Journal of Advanced Research in Natural and Applied Sciences
dergipark.org.tr/tr/pub/jarnas
Open Access

ISSN: 2757-5195 11 (1) (2025) 12-26 doi.org/10.28979/jarnas.1611875

Varying Capacity and Harvesting in a Prey-Predator System with Memory Effect

Ercan Balcı[1]

Article Info
Received: 02 Jan 2025
Accepted: 13 Mar 2025
Published: 31 Mar 2025
Research Article

Abstract— This paper investigates a fractional-order prey-predator model with varying prey-carrying capacity and the inclusion of harvesting in both populations. The model uses fractional derivatives to include memory effects, aiming to capture ecological dynamics better. Moreover, it considers how prey can alter its carrying capacity by modifying the environment. The stability and Hopf bifurcation analyses are used to study population cycles and equilibrium states. Numerical simulations reveal key biological insights, emphasizing the need for sustainable harvesting and the influence of past interactions on ecosystem balance.

Keywords — *Prey-predator, varying carrying capacity, Caputo fractional derivative, harvesting*

1. Introduction

Understanding prey-predator interactions is key to studying ecosystem dynamics, as they play a vital role in shaping biodiversity, maintaining population balance, and ensuring species survival. Classical mathematical models like Lotka-Volterra have been important in exploring these relationships, offering valuable insights into how such dynamics operate across ecosystems. Over time, these models have been refined to include elements like adaptive behaviors, environmental changes, and intricate biological interactions, making them more representative of real-world circumstances. These models focus on addressing the rapid loss of biodiversity caused by habitat destruction, climate shifts, and the overuse of natural resources [1].

Fractional calculus has been emerging as an essential tool for improving the realism of mathematical models, especially in biological systems where memory and inherited traits are significant. Unlike standard integer-order models, fractional differential equations (FDEs) offer a more detailed depiction of processes influenced by past events [2]. Using fractional-order derivatives introduces memory effects in prey-predator dynamics, enabling a deeper understanding of long-term population interactions and ecosystem behavior [3,4]. Caputo and Riemann-Liouville derivatives are commonly used in such models. The Caputo fractional derivative (CFD) is used widely because it is compatible with traditional initial conditions and still captures the non-local, history-dependent nature of biological processes [2,5]. Fractional-order models provide a deeper insight into population dynamics, particularly in intricate biological systems where interactions may span over extended periods [3,4,6,7].

---

[1] ercanbalci@erciyes.edu.tr (Corresponding Author)
[1] Department of Mathematics, Faculty of Science, Erciyes University, Kayseri, Türkiye

Numerous biological aspects, including various functional responses, the Allee effect, refuges, supplementary food sources, fear effects, and herd behavior, have been investigated in predator-prey systems utilizing FDEs. Furthermore, several studies have also focused on the role of harvesting [8–12]. Harvesting significantly affects prey-predator dynamics, ecosystem balance, and long-term sustainability. It involves the removal of individuals, either prey or predator, for activities like fishing, hunting, and forestry. While harvesting is essential for economic and subsistence needs, excessive exploitation can disrupt the ecological balance, causing population declines or extinction. Harvesting can influence population cycles in these systems, potentially destabilizing ecosystems if not carefully regulated. Therefore, sustainable harvesting practices are critical to prevent over-exploitation, ensure species recover, and continue contributing to biodiversity and ecosystem health.

In this study, we consider a prey-predator system with prey that can impact its carrying capacity, and both populations are subject to harvesting. Since pioneering the Lotka-Volterra system, the prey-predator model has advanced significantly. In traditional predator-prey models, the carrying capacity of the prey population is often treated as a fixed constant, representing the maximum population size that the environment can support. However, in more realistic scenarios, the carrying capacity can vary due to environmental factors like seasonal changes, resource availability, or habitat quality [13–15]. Another significant consideration in prey-predator dynamics is the role of prey in shaping its carrying capacity [16, 17]. In many ecosystems, prey species actively modify their environment through resource consumption, habitat alteration, or even social behaviors, which can impact the resources available for future generations. For example, overgrazing by herbivores may degrade vegetation, reducing the future carrying capacity of the environment. The following section explains how this aspect is incorporated into the model.

This study introduces a novel approach by incorporating a varying carrying capacity into a prey-predator system formulated by fractional differential equations (FDEs). Unlike previous works, which employed a discrete delay in the carrying capacity terms to account for the time-lagged influence of prey on its environment [16, 17], we take a different route. Rather than assuming a strictly delayed response, our model integrates a generalized memory effect by leveraging the hereditary characteristics of the Caputo fractional derivative. Additionally, including both prey and predator harvesting and its interaction with variable carrying capacity further distinguishes this study and expands its scope.

The structure of the paper is as follows: The next section is devoted to the model formulation and its fractional version, providing some preliminary information on CFDs and FDEs. Following this, we perform stability and Hopf bifurcation analysis of the proposed system. Finally, we conclude with numerical simulations and conclusions, where we provide numerical simulations and their biological interpretations.

Table 1. Variables and parameters of (2.3) - (2.5) with descriptions

Variable/Parameter	Biological Meaning	Dimension	Dimensionless Representation
U	Prey density	*biomass*	$u = \frac{U}{B}$
V	Predator density	*biomass*	$v = \frac{aV}{rb}$
T	Time	*time*	$t = rT$
r	Prey growth rate	$time^{-1}$	
K	Carrying capacity	*biomass*	$k = \dfrac{K}{b}$
c	Constructive impact of prey	*dimensionless*	c
a	Consumption rate	$time^{-1}$	
b	Half saturation constant	*biomass*	
q_1	Harvesting rate of prey	$time^{-1}$	$h_1 = \frac{q_1}{r}$
e	Conversion rate	*dimensionless*	$\alpha = \frac{ae}{r}$
d	Predator mortality rate	$time^{-1}$	$\delta = \frac{d+q_2}{r}$
q_2	Harvesting rate of predator	$time^{-1}$	Included in δ

2. Model Construction

This section introduces the predator-prey model that will be studied and provides preliminary information on CFDs and FDEs.

2.1. Formulation of Model

Consider a traditional Lotka-Volterra type prey-predator system with Holling-type II functional response:

$$\begin{cases} \dfrac{dU}{dT} = rU\left(1 - \dfrac{U}{K}\right) - \dfrac{aUV}{b+U} \\ \dfrac{dV}{dT} = \dfrac{eaUV}{b+U} - dV \end{cases} \tag{2.1}$$

where U and V denotes prey and predator population densities at time T respectively. All the parameters are positive, and their biological meaning are given in Table 1. Instead of taking constant carrying capacity K for prey, we assume that the prey population can influence its carrying capacity with impact parameter c. Then, we replace the constant K in (2.1) with a new carrying capacity function $\kappa(T) = K + c\,U(T)$ to get

$$\begin{cases} \dfrac{dU}{dT} = rU\left(1 - \dfrac{U}{K+cU}\right) - \dfrac{aUV}{b+U} \\ \dfrac{dV}{dT} = \dfrac{eaUV}{b+U} - dV \end{cases} \tag{2.2}$$

In this study, we assume that prey impacts on its carrying capacity are positive, and we call this parameter c a constructive impact parameter. For instance, some species contribute to biodiversity through seed dispersal and nutrient cycling. This contribution can result in greater availability of food resources, enhancing their carrying capacity, represented by $c \geq 0$. Ecosystem engineer species provide a compelling example of prey impacting their carrying capacity. These organisms actively shape their environment by modifying resource availability, such as by altering soil composition, water flow, or vegetation, which can enhance their habitat and the habitats of other species. In doing so, they improve their chances of survival and promote biodiversity within the ecosystem. This modification of the environment can lead to an increase in their carrying capacity as they create more favorable conditions for themselves and others. For more in-depth information, we refer to [18, 19].

The predator-prey dynamics between beavers and wolves can be a relevant example for our model. As ecosystem engineers, Beavers significantly alter their environment by building dams that create wetlands, providing more aquatic plants they consume [19]. Wolves, as predators, rely on beavers for food in areas where both species coexist [20]. In such regions, beavers and wolves influence each other, with both species also being subject to harvesting. These interactions make this system a relevant biological scenario for our model. Moreover, as described in [16], we assume $c < 1$; otherwise, the prey population would grow unbounded, and the boundary equilibrium would no longer exist. Next, we incorporate harvesting for both prey and predator populations. Key factors influencing predator-prey dynamics also include harvesting one or both populations. Harvesting ecological resources is common in fisheries, forestry, and wildlife management. In this study, we incorporate linear harvesting, where predator and prey populations are harvested at rates proportional to their respective sizes. With q_1 and q_2 stands for harvesting rate of prey and predator populations, respectively, (2.2) turns into the following system:

$$\begin{cases} \dfrac{dU}{dT} = rU\left(1 - \dfrac{U}{K + cU}\right) - \dfrac{aUV}{b+U} - q_1U \\ \dfrac{dV}{dT} = \dfrac{eaUV}{b+U} - dV - q_2V \end{cases} \tag{2.3}$$

We will now apply a non-dimensionalization process to reduce the number of parameters, simplifying the analysis. Applying the given change of variables $u = \dfrac{U}{b}$, $v = \dfrac{aV}{rb}$, and $t = rT$, (2.3) transforms into the following:

$$\begin{cases} \dfrac{du}{dt} = u\left(1 - \dfrac{u}{k + cu}\right) - \dfrac{uv}{1+u} - h_1u \\ \dfrac{dv}{dt} = \dfrac{\alpha uv}{1+u} - \delta v \end{cases} \tag{2.4}$$

where $k = \dfrac{K}{b}$, $h_1 = \dfrac{q_1}{r}$, $\alpha = \dfrac{ae}{r}$, $d_2 = \dfrac{d}{r}$, $h_2 = \dfrac{q_2}{r}$, and $\delta = d_2 + h_2$. Here, we combine predator death rate d_2 and predator harvesting rate h_2 as δ for simplification.

Here, we replace left-hand side ordinary derivatives in (2.4) with Caputo fractional derivatives to get

$$\begin{cases} {}_{t_0}^{C}D_t^{\gamma}u(t) = u\left(1 - \dfrac{u}{k + cu}\right) - \dfrac{uv}{1+u} - h_1u \\ {}_{t_0}^{C}D_t^{\gamma}v(t) = \dfrac{\alpha uv}{1+u} - \delta v \end{cases} \tag{2.5}$$

Here, ${}_{t_0}^{C}D_t^{\gamma}$ is Caputo fractional differentiation of order γ with $\gamma \in (0,1)$, and is defined as [2]

$$ {}_{t_0}^{C}D_t^{\gamma}F(t) = \frac{1}{\Gamma(1-\gamma)} \int_{t_0}^{t} \frac{F'(\gamma)}{(t-\gamma)^{\gamma}} d\gamma$$

For more detail about taking the fractional order version of the given system, we refer to [6]. With this definition, as γ approaches 1, the influence of past events on future outcomes diminishes, resulting in a short memory effect [6]. On the other hand, when γ nears 0, the impact of previous events becomes more significant, leading to a stronger memory effect.

2.2. Existence-Uniqueness of Solutions

In this subsection, we prove the existence and uniqueness of solutions of (2.5) after providing necessary theorems.

Lemma 2.1. [21] Let $f \in C^q([t_0, T], \mathbb{R})$. Suppose that for any $t_1 \in (t_0, T]$, one has $g(t_1) = 0$ and $f(t) < 0$, for $t_0 \le t < t_1$, then it follows that ${}_{t_0}^{C}D_t^{\gamma}f(t_1) > 0$.

Lemma 2.2. [22] Consider the following system

$$ {}_0^C D_t^\gamma Y(t) = F(t, Y),\ t \geq 0 $$

with initial condition $Y(0) = (Y_1(0), \dots, Y_n(0))$, where $\gamma \in (0,1]$, $F : [0,\infty) \times \Delta \to \mathbb{R}^n$, $\Delta \subseteq \mathbb{R}^n$. If $F(t, Y)$ fulfills the local Lipschitz condition with respect to $Y \in \mathbb{R}^n$, i.e.,

$$ \|F(t, Y) - F(t, \tilde{Y})\| \leq N \|Y - \tilde{Y}\| $$

there exists a unique solution for (2.2) on $[0, \infty) \times \Delta$.

Theorem 2.3. The solution for (2.5) with $t \geq 0$ remains in $\mathbb{R}^2_+$.

Proof. Consider the solutions starting from $Y(t_0) = (u_{t_0}, v_{t_0})$. Suppose there exists a constant t_1 such that $t_0 \leq t < t_1$ and

$$ \begin{cases} u(t) > 0, & t_0 < t < t_1 \\ u(t_1) = 0 \\ u(t_1^+) < 0 \end{cases} $$

Since ${}_{t_0}^C D_t^\gamma u(t)|_{x(t_1)=0} = 0$, Lemma 2.1 tells that $u(t_1^+) = 0$ which contradicts $u(t_1^+) < 0$. Hence, $u(t) \geq 0$ for all $t \in [t_0, \infty)$. Similarly, $v(t) \geq 0$, for all $t \in [t_0, \infty)$. $\square$

The existence-uniqueness of the solutions of the system (2.5) will be investigated considering the region $\Lambda \times [t_0, T)$ where

$$ \Lambda = \{(u, v) \in \mathbb{R}^2_+ : \max\{|u|, |v|\} < \mu\} $$

$T < \infty$, μ is large, and $\mathbb{R}^2_+ = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0, y_2 \geq 0\}$.

Theorem 2.4. The Caputo fractional order system (2.5) admits a unique solution for any nonnegative initial conditions.

Proof. Let $Y_0, \tilde{Y}_0 \in \Lambda$ and $Y(t), \tilde{Y}(t)$ be two solutions of the system $D^\gamma Y = F(Y)$ initiating from $Y_0, \tilde{Y}_0$ where $Y = (u, v)^T$, $F(Y) = (F_1(Y), F_2(Y))^T$ with

$$ F_1(Y) = u\left(1 - \frac{u}{k + cu}\right) - \frac{uv}{1 + u} - h_1 u \quad \text{and} \quad F_2(Y) = \frac{\alpha uv}{1 + u} - \delta v $$

Thus,

$$ \begin{aligned} \|F(Y) - F(\tilde{Y})\| &= |F_1(Y) - F_1(\tilde{Y})| + |F_2(Y) - F_2(\tilde{Y})| \\ &= \left| u\left(1 - \frac{u}{k + cu}\right) - \tilde{u}\left(1 - \frac{\tilde{u}}{k + c\tilde{u}}\right) - \frac{uv}{1 + u} + \frac{\tilde{u}\tilde{v}}{1 + \tilde{u}} - h_1 u + h_1 \tilde{u} \right| \\ &\quad + \left| \frac{\alpha uv}{1 + u} - \frac{\alpha \tilde{u}\tilde{v}}{1 + \tilde{u}} - \delta v + \delta \tilde{v} \right| \\ &\leq |u - \tilde{u}| + \left| \frac{k(\tilde{u}^2 - u^2) + cu\tilde{u}(\tilde{u} - u)}{(k + cu)(k + c\tilde{u})} \right| + \left| \frac{\tilde{u}(\tilde{v} - v) + v(\tilde{u} - u)}{(1 + u)(1 + \tilde{u})} \right| + |h_1(\tilde{u} - u)| \\ &\quad + \left| \alpha \frac{\tilde{u}(v - \tilde{v}) + v(u - \tilde{u})}{(1 + u)(1 + \tilde{u})} \right| + |\delta(\tilde{v} - v)| \\ &\leq |u - \tilde{u}| + \frac{1}{k}|u^2 - \tilde{u}^2| + \frac{c}{k^2} u\tilde{u}|u - \tilde{u}| + \tilde{u}|v - \tilde{v}| + v|u - \tilde{u}| + h_1|u - \tilde{u}| \\ &\quad + \alpha\tilde{u}|v - \tilde{v}| + \alpha v|u - \tilde{u}| + \delta|v - \tilde{v}| \\ &\leq \left(1 + \frac{2\mu}{k} + \frac{c\mu^2}{k^2} + \mu(1 + \alpha) + h_1\right) |u - \tilde{u}| + (\mu(1 + \alpha) + \delta)\, |v - \tilde{v}| \\ &= N_1\, |u - \tilde{u}| + N_2\, |v - \tilde{v}| \end{aligned} $$

where $N_1 = 1 + \dfrac{2\mu}{k} + \dfrac{c\mu^2}{k^2} + \mu(1+\alpha) + h_1$ and $N_2 = \mu(1+\alpha) + \delta$. Defining $N = \max\{N_1, N_2\}$, the last expression is smaller than $N \, \|Y - \tilde{Y}\|$. Hence, the result is obtained using Lemma 2.2. □

3. Dynamical Analysis

In this section, we perform stability and bifurcation analysis of the equilibrium points of (2.5). To achieve this, we will utilize the following theorems.

Theorem 3.1. [5] Consider the autonomous nonlinear system with CFD

$$ {}^{C}_{t_0}D^{\gamma}_{t} Y(t) = F(Y(t)), \quad Y(0) = Y_0 \in \mathbb{R}^n, \quad \gamma \in (0,1) \tag{3.1}$$

where $Y(t) = (Y_1(t), \ldots, Y_n(t)) \in \mathbb{R}^n$, $F : [F_1, \ldots, F_n] : \mathbb{R}^n \to \mathbb{R}^n$. A point E^* is called equilibrium point of (3.1) if it satisfies $F(E^*) = 0$. If all eigenvalues λ_i, $i \in \{1, 2, 3, \ldots, n\}$ of the Jacobian matrix $J_{E*} = \dfrac{\partial F}{\partial Y}\Big|_{Y=E^*}$ meet following conditions

$$|\arg(\lambda_i)| > \frac{\gamma\pi}{2}$$

Then, Y^* is locally asymptotically stable (LAS).

Theorem 3.2. [23] Consider the following system with CFD

$$D^{\gamma} Y(t) = F(Y), \quad Y(0) = Y_0 \in \mathbb{R}^2 \tag{3.2}$$

with $\gamma \in (0,1)$. There exists a Hopf bifurcation about E^*, if a critical $\gamma = \gamma_h$ value exist and satisfies the conditions:

i. The Jacobian at E^* has a pair of complex conjugate eigenvalues $\lambda_{1,2} = p \pm iq$ (where $p > 0$)

ii. $m(\gamma_h) = 0$ where $m(\gamma) = \dfrac{\gamma\pi}{2} - \min_{1 \le j \le 2} |\arg(\lambda_j)|$

iii. $\dfrac{dm(\gamma)}{d\gamma}\Big|_{\gamma=\gamma_h} \neq 0$

Setting both equations in (2.5) equal to zero, we obtain three equilibrium points of (2.5):

i. The extinction state of both populations $E_0 = (0,0)$ which is always feasible.

ii. The predator-free boundary equilibrium $E_1 = \left(\dfrac{k(1-h_1)}{1-c(1-h_1)}, 0 \right)$. E_1 exists if $h_1 < 1$.

iii. The interior equilibrium point $E^* = (u^*, v^*)$ where

$$u^* = \frac{\delta}{e - \delta} \quad \text{and} \quad v^* = \frac{e((1-h_1)(ek + (c-k)\delta) - \delta)}{(e-\delta)(ek + (c-k)\delta)}$$

The existence conditions $(u^* > 0, v^* > 0)$ of the interior equilibrium point $E^* = (u^*, v^*)$ can be summarized as follows:

i. $e > \delta$

ii. $k > \dfrac{\delta(1-c)}{e-\delta} = (1-c)u^*$

iii. $h_1 < 1 - \dfrac{u^*}{k + cu^*}$

The first condition ensures that $u^* > 0$, while the last two conditions are included to guarantee the positivity of v^*. Note that the second condition can be expressed as $\dfrac{k}{(1-c)} > u^*$. If there is no positive influence from the prey (i.e., $c = 0$), this simplifies to $k > u^*$. Therefore, with this varying carrying capacity, the prey's carrying capacity k can be smaller than the prey component u^* of the E^*, and an interior equilibrium can still exist depending on the strength of the constructive impact. The third condition sets an upper limit for the prey harvesting rate, h_1, to ensure the existence of an interior equilibrium. If the constructive impact parameter c is present or significant, the prey population can sustain the predator population even under higher prey harvesting rates.

Theorem 3.3. *i.* E_0 is LAS if $h_1 > 1$, otherwise it is a saddle point.

ii. E_1 is LAS if $1 - \dfrac{u^*}{k + cu^*} < h_1 < 1$ where $u^* = \dfrac{\delta}{e - \delta}$

Proof. *i.* The eigenvalues of J_{E_0} are $\lambda_1 = -\delta < 0$ and $\lambda_2 = 1 - h_1$. If $h_1 > 0$ both eigenvalues satisfies LAS condition $|\arg(\lambda_i)| > \dfrac{\gamma\pi}{2}$ for any choice of $\gamma \in (0, 1]$.

ii. The eigenvalues of J_{E_1} are both real numbers and calculated as

$$\lambda_1 = (h_1 - 1)(1 + c(h_1 - 1)) \text{ and } \lambda_2 = \frac{ek + (-1 + c - k)\delta - (ek + (c - k)\delta)h_1}{1 - c + k + (c - k)h_1}$$

The condition $h_1 < 1$ guarantees $\lambda_1 < 0$. In addition to this condition if we have

$$e < \frac{-\delta + c\delta(1 - h_1) + \delta(-k + kh_1)}{-k(1 - h_1)}$$

λ_2 is also negative real number and both eigenvalues satisfy $|\arg(\lambda_{1,2})| > \dfrac{\gamma\pi}{2}$ for any choice of $\gamma \in (0, 1]$. These two conditions can be combined as $1 - \dfrac{u^*}{k + cu^*} < h_1 < 1$.

□

For the LAS of the interior equilibrium point E^*, we firstly evaluate Jacobian about $E^* = (u^*, v^*)$ and we obtain

$$J_{E^*} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

where

i. $p_{11} = -\dfrac{\delta(-e^2(-1 + k)k + 2ek(-c + k)\delta - (-c + c^2 + k - 2ck + k^2)\delta^2 + (ek + (c - k)\delta)^2 h_1}{e(ek + (c - k)\delta)^2}$

ii. $p_{12} = \dfrac{-\delta}{e}$

iii. $p_{21} = -\dfrac{(e - \delta)(-ek + (1 - c + k)\delta + (ek + (c - k)\delta)h_1)}{ek + (c - k)\delta}$

iv. $p_{22} = 0$

Since that system parameters are positive, p_{12} is a negative real number. The existence conditions imply that p_{21} is a positive real number. The corresponding characteristic is given as

$$h(\lambda) = \lambda^2 + \varphi_1\lambda + \varphi_0 = 0 \tag{3.3}$$

where $\varphi_1 = -p_{11}$, $\varphi_0 = -p_{12}p_{21} > 0$. Using Theorem 3.1 gives rise to the following conclusion:

Theorem 3.4. The interior equilibrium point $E^* = (u^*, v^*)$ is LAS if one of the following conditions satisfied:

i. $\varphi_1 \geq 0$

ii. $\varphi_1 < 0$, $\varphi_1^2 - 4\varphi_0 < 0$, $\left|\tan^{-1}\left(\dfrac{\sqrt{4\varphi_0 - \varphi_1^2}}{\varphi_1}\right)\right| > \dfrac{\gamma\pi}{2}$

Proof. The roots of (3.3) giving us the eigenvalues of J_{E^*} as

$$\lambda_{1,2} = \frac{-\varphi_1 \pm \sqrt{\varphi_1^2 - 4\varphi_0}}{2}$$

For $\varphi_1 = 0$, $\lambda_{1,2}$ are complex conjugates satisfying $|\arg(\lambda_{1,2})| = \frac{\pi}{2} > \frac{\gamma\pi}{2}$, $0 < \gamma < 1$. The inequalities $\varphi_1 > 0$ and $\varphi_1^2 - 4\varphi_0 \geq 0$ implies that both $\lambda_{1,2}$ are negative real numbers and the LAS condition $|\arg(\lambda_{1,2})| = \pi > \frac{\gamma\pi}{2}$ is again satisfied. Lastly, if $\varphi_1 > 0$ and $\varphi_1^2 - 4\varphi_0 < 0$ holds, the eigenvalues $\lambda_{1,2} \in \mathbb{C}$ with negative real parts, and we have $|\arg(\lambda_{1,2})| > \frac{\gamma\pi}{2}$.

The inequalities $\varphi_1 < 0$, $\varphi_1^2 - 4\varphi_0 < 0$ imply that the eigenvalues are complex conjugate numbers with strictly positive real parts. Then, the condition

$$\left| \tan^{-1}\left(\frac{\sqrt{4\varphi_0 - \varphi_1^2}}{\varphi_1} \right) \right| > \frac{\gamma\pi}{2}$$

ensures that $|\arg(\lambda_{1,2})| > \frac{\gamma\pi}{2}$. □

Note that no bistability situation exists for the system (2.5). The LAS condition $h_1 > 1$ of E_0 violates the existence conditions of boundary equilibrium E_1 and interior equilibrium E^*. Moreover, the existence conditions of E^* contradict the LAS condition of E_1. The theorem below explains Hopf bifurcation situation of the system (2.5) around E^*.

Theorem 3.5. Suppose that interior equilibrium exists and the inequalities $\varphi_1 < 0$, $\varphi_1^2 - 4\varphi_0 < 0$ holds. Then, as fractional order parameter γ passes through the critical

$$\gamma_h = \frac{2}{\pi} \left| \tan^{-1}\left(\frac{\sqrt{4\varphi_0 - \varphi_1^2}}{\varphi_1} \right) \right|$$

value, a Hopf bifurcation exists around E^* of the system (2.5).

Proof. $\varphi_1 < 0$, $\varphi_1^2 - 4\varphi_0 < 0$ implies that the eigenvalues $\lambda_{1,2}$ of the Jacobian J_{E^*} are complex conjugates with positive real parts. Therefore, the stability of E^* is depending on the choice of fractional order parameter $\gamma \in (0, 1)$ and we have

$$\min_{1\leq i\leq 2} |\arg(\lambda_i)| = \left| \tan^{-1}\left(\frac{\sqrt{4\varphi_0 - \varphi_1^2}}{\varphi_1} \right) \right|$$

For

$$\gamma = \gamma_h = \frac{2}{\pi} \left| \tan^{-1}\left(\frac{\sqrt{4\varphi_0 - \varphi_1^2}}{\varphi_1} \right) \right|$$

we have $m(\gamma_h) = 0$. Finally, the transversality condition

$$\left.\frac{dm(\gamma)}{d\gamma}\right|_{\gamma=\gamma_h} = \frac{\pi}{2} \neq 0$$

guarantees the existence of Hopf bifurcation as γ passes through γ_h. □

4. Numerical Simulations

In this section, we explore the dynamic behavior of the fractional-order prey-predator system (2.5) through numerical simulations. The previously derived analytical results are validated using the fractional Adams-Bashforth-Moulton method, also known as the PECE algorithm [24,25], implemented in MATLAB R2016b. The parameter values used for numerical examples are given in Table 2. Given the importance of the fractional order in capturing memory effects and its influence on the entire system, it has been thoroughly examined in the numerical analysis. Additionally, the constructive impact parameter and harvesting rates, unique to this study, are analyzed in detail. The numerical outputs are given with their biological and ecological interpretations.

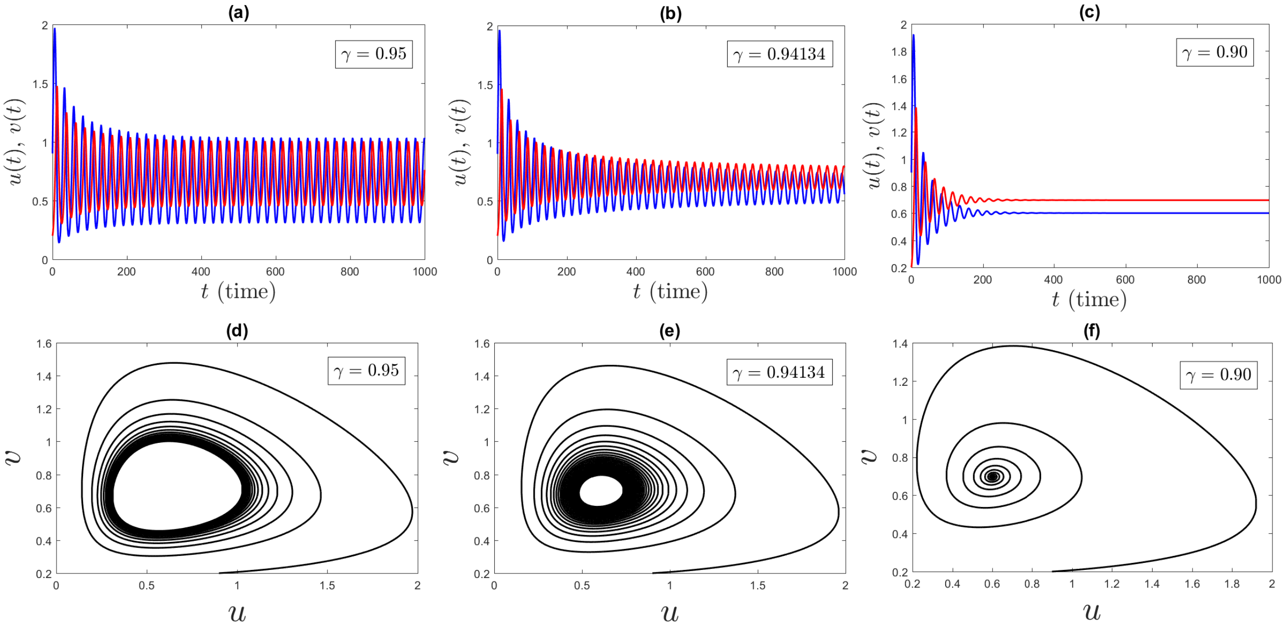

Figure 1. Prey (blue) and predator (red) populations concerning time and phase portraits for $\gamma = 0.95$ (a)(d); $\gamma = 0.94134$ (b)(e); $\gamma = 0.90$ (c)(f)

Firstly, we investigate the impact of the fractional order γ. For chosen parameter values given in Table 2, the existence conditions of the interior equilibrium point are satisfied, and it is calculated as $E^* = (u^*, v^*) = (0.8, 0.560377)$. Moreover, the eigenvalues of J_{E^*} are complex conjugate numbers $\lambda_{1,2} = 0.0262713 \pm 0.284327i$ with positive real parts. By Theorem 3.4, the LAS depends on the fractional order parameter γ. The critical value of γ is calculated as $\gamma_h = 0.941344$ on which Hopf bifurcation occurs about E^* (Figure 1 (b)(e) and Figure 2). Above this value, the oscillations gets higher and E^* is not LAS (Figure 1 (a)(d) and Figure 2). This indicates that predator-prey interactions become more unsettled when historical effects weaken due to a higher fractional order, leading to sustained population cycles or destabilization. However, below this value (i.e., $\gamma < \gamma_h$), the memory effect is sufficient to dampen the oscillations, allowing the populations to stabilize at the interior equilibrium $E^* = (u^*, v^*)$ (Figure 1 (c)(f) and Figure 2). Biologically, this suggests that both species regulate their populations based on immediate environmental conditions and past population dynamics. The stronger memory effect introduces ecological resilience, enabling a more stable coexistence by smoothing out extreme fluctuations in population densities.

Table 2. Parameter values used for numerical examples

k	c	h_1	α	δ
5	0.35	0.45	0.8	0.3

In Figure 3, time-series solutions of the system (2.5) are given for four different values of h_1. For $h_1 = 1$, the equilibriums E_1 and E^* do not exist, and the extinction state is LAS (Figure 3 (a)(e) and Figure

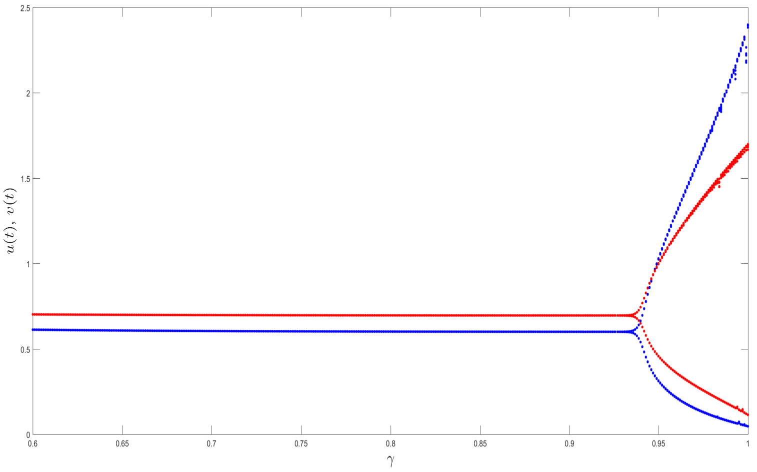

Figure 2. Bifurcation diagram concerning fractional order parameter γ

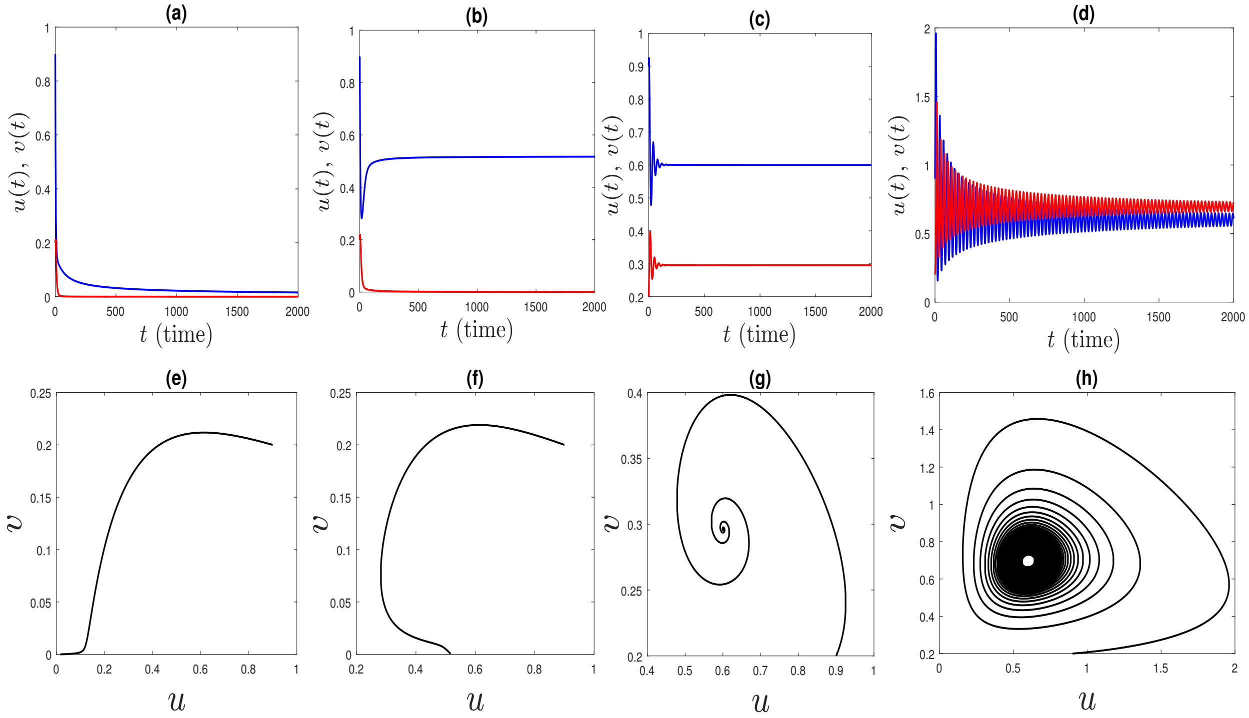

Figure 3. Time-series solutions with phase portraits where $h_1 = 1$ (a)(e); $h_1 = 0.90$ (b)(f); $h_1 = 0.70$ (c)(g); $h_1 = 0.45$ (d)(h)

4). For lower values of h_1, we always have the boundary equilibrium E_1. For given parameter values in Table 2, the critical value for the LAS of E_1 is calculated as $h = 0.88417$; this is also the existence threshold for interior equilibrium E^*. For $0.88417 < h_1 < 1$, we have LAS predator-free equilibrium E^*. For $h < 0.88417$, E_1 loses its stability while the interior equilibrium E^* exist (Figure 3 (c)(g) and Figure 4). While h_1 gets lower, E_1 is LAS since the Jacobian J_{E^*} has negative real eigenvalues until $h_1 \approx 0.8705$. After this value until $h_1 \approx 0.5905$, the eigenvalues of J_{E^*} are complex numbers with negative real parts. Hence, despite periodic oscillations have seen, E^* is LAS irrespective of fractional order parameter $\gamma \in (0, 1]$ since $|\arg(\lambda_{1,2})| > \dfrac{\pi}{2} \geq \dfrac{\gamma\pi}{2}$. Below the value $h_1 \approx 0.5905$, the intensifying memory effect (i.e., decreasing the fractional order γ) can turn an unstable E^* into LAS one or vice versa. Even for $h_1 = 0$, the critical value of fractional order γ is calculated as $\gamma_h = 0.826042$ at which Hopf bifurcation happens while below it E^* is LAS. As shown in Figure 4, a lower harvesting rate h_1

results in a higher predator population v^* of the equilibrium point E^*. Biologically, this highlights that reducing prey exploitation increases available resources, promoting predator survival and growth. If prey harvesting is excessive, the predator population collapses due to insufficient food, whereas lower harvesting levels enable a more sustainable predator-prey balance.

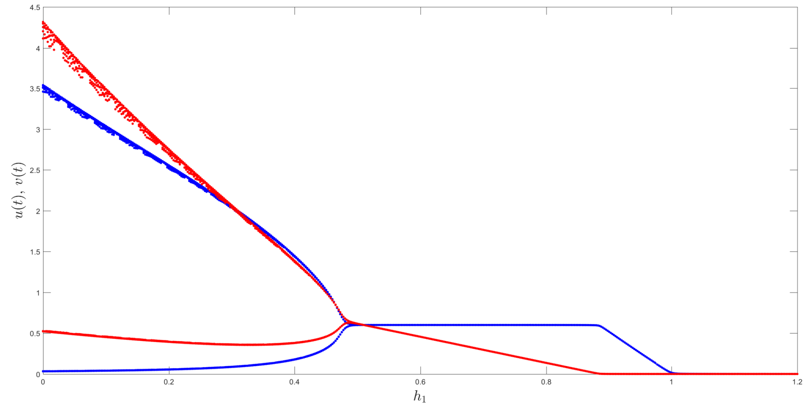

Figure 4. Bifurcation diagram with respect to harvesting rate h_1 of prey

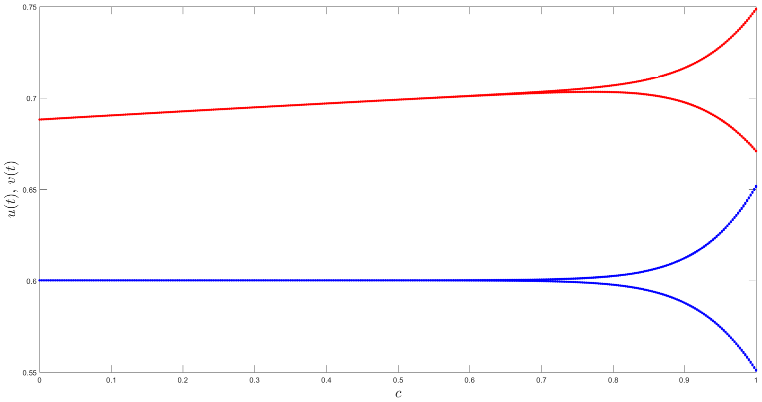

Figure 5. Bifurcation diagram with varying c for $\gamma = 0.92$

Figure 5 presents a bifurcation diagram concerning constructive impact parameter c. What we observe here can be seen as an alternative perspective on the paradox of enrichment, a concept introduced by Rosenzweig in predator-prey dynamics. This phenomenon describes how increasing the food supply available to the prey can destabilize the ecosystem [26]. Biologically, as c increases, it enhances the prey's access to resources, allowing the prey population to grow beyond sustainable levels. This, in turn, fuels a rapid rise in the predator population, leading to intensified predator-prey interactions. The resulting instability manifests as persistent population oscillations or even chaotic dynamics, where population crashes and booms occur unpredictably. Therefore, an increase in c leads to increased environmental resources for prey, causing destabilization for the system (2.5). This finding aligns with ecological observations that over-enrichment of an ecosystem, such as through excessive nutrient input, can lead to severe fluctuations in species populations, increasing the risk of extinction for one or both species.

For larger values of the constant carrying capacity term k, the constructive impact parameter c does not influence the existence of the interior equilibrium E^* (See Figure 5). This suggests that additional environmental enhancements do not significantly alter species coexistence when the ecosystem has a sufficiently high baseline carrying capacity. However, lower values of k can influence that context. We take $k = 0.8$ for Figure 6 and again give the bifurcation diagram concerning c. Initially, for $c = 0$, the third condition $h_1 < 1 - \dfrac{u^*}{k + cu^*}$ of existence of E^* is not satisfied and the system (2.5) goes to predator-free equilibrium. Biologically, this means that when the constant carrying capacity k and constructive impact parameter c are too low, the prey population cannot sustain predators, leading to predator extinction. Then, for $c \approx 0.4848$, transcritical bifurcation occurs where E_1 loses its stability, and E^* becomes to appear. At this point, the prey population has grown sufficiently due to increased resource availability, allowing the predator population to establish itself. This transition represents shifting from a prey-only ecosystem to a balanced predator-prey coexistence. Beyond that, with c increasing, prey component u^* of E^* remains constant, while predator component v^* of E^* increases due to the increased available resources.

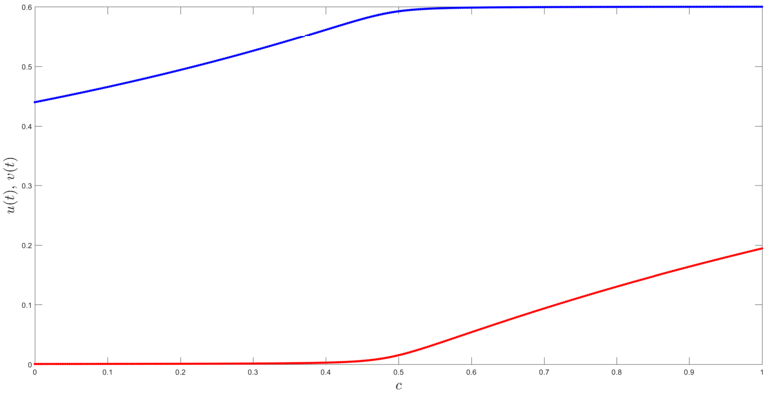

Figure 6. Bifurcation diagram with varying c for $k = 0.8$ and $\gamma = 0.95$

In Figure 7, we illustrate the bifurcation diagram concerning $\delta \in (0.5, 0.8)$. Remember that this parameter contains the predator death rate and predator harvesting rate. This figure can be taught as a bifurcation diagram concerning predator harvesting. If predator harvesting is low, prey populations are at low levels but can sustain predator populations. While predator harvesting rates get higher, two Hopf bifurcations occur for the system (2.5) around the interior equilibrium point E^*: $\delta_{hopf1} \approx 0.1225$ and $\delta_{hopf2} \approx 0.2945$. This situation is specific to using the fractional derivative since we obtain complex eigenvalues with positive real parts until $\delta \approx 0.409$. After $\delta > 0.619$, the third condition for the existence of the interior equilibrium point is not satisfied, and (2.5) converges to boundary equilibrium E_1.

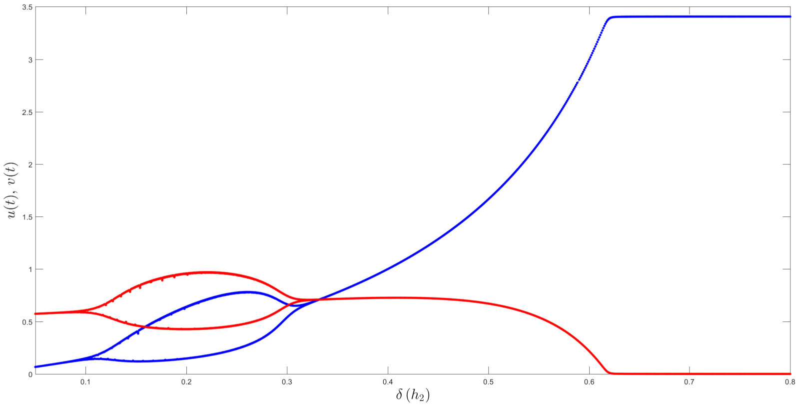

Figure 7. Bifurcation diagram for $\delta \in [0.05, 0.8]$ (or $h_2 \in [0, 0.75]$) where $\gamma = 0.94$

5. Conclusion

This study investigates a predator-prey system within a fractional-order framework, incorporating the unique aspect of the prey's constructive impact on its carrying capacity. The model follows a Holling type-II functional response and includes linear harvesting for both populations. The existence, uniqueness, and non-negativity of solutions are established. A thorough analysis of local stability for all equilibrium points is conducted. For the extinction and predator-free states, stability properties align with those in ordinary systems, as the eigenvalues of the associated Jacobian matrix are real. However, for the coexistence equilibrium, the situation differs, as the eigenvalues may be complex. Additionally, the occurrence of Hopf bifurcation, influenced by the order of the fractional derivative, is proven. To support the theoretical results, numerical simulations using the predictor-corrector scheme demonstrate the dynamics of prey and predator populations. These simulations highlight the effects of the fractional-order parameter γ, the constructive impact parameter c, and the harvesting rates h_1 and h_2.

Future studies could delve into more complex prey-predator dynamics involving variable carrying capacities. For instance, investigating varying carrying capacities alongside other biological factors, such as different functional responses or the Allee effect, could lead to intriguing findings. Prey species with a constructive influence often exhibit collaborative behavior, which motivates us to study them in conjunction with the Allee effect. Similarly, as many ecosystem-engineering species rely on refuge, the role of refuge could also be examined alongside variable carrying capacity. Moreover, delayed differential equations could explore varying carrying capacity, while fractional derivatives might also be incorporated as another memory mechanism. Adding fractional-order derivatives into the model could significantly enhance the system's dynamic complexity.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] S. Diaz, J. Settele, E. Brondizio, H. T. Ngo, M. Guezi, J. Agard, A. Arnith, P. Balvanira, K. Brauman, S. Butchart, K. Chan, L. Garibaldi, K. Ichii, J. Liu, S. M. Subramanian, G. Midgliy, P. Miloslavich, Z. Molnar, D. Obura, A. Pfaff, ..., C. Zayas, *Summary for policymakers of the global assessment report on biodiversity and ecosystem services of the Intergovernmental Science-Policy Platform on Biodiversity and Ecosystem Services*, IPBES Secretariat: Bonn, Germany (2019) 22–47.

[2] I. Podlubny, Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Elsevier, 1998.

[3] B. Xie, Z. Zhang, *Impact of Allee and fear effects in a fractional order prey–predator system incorporating prey refuge*, Chaos: An Interdisciplinary Journal of Nonlinear Science 33 (1) (2023) 013131.

[4] E. Balcı, *Predation fear and its carry-over effect in a fractional order prey–predator model with prey refuge*, Chaos, Solitons and Fractals 175 (2023) 114016.

[5] I. Petráš, Fractional-order nonlinear systems: Modeling, analysis and simulation, Springer Science and Business Media, 2011.

[6] P. Majumdar, B. Mondal, S. Debnath, U. Ghosh, *Controlling of periodicity and chaos in a three dimensional prey predator model introducing the memory effect*, Chaos, Solitons and Fractals 164 (2022) 112585.

[7] S. Işık, F. Kangalgil, *Dynamical analysis and chaos control of a fractional-order Leslie-type predator-prey model with Caputo derivative*, International Journal of Biomathematics 17 (04) (2024) 2350035.

[8] G. Mahapatra, P. Santra, *Prey–predator model for optimal harvesting with functional response incorporating prey refuge*, International Journal of Biomathematics 9 (1) (2016) 1650014.

[9] M. Mandal, S. Jana, S. K. Nandi, T. Kar, *Modeling and analysis of a fractional-order prey-predator system incorporating harvesting*, Modeling Earth Systems and Environment 7 (2021) 1159–1176.

[10] D. Elmacı, F. Kangalgil, *Complex dynamics of a discrete prey-predator model exposing to harvesting and Allee effect on the prey species with chaos control*, International Journal of Bifurcation and Chaos 34 (09) (2024) 2450114.

[11] M. J. Uddin, S. M. S. Rana, S. Işık, F. Kangalgil, *On the qualitative study of a discrete fractional order prey-predator model with the effects of harvesting on predator population*, Chaos, Solitons and Fractals 175 (2023) 113932.

[12] N. A. Khan, O. A. Razzaq, S. P. Mondal, Q. Rubbab, *Fractional order ecological system for complexities of interacting species with harvesting threshold in imprecise environment*, Advances in Difference Equations 2019 (2019) 405.

[13] V. I. Yukalov, E. Yukalova, D. Sornette, *Punctuated evolution due to delayed carrying capacity*, Physica D: Nonlinear Phenomena 238 (17) (2009) 1752–1767.

[14] H. M. Safuan, H. S. Sidhu, Z. Jovanoski, I. N. Towers, *A two-species predator-prey model in an environment enriched by a biotic resource*, Australian and New Zealand Industrial and Applied Mathematics Journal 54 (2012) C768–C787.

[15] J. J. Shepherd, L. Stojkov, *The logistic population model with slowly varying carrying capacity*, Australian and New Zealand Industrial and Applied Mathematics Journal 47 (2005) C492–C506.

[16] S. Pal, A. Gupta, A. Misra, B. Dubey, *Complex dynamics of a predator–prey system with fear and memory in the presence of two discrete delays*, The European Physical Journal Plus 138 (2023) 984.

[17] N. Pati, B. Ghosh, *Delayed carrying capacity induced subcritical and supercritical Hopf bifurcations in a predator–prey system*, Mathematics and Computers in Simulation 195 (2022) 171–196.

[18] C. Jones, J. Lawton, M. Shachak, *Organisms as ecological engineers*, Oikos 69 (1994) 373–386.

[19] D. Müller-Schwarze, The beaver: Natural history of a wetlands engineer, Cornell University Press, 2017.

[20] T. D. Gable, S. M. Johnson-Bice, A. T. Homkes, S. K. Windels, J. K. Bump, *Outsized effect of predation: Wolves alter wetland creation and recolonization by killing ecosystem engineers*, Science Advances 6 (46) (2020) eabc5439.

[21] M. Çiçek, C. Yakar, B. Oğur, *Stability, boundedness, and Lagrange stability of fractional differential equations with initial time difference*, The Scientific World Journal 2014 (2014) 939027.

[22] Z. M. Odibat, N. T. Shawagfeh, *Generalized Taylors formula*, Applied Mathematics and Computation 186 (1) (2007) 286–293.

[23] Y. Li, Z. Liu, Z. Zhang, *Hopf bifurcation for an age-structured predator–prey model with Crowley–Martin functional response and two delays*, Qualitative Theory of Dynamical Systems 22 (2023) 66.

[24] K. Diethelm, N. J. Ford, A. D. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations*, Nonlinear Dynamics 29 (2002) 3–22.

[25] R. Garrappa, *On linear stability of predictor–corrector algorithms for fractional differential equations*, International Journal of Computer Mathematics 87 (10) (2010) 2281–2290.

[26] S. Roy, J. Chattopadhyay, *The stability of ecosystems: A brief overview of the paradox of enrichment*, Journal of Biosciences 32 (2007) 421–428.