})]^\sigma}
\end{aligned}$$

On taking limit when $\mathfrak{u}\to\infty$, we have $Dw^{\sigma,(S,\Omega_{\mathbb{O}})}_{\mathfrak{m}',\mathfrak{n}'} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$. □

Let us consider $\mathfrak{m}' = \{\mathfrak{m}'(\mathfrak{u})\}$ and $\mathfrak{n}' = \{\mathfrak{n}'(\mathfrak{u})\}$ which are satisfying

$$\mathfrak{m}(\mathfrak{u}) \le \mathfrak{m}'(\mathfrak{u}) < \mathfrak{n}'(\mathfrak{u}) \le \mathfrak{n}(\mathfrak{u}) \tag{7}$$

for every $\mathfrak{u}\in\mathbb{N}$ besides (2). Denote by the associated set $E := \{\mathfrak{m}(\mathfrak{u}) : \mathfrak{u}\in\mathbb{N}\}$, $E' := \{\mathfrak{m}'(\mathfrak{u}) : \mathfrak{u}\in\mathbb{N}\}$, $F := \{\mathfrak{n}(\mathfrak{u}) : \mathfrak{u}\in\mathbb{N}\}$ and $F' := \{\mathfrak{n}'(\mathfrak{u}) : \mathfrak{u}\in\mathbb{N}\}$.

Theorem 11. *If the set $F'\diagdown F$ is finite and $\lim_{\mathfrak{u}\to\infty}\frac{\mathfrak{n}(\mathfrak{u})-\mathfrak{n}'(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})} < \infty$ holds. Then, $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$ implies $DS^{(S,\Omega_{\mathbb{O}})}_{\mathfrak{m},\mathfrak{n}'} - \lim_{\alpha\to\infty}\omega_\alpha = \omega$.*

Proof. Since $F'\diagdown F$ is finite, then there is an $\mathfrak{u}_0\in\mathbb{N}$ such that the inclusion $\{\mathfrak{n}'(\mathfrak{u}) : \mathfrak{u} > \mathfrak{u}_0\} \subset \{\mathfrak{n}(\mathfrak{u}) : \mathfrak{u}\in\mathbb{N}\}$ holds. So, there is a strictly increasing sequence $t = \{t(\mathfrak{u})\}$ such that $\mathfrak{n}'(\mathfrak{u}) = \mathfrak{n}(t(\mathfrak{u}))$ for every $\mathfrak{u}\ge\mathfrak{u}_0$. Therefore, sufficiently large $\mathfrak{u}\in\mathbb{N}$, following inequality

$$\begin{aligned}
&\tfrac{1}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&= \tfrac{1}{\mathfrak{n}(t(\mathfrak{u}))-\mathfrak{m}(\mathfrak{u})}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(t(\mathfrak{u})) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&\le \tfrac{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})}\tfrac{1}{\varpi(\mathfrak{u})}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right| \\
&= \left(\tfrac{\mathfrak{n}(\mathfrak{u})-\mathfrak{n}'(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})} + 1\right)\tfrac{1}{\varpi(\mathfrak{u})}\left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \le \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha,\omega) \not\prec \varrho\}\right|
\end{aligned}$$

holds. Under the assumption we have desired result. □

Theorem 12. *If the set* $F \diagdown F'$ *is finite and* $\lim_{\mathfrak{u}\to\infty} \inf \frac{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})}{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})} > 0$ *hold. Then,* $DS_{\mathfrak{m},\mathfrak{n}'}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. This can be demonstrated by applying Theorem 11. Therefore, the proof is omitted here. □

Corollary 3. *If* $F \triangle F'$ *is finite, then* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *iff* $DS_{\mathfrak{m},\mathfrak{n}'}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Theorem 13. *If* $E' \diagdown E$ *is a finite set and* $\lim_{\mathfrak{u}\to\infty} \inf \frac{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})}{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})} > 0$ *hold. Then,* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *iff* $DS_{\mathfrak{m}',\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. The same concept used in Theorem 11 may be used to prove it. So, it is omitted here. □

Theorem 14. *The sequence* $\mathfrak{m}'(n)$ *and* $\mathfrak{n}'(n)$ *are satisfying (7) such that the set*

$$\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u})\}$$

and

$$\{\alpha : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u})\}$$

are finite. Then, $DS_{\mathfrak{m}',\mathfrak{n}'}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Assume that $DS_{\mathfrak{m}',\mathfrak{n}'}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. So, for an arbitrary $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$, we have the following inequality

$$\begin{aligned}
&\tfrac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad \leq \tfrac{1}{\varpi'(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad + \tfrac{1}{\varpi'(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad + \tfrac{1}{\varpi'(\mathfrak{u})} \left|\{\alpha : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad \leq \tfrac{w_1}{\varpi'(\mathfrak{u})} + \tfrac{1}{\varpi'(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| + \tfrac{w_2}{\varpi'(\mathfrak{u})},
\end{aligned}$$

where

$$w_1 := \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{m}'(\mathfrak{u})\}\right|,$$

and

$$w_2 := \left|\{\alpha : \mathfrak{n}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u})\}\right|.$$

On taking limit when $\mathfrak{u} \to \infty$ we have

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| = 0,$$

thus $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. □

Theorem 15. *If the sequence* $\mathfrak{m}'(\mathfrak{u})$ *and* $\mathfrak{n}'(\mathfrak{u})$ *are satisfying (7) such that*

$$\lim_{\mathfrak{u}\to\infty} \frac{\mathfrak{n}(\mathfrak{u}) - \mathfrak{m}(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u}) - \mathfrak{m}'(\mathfrak{u})} = 0 \tag{8}$$

then, $DS_{\mathfrak{m},\mathfrak{n}}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS_{\mathfrak{m}',\mathfrak{n}'}^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. It is clear from (7) that the inclusion

$$\begin{aligned}
&\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\} \\
&\quad \subseteq \{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}
\end{aligned}$$

and the inequality

$$\begin{aligned}
&\tfrac{1}{\varpi'(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}'(\mathfrak{u}) < \alpha \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right| \\
&\quad \leq \tfrac{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})} \cdot \tfrac{1}{\varpi(\mathfrak{u})} \left|\{\alpha : \mathfrak{m}(\mathfrak{u}) < \alpha \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\prec \varrho\}\right|
\end{aligned}$$

hold. After taking limit when $\mathfrak{u} \to \infty$ and (8) the desired result is obtained. □

Theorem 16. *Under the assumption of Theorem 14,* $Dw_{\mathfrak{m}',\mathfrak{n}'}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $Dw_{\mathfrak{m},\mathfrak{n}}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *for any bounded* (ω_α).

Proof. Given that $(\omega_\alpha) \in \ell_\infty^{(S,\Omega_{\mathbb{O}})}$, we can express $\Omega_{\mathbb{O}}(\omega_\alpha, \omega) \not\precsim Q$ for any $Q \in \mathbb{R}^+$. Then, we obtain

$$\frac{1}{\varpi(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}} = \frac{1}{\varpi(\mathfrak{u})} \begin{bmatrix} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{m}'(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}} \\ \sum_{\alpha=\mathfrak{m}'(\mathfrak{u})+1}^{\mathfrak{n}'(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}} \\ \sum_{\alpha=\mathfrak{n}'(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}} \end{bmatrix}$$

$$= \frac{2}{\varpi'(\mathfrak{u})} Q^{\mathfrak{s}} O(1) + \frac{1}{\varpi'(\mathfrak{u})} \sum_{\alpha=\mathfrak{n}'(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}}.$$

Hence, we get $Dw_{\mathfrak{m},\mathfrak{n}}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. □

Theorem 17. *Let $\{\mathfrak{m}(\mathfrak{u})\}, \{\mathfrak{n}(\mathfrak{u})\}, \{\mathfrak{m}'(\mathfrak{u})\}$ and $\{\mathfrak{n}'(\mathfrak{u})\}$ be sequences of non-negative integers satisfying (7) and (8), then $Dw_{\mathfrak{m},\mathfrak{n}}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ implies $Dw_{\mathfrak{m}',\mathfrak{n}'}^{\mathfrak{s},(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.*

Proof. It is evident that the inequality

$$\frac{1}{\varpi(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}} \geq \frac{1}{\varpi(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}'(\mathfrak{u})+1}^{\mathfrak{n}'(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}}$$

$$\geq \frac{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})}{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})} \frac{1}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})} \sum_{\alpha=\mathfrak{m}'(\mathfrak{u})+1}^{\mathfrak{n}'(\mathfrak{u})} [\Omega_{\mathbb{O}}(\omega_\alpha, \omega)]^{\mathfrak{s}}$$

holds true. Therefore, by taking the limit as $\mathfrak{u}$ approaches infinity, the desired result is achieved. □

4. Deferred Invariant and Deferred Invariant Statistical Convergence in Octonion-valued Metric Space

We present the ideas of strongly deferred invariant convergence and deferred invariant statistical convergence in an octonion-valued metric space. We also explore how these concepts relate to one another.

Definition 12. *Let $(S, \Omega_{\mathbb{O}})$ be an octonion-valued metric space, $\omega \in S$ be a point, and $(\omega_\alpha) \subseteq S$ be a sequence. A sequence (ω_α) is said to be strongly deferred $\Omega_{\mathbb{O}}$-invariant convergent to ω if*

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \sum_{\mathfrak{h}=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) = 0,$$

uniformly in $\mathfrak{t}$. Here, we write $D_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ or $\omega_\alpha \to \omega \left(D_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$.

Definition 13. *A sequence (ω_α) is said to be strongly $\mathfrak{f}$-deferred invariant convergent to ω if*

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \sum_{\mathfrak{h}=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \left[\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega)\right]^{\mathfrak{f}} = 0,$$

uniformly in $\mathfrak{t}$ where $0 < \mathfrak{f} < \infty$. In this case, we write $D_\sigma^{\mathfrak{f},(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ or $\omega_\alpha \to \omega \left(D_\sigma^{\mathfrak{f},(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$.

The following can be observed:
(a) When $\mathfrak{n}(\mathfrak{u}) = \mathfrak{u}$ and $\mathfrak{m}(\mathfrak{u}) = 0$, then Definition 12 aligns with the concept of strong invariant convergence in an octonion-valued metric space,
(b) If we consider $\mathfrak{n}(\mathfrak{u}) = k_{\mathfrak{u}}$ and $\mathfrak{m}(\mathfrak{u}) = k_{\mathfrak{u}-1}$, then Definition 12 corresponds to strong lacunary invariant convergence in octonion-valued metric space,
(c) If $\mathfrak{n}(\mathfrak{u}) = \mathfrak{u}$ and $\mathfrak{m}(\mathfrak{u}) = \mathfrak{u} - \lambda_{\mathfrak{u}}$, then Definition 12 matches the definition of strong λ-invariant convergence in ctonion-valued metric space.

Definition 14. *A sequence* (ω_α) *is said to be* $\Omega_{\mathbb{O}}$*-invariant statistically convergent to* ω *if for every* $\varrho \in \mathbb{O}$ *with* $0_{\mathbb{O}} \prec \varrho$,

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\mathfrak{u}} \left|\left\{\mathfrak{h} : \mathfrak{h} \leq \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right| = 0,$$

uniformly in $\mathfrak{t}$*. In this case we write* $S_\sigma^{(S,\Omega_{\mathbb{O}})} - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *or* $\omega_\alpha \to \omega \left(S_\sigma^{(S,\Omega_{\mathbb{O}})}\right)$.

Definition 15. *A sequence* (ω_α) *is said to be deferred* $\Omega_{\mathbb{O}}$*-invariant statistically convergent to* ω *if for every* $\varrho \in \mathbb{O}$ *with* $0_{\mathbb{O}} \prec \varrho$,

$$\lim_{\mathfrak{u}\to\infty} \frac{1}{\varpi(\mathfrak{u})} \left|\left\{\mathfrak{h} : \mathfrak{m}(\mathfrak{u}) < \mathfrak{h} \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right| = 0,$$

uniformly in $\mathfrak{t}$*. In this case we write* $DS_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *or* $\omega_\alpha \to \omega \left(DS_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$.

Clearly;

($\mathfrak{I}$) If $\mathfrak{n}(\mathfrak{u}) = \mathfrak{u}$ and $\mathfrak{m}(\mathfrak{u}) = 0$, then Definition 15 aligns with the definition of invariant statistical convergence in octonion-valued metric space,

($\mathfrak{II}$) If we consider $\mathfrak{n}(\mathfrak{u}) = k_{\mathfrak{u}}$ and $\mathfrak{m}(\mathfrak{u}) = k_{\mathfrak{u}-1}$, then Definition 15 corresponds to the lacunary invariant statistical convergence in octonion-valued metric space,

($\mathfrak{III}$) If $\mathfrak{n}(\mathfrak{u}) = \mathfrak{u}$ and $\mathfrak{m}(\mathfrak{u}) = \mathfrak{u} - \lambda_{\mathfrak{u}}$, then Definition 15 is equivalent to the invariant λ-statistical convergence of sequences in octonion-valued metric space.

Theorem 18. *Let* $\{\mathfrak{m}(\mathfrak{u})\}$, $\{\mathfrak{n}(\mathfrak{u})\}$, $\{\mathfrak{m}'(\mathfrak{u})\}$ *and* $\{\mathfrak{n}'(\mathfrak{u})\}$ *be sequences of non-negative integers satisfying* $\mathfrak{m}(\mathfrak{u}) \leq \mathfrak{m}'(\mathfrak{u}) < \mathfrak{n}'(\mathfrak{u}) \leq \mathfrak{n}(\mathfrak{u})$ *for all* $\mathfrak{u} \in \mathbb{N}$ *and*

$$\limsup_{\mathfrak{u}\to\infty} \frac{\mathfrak{n}(\mathfrak{u}) - \mathfrak{m}(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u}) - \mathfrak{m}'(\mathfrak{u})} < \infty$$

then $DS_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $DS_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m}',\mathfrak{n}'] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. From the inclusion

$$\begin{aligned}&\left\{\mathfrak{h} : \mathfrak{m}'(\mathfrak{u}) < \mathfrak{h} \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\\ &\quad\subseteq \left\{\mathfrak{h} : \mathfrak{m}(\mathfrak{u}) < \mathfrak{h} \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\},\end{aligned}$$

we can write

$$\begin{aligned}&\tfrac{1}{\varpi'(\mathfrak{u})} \left|\left\{\mathfrak{h} : \mathfrak{m}'(\mathfrak{u}) < \mathfrak{h} \leq \mathfrak{n}'(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right|\\ &\quad\leq \tfrac{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u})-\mathfrak{m}'(\mathfrak{u})} \tfrac{1}{\varpi(\mathfrak{u})} \left|\left\{\mathfrak{h} : \mathfrak{m}(\mathfrak{u}) < \mathfrak{h} \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right|.\end{aligned}$$

Following the take-limit at $\mathfrak{u} \to \infty$, the intended outcome is achieved. □

Theorem 19. *Let* $\{\mathfrak{m}(\mathfrak{u})\}$, $\{\mathfrak{n}(\mathfrak{u})\}$, $\{\mathfrak{m}'(\mathfrak{u})\}$ *and* $\{\mathfrak{n}'(\mathfrak{u})\}$ *be sequences of of non-negative integers satisfying* $\mathfrak{m}(\mathfrak{u}) \leq \mathfrak{m}'(\mathfrak{u}) < \mathfrak{n}'(\mathfrak{u}) \leq \mathfrak{n}(\mathfrak{u})$ *for all* $\mathfrak{u} \in \mathbb{N}$ *and*

$$\limsup_{\mathfrak{u}\to\infty} \frac{\mathfrak{n}(\mathfrak{u}) - \mathfrak{m}(\mathfrak{u})}{\mathfrak{n}'(\mathfrak{u}) - \mathfrak{m}'(\mathfrak{u})} < \infty$$

then $D_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$ *implies* $D_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m}',\mathfrak{n}'] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. We omit the proof since it is identical to the Theorem 18's proof. □

Theorem 20. *If* (ω_α) *is strongly deferred* $\Omega_{\mathbb{O}}$*-invariant convergent to* ω*, then* (ω_α) *is deferred* $\Omega_{\mathbb{O}}$*-invariant statistically convergent to* ω*, that is, if* $D_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$*, then* $DS_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$.

Proof. Assume $D_\sigma^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}] - \lim_{\alpha\to\infty} \omega_\alpha = \omega$. For an arbitrary $\varrho \in \mathbb{O}$ with $0_{\mathbb{O}} \prec \varrho$, we have

$$\begin{aligned}&\tfrac{1}{\varpi(\mathfrak{u})} \sum_{\mathfrak{h}=\mathfrak{m}(\mathfrak{u})+1}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) = \tfrac{1}{\varpi(\mathfrak{u})} \sum_{\substack{\mathfrak{h}=\mathfrak{m}(\mathfrak{u})+1\\ \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\not\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega)\\ &+\tfrac{1}{\varpi(\mathfrak{u})} \sum_{\substack{\mathfrak{h}=\mathfrak{m}(\mathfrak{u})+1\\ \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \geq \tfrac{1}{\varpi(\mathfrak{u})} \sum_{\substack{\mathfrak{h}=\mathfrak{m}(\mathfrak{u})+1\\ \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\not\prec\varrho}}^{\mathfrak{n}(\mathfrak{u})} \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega)\\ &\geq \tfrac{\varrho}{\varpi(\mathfrak{u})} \left|\left\{\mathfrak{h} : \mathfrak{m}(\mathfrak{u}) < \mathfrak{h} \leq \mathfrak{n}(\mathfrak{u}) : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right|\end{aligned}$$

for all $\mathfrak{t}$. So, we obtain

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\varpi(\mathfrak{u})}\left|\left\{\mathfrak{h}:\mathfrak{m}(\mathfrak{u})<\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|=0$$

uniformly in $\mathfrak{t}$, that is $DS_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]-\lim_{\alpha\to\infty}\omega_{\alpha}=\omega$. □

Theorem 21. *If (ω_α) is bounded and deferred $\Omega_{\mathbb{O}}$-invariant statistically convergent to ω, then (ω_α) is strongly deferred $\Omega_{\mathbb{O}}$-invariant convergent to ω, that is, if (ω_α) bounded and $\omega_\alpha\to\omega\left(DS_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$, then $\omega_\alpha\to\omega\left(D_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$.*

Proof. Omitted. □

Theorem 22. *If $\lim_{\mathfrak{u}\to\infty}\frac{\mathfrak{m}(\mathfrak{u})}{\varpi(\mathfrak{u})}=\mathfrak{s}>0$ and $\mathfrak{n}(\mathfrak{u})<\mathfrak{u}$, then $\omega_\alpha\to\omega\left(S_{\sigma}^{(S,\Omega_{\mathbb{O}})}\right)$ implies $\omega_\alpha\to\omega\left(DS_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$.*

Proof. Let $\omega_\alpha\to\omega\left(S_{\sigma}^{(S,\Omega_{\mathbb{O}})}\right)$ then for every $\varrho\in\mathbb{O}$ with $0_{\mathbb{O}}\prec\varrho$,

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\mathfrak{u}}\left|\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|=0,$$

uniformly in $\mathfrak{t}$. So, for all $\varrho\in\mathbb{O}$ with $0_{\mathbb{O}}\prec\varrho$,

$$\lim_{\mathfrak{u}\to\infty}\frac{1}{\mathfrak{n}(\mathfrak{u})}\left|\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|=0$$

uniformly in $\mathfrak{t}$. From the inclusion

$$\begin{aligned}&\left\{\mathfrak{h}:\mathfrak{m}(\mathfrak{u})<\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\\&\subseteq\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\end{aligned}$$

and the inequality

$$\begin{aligned}&\left|\left\{\mathfrak{h}:\mathfrak{m}(\mathfrak{u})<\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|\\&\leq\left|\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|\end{aligned}$$

we have

$$\begin{aligned}&\tfrac{1}{\varpi(\mathfrak{u})}\left|\left\{\mathfrak{h}:\mathfrak{m}(\mathfrak{u})<\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|\\&=\tfrac{\mathfrak{n}(\mathfrak{u})-\mathfrak{m}(\mathfrak{u})+\mathfrak{m}(\mathfrak{u})}{\varpi(\mathfrak{u})}\tfrac{1}{\mathfrak{n}(\mathfrak{u})}\left|\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|\\&\leq\left(1+\tfrac{\mathfrak{m}(\mathfrak{u})}{\varpi(\mathfrak{u})}\right)\tfrac{1}{\mathfrak{n}(\mathfrak{u})}\left|\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{n}(\mathfrak{u}):\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\right|\end{aligned}$$

for all $\mathfrak{t}$ and we deduce $\omega_\alpha\to\omega\left(DS_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$. □

Theorem 23. *Assume $\mathfrak{n}(\mathfrak{u})=\mathfrak{u}$ for all $\mathfrak{u}\in\mathbb{N}$. Then, $\omega_\alpha\to\omega\left(DS_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$ iff $\omega_\alpha\to\omega\left(S_{\sigma}^{(S,\Omega_{\mathbb{O}})}\right)$.*

Proof. Assume that $\omega_\alpha\to\omega\left(DS_{\sigma}^{(S,\Omega_{\mathbb{O}})}[\mathfrak{m},\mathfrak{n}]\right)$. Using the method that Agnew in [1] used, we can write

$$\begin{aligned}\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}&=\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(1)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\\&\cup\left\{\mathfrak{h}:\mathfrak{u}^{(1)}<\mathfrak{h}\leq\mathfrak{u}^{(2)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\},\\ \left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(1)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}&=\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(2)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\\&\cup\left\{\mathfrak{h}:\mathfrak{u}^{(2)}<\mathfrak{h}\leq\mathfrak{u}^{(1)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\},\end{aligned}$$

for each $\mathfrak{t}\in\mathbb{N}$, by allowing $\mathfrak{m}(\mathfrak{u})=\mathfrak{u}^{(1)}$, $\mathfrak{m}\left(\mathfrak{u}^{(1)}\right)=\mathfrak{u}^{(2)}$, $\mathfrak{m}\left(\mathfrak{u}^{(2)}\right)=\mathfrak{u}^{(3)}$, and

$$\begin{aligned}\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(2)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}&=\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(3)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\\&\cup\left\{\mathfrak{h}:\mathfrak{u}^{(3)}<\mathfrak{h}\leq\mathfrak{u}^{(2)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\},\end{aligned}$$

for each $\mathfrak{t}\in\mathbb{N}$. This procedure may be carried out again until, for each positive integer $\mathfrak{g}$ that depends on $\mathfrak{u}$,

$$\begin{aligned}\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(\mathfrak{g}-1)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}&=\left\{\mathfrak{h}:\mathfrak{h}\leq\mathfrak{u}^{(\mathfrak{g})}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\}\\&\cup\left\{\mathfrak{h}:\mathfrak{u}^{(\mathfrak{g})}<\mathfrak{h}\leq\mathfrak{u}^{(\mathfrak{g}-1)}:\Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})},\omega)\nprec\varrho\right\},\end{aligned}$$

is obtained for any $\mathfrak{t}$ in which $\mathfrak{u}^{(\mathfrak{g})} \geq 1$ and $\mathfrak{u}^{(\mathfrak{g}+1)} = 0$. Therefore, we can write

$$\frac{1}{\mathfrak{u}}\left|\left\{\mathfrak{h} : \mathfrak{h} \leq \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right| = \sum_{\mathfrak{r}=0}^{\mathfrak{g}} \frac{\mathfrak{u}^{(\mathfrak{r})} - \mathfrak{u}^{(\mathfrak{r}+1)}}{\mathfrak{u}} y_{\mathfrak{r}\mathfrak{t}}$$

for every $\mathfrak{u}$ and $\mathfrak{t}$, where

$$y_{\mathfrak{r}\mathfrak{t}} := \frac{1}{\mathfrak{u}^{(\mathfrak{r})} - \mathfrak{u}^{(\mathfrak{r}+1)}}\left|\left\{\mathfrak{u}^{(\mathfrak{r}+1)} < \mathfrak{h} \leq \mathfrak{u}^{(\mathfrak{r})} : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right|.$$

If we consider a matrix $A = (a_{\mathfrak{u}\mathfrak{r}})$ as

$$a_{\mathfrak{u}\mathfrak{r}} = \begin{cases} \frac{\mathfrak{u}^{(\mathfrak{r})} - \mathfrak{u}^{(\mathfrak{r}+1)}}{\mathfrak{u}}, & \mathfrak{r} = 0, 1, 2, \ldots, \mathfrak{g} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathfrak{u}^{(0)} = \mathfrak{u}$, then the sequence

$$\left\{\frac{1}{\mathfrak{u}}\left|\left\{\mathfrak{h} : \mathfrak{h} \leq \mathfrak{u} : \Omega_{\mathbb{O}}(\omega_{\sigma^{\mathfrak{h}}(\mathfrak{t})}, \omega) \not\prec \varrho\right\}\right|\right\}_{\mathfrak{u}\in\mathbb{N}}$$

is the $(a_{\mathfrak{u}\mathfrak{r}})$ transformation of the sequence $(y_{\mathfrak{r}\mathfrak{t}})$. Since the matrix $A = (a_{\mathfrak{u}\mathfrak{r}})$ satisfies Silverman -Toeplitz Theorem (see in [23]) and from the assumption on $(a_{\mathfrak{u}\mathfrak{r}})$, then we have the desired result. □

5. Conclusion

In this work, we have presented and examined two new ideas in octonion-valued metric space: deferred statistical convergence and deferred strong Cesàro summability. Understanding the relationships between these concepts inside these specific regions was the main goal of our investigation. First, we defined deferred statistical convergence and deferred strong Cesàro summability in the framework of octonion-valued metric spaces. After that, we examined the relationships between these ideas, emphasizing both their similarities and contrasts. We further expanded our study by presenting and studying the concepts of deferred invariant convergence in octonion-valued metric spaces, strongly deferred invariant convergence, and deferred invariant statistical convergence. Our comprehension of convergence qualities in this particular context was enhanced by these extensions. To demonstrate the significance and applicability of these ideas, we have included theoretical explanations and computational examples throughout our investigation. In addition to laying the theoretical groundwork for octonion-valued metric spaces, the ideas and findings provided here open up new avenues for investigation and use in related domains. To sum up, our results highlight the depth and intricacy of convergence ideas in octonion-valued metric spaces, providing fresh viewpoints and directions for further study in mathematical analysis and related fields.

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References

[1] Agnew, R. P., On deferred Cesàro means, *Ann. Math.*, 33(3) (1932), 413–421. https://doi.org/10.2307/1968524.
[2] Albert, A. A., On a certain algebra of quantum mechanics, *Ann. Math.*, 35(1) (1934), 65–73. https://doi.org/10.2307/1968118.
[3] Azam, A., Fisher, B., Khan, M., Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, 32(3) (2011), 243–253. https://doi.org/10.1080/01630563.2011.533046.
[4] Baez, J. C., The octonions, *Bull. Amer. Math. Soc.*, 39 (2002), 145–205. https://doi.org/10.1090/S0273-0979-01-00934-X.
[5] Belen, C., Mohiuddine, S. A., Generalized weighted statistical convergence and application, *Appl. Math. Comput.*, 219(18) (2013), 9821–9826. https://doi.org/10.1016/j.amc.2013.03.115.
[6] Connor, J., The statistical and strongly p Cesàro convergence of sequences, *Analysis*, 8 (1988), 47–63. https://doi.org/10.1524/anly.1988.8.12.47.

[7] Conway, J. H., Smith, D. A., On Quaternions and Octonions: Their geometry, arithmetic, and symmetry, *Bull. Amer. Math. Soc.*, 42 (2005), 229–243.
[8] Çetin, S., Mursaleen, M., Kişi, Ö., Gürdal, M., Octonion-valued metric spaces and the concept of convergence in these spaces. Preprint. 2024.
[9] Dray, T., Manogue, C., The Geometry of the Octonions, World Scientific, 2015. https://doi.org/10.1142/8456.
[10] El-Sayed, A. A., Omran, S., Asad, A. J., Fixed point theorems in quaternion valued metric spaces, *Abstr. Appl. Anal.*, Article ID 258958, (2014), 1–9. https://doi.org/10.1155/2014/258985.
[11] Et, M., Cinar, M., Kandemir, H. S., Deferred statistical convergence of order α in metric spaces, *AIMS Mathematics* 5(4) (2020), 3731–3740. https://doi.org/10.3934/math.2020241.
[12] Fast, H., Sur la convergence statistique, *Colloq. Math.*, 2(3-4) (1951), 241–244.
[13] Fiorenza, D., Sati, H., Schreiber, U., Super-exceptional embedding construction of the heterotic M5: Emergence of SU(2)-flavor sector, *J. Geom. Phys.*, 170 (2021), 104349. https://doi.org/10.1016/j.geomphys.2021.104349.
[14] Freedman, A. R., Sember, J., Raphael, M., Some Cesàro-type summability spaces, *Proc. London Math. Soc.*, 37(3) (1978), 508–520. https://doi.org/10.1112/PLMS2FS3-37.3.508.
[15] Gürdal, M., Şahiner, A., Açık, I., Approximation theory in 2-Banach spaces, *Nonlinear Anal.*, 71(5–6) (2009), 1654–1661. https://doi.org/10.1016/j.na.2009.01.030.
[16] Gürdal, M., Şahiner, A., Statistical approximation with a sequence of 2-Banach spaces, *Math. Comput. Model.*, 55(3-4) (2012), 471–479. https://doi.org/10.1016/j.mcm.2011.08.026.
[17] Gürdal, M., Sarı, N., Savaş, E., *A*-statistically localized sequences in n-normed spaces, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 69(2) (2020), 1484–1497. https://doi.org/10.31801/cfsuasmas.704446.
[18] Gürdal, M., Yamancı, U., Statistical convergence and some questions of operator theory, *Dynam. Systems Appl.*, 24(7) (2015), 305–311.
[19] Kadak, U., Mohiuddine, S. A., Generalized statistically almost convergence based on the difference operator which includes the (p,q)-gamma function and related approximation theorems, *Results Math.*, 73(9) (2018), https://doi.org/10.1007/s00025-018-0789-6.
[20] Kansu, M., Tanışlı, E. M., Demir, S., Octonion form of duality-invariant field equations for dyons, *Turk. J. Phys.*, 44(1) (2020), 10–23. https://doi.org/10.3906/fiz-1910-7.
[21] Koşar, C., Küçükaslan, M., Et, M., On asymptotically deferred statistical equivalence of sequences, *Filomat*, 31(16) (2017), 5139–5150. https://doi.org/10.2298/FIL1716139K.
[22] Küçükaslan, M., Yılmaztürk, M., On deferred statistical convergence of sequences, *Kyungpook Math. J.*, 56(2) (2016), 357–366. https://doi.org/10.5666/KMJ.2016.56.2.357.
[23] Maddox, I. J., Elements of Functional Analysis, Cambridge at the University Press, 1970.
[24] Mohiuddine, S. A., Alamri, B. A. S., Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM*, 113(3) (2019), 1955–1973. https://doi.org/10.1007/s13398-018-0591-z.
[25] Mohiuddine, S. A., Statistical weighted *A*-summability with application to Korovkin's type approximation theorem, *J. Inequal. Appl.*, 2016(101) (2016). https://doi.org/10.1186/s13660-016-1040-1.
[26] Mursaleen, M., Çetin, S., Kişi, Ö., Gürdal, M., An analysis of ideal convergence of sequences in Octonion-valued metric spaces. Preprint. 2024.
[27] Mursaleen, M., Edely, O. H. H., On the invariant mean and statistical convergence, *Appl. Math. Lett.*, 22(11) (2009), 1700–1704. https://doi.org/10.1016/j.aml.2009.06.005.
[28] Mursaleen, M., Invariant mean and some matrix transformations, *Thamkang J. Math.*, 10 (1979), 183–188.
[29] Mursaleen, M., On some new invariant matrix methods of summability, *Quart. J. Math. Oxford*, 34(2) (1983), 77–86. http://dx.doi.org/10.1093/qmath/34.1.77.
[30] Nabiev, A. A., Savaş, E., Gürdal, M., Statistically localized sequences in metric spaces, *J. Appl. Anal. Comput.*, 9(2) (2019), 739–746. https://doi.org/10.11948/2156-907X.20180157.
[31] Nuray, F., Strongly deferred invariant convergence and deferred invariant statistical convergence, *J. Comput. Sci. Comput. Math.*, 10(1) (2020), 1–6. http://dx.doi.org/10.20967/jcscm.2020.01.001.
[32] Okubo, S., Introduction to Octonion and Other Non-Associative Algebras in Physics, Cambridge University Press, Cambridge, 1995.
[33] Quan, J. J., Çetin, S., Kişi, Ö., Gürdal, M., On certain results of statistical convergence in octonion-valued metric spaces, 2020, Preprint.
[34] Raimi, R. A., Invariant means and invariant matrix methods of summability, *Duke Math. J.*, 30(1) (1963), 81–94. http://dx.doi.org/10.1215/S0012-7094-63-03009-6.
[35] Savaş, E., Kişi, Ö., Gürdal, M., On statistical convergence in credibility space, *Numer. Funct. Anal. Optim.*, 43 (2022), 987–1008. https://doi.org/10.1080/01630563.2022.2070205.
[36] Savaş, E., Nuray, F., On σ-statistically convergence and lacunary σ-statistically convergence, *Math. Slovaca*, 43(3) (1993), 309–315.
[37] Savaş, E., On lacunary strong σ-convergence, *Indian J. Pure Appl. Math.*, 21 (1990), 359–365.
[38] Savaş, E., Some sequence spaces involving invariant means, *Indian J. Math.*, 31 (1989), 1–8.
[39] Schaefer, P., Infinite matrices and invariant means, *Proc. Amer. Math. Soc.*, 36(1) (1972), 104–110. https://doi.org/10.2307/2039044.
[40] Takahashi, K., Fujita, M., Hashimoto, M., Remarks on Octonion-valued neural networks with application to robot manipulator control, (2021), 1–6, IEEE International Conference on Mechatronics (ICM), Kashiwa, Japan, 2021.

[41] Wu, J., Xu, L., Wu, F., Kong, Y., Senhadji, L., Shu, H., Deep octonion networks, *Neurocomputing*, 397 (2020), 179–191. https://doi.org/10.1016/j.neucom.2020.02.053.
[42] Yamancı, U., Gürdal, M., Statistical convergence and operators on Fock space, *New York J. Math.*, 22 (2016), 199–207.