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Research Article

Overcoming the intuitive rule of three: improving the false direct and inverse proportion[1]

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Abstract

This article analyzes the teaching of the rule of three and its application in solving mathematical problems involving magnitudes. It is also called direct and inverse proportion. It questions traditional didactics, which rely on intuition, arguing that they omit mathematical justification and lead to conceptual errors. It is proposed an alternative method based on the "First Algebra of Magnitudes", introducing a more rigorous and logical approach. The study highlights how current teaching methods fail to adequately operate physical magnitudes by deleting the magnitudes of operations, leading to the "arithmetization of physics". Traditional resolution methods (unit reduction, proportions, and practical methods) are criticized for their lack of logical foundation. The proposed new method structures reasoning through the algebraic formalization of magnitudes and demonstrates its effectiveness with an empirical study conducted on 97 students. A compound rule of three problem is presented, and different approaches are compared, concluding that the method based on the First Algebra of Magnitudes is more precise and logical. The article emphasizes the need to rethink the teaching of proportionality in mathematics to avoid mechanical and intuitive procedures and promote learning based on deduction and a deep understanding of the relationships between magnitudes.

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Introduction

The rule of three has always been a problem in teaching for student learning. Throughout my teaching career at the university level I have observed the common confusion among students regarding the correct approach to proportionality arithmetic and magnitudes according to the different methodologies taught when the problem poses different dimensions, leading to solutions that do not make logical sense, an issue demonstrated in the present statistical study.

Martínez, Muñoz and Oller (2015, p. 104) point out that "the general practice when presenting resolution methods is the complete absence of justifications. Most of the texts limit themselves to giving a series of instructions to outline the steps to follow for the construction of the solution", something I am in agreement with and which is precisely the reason that led to this research.

[1] This article is a translation of the following publication: Arnaiz Boluda, D. (2024). Método didáctico avanzado para superar la arcaica "regla de tres" clásica, perfeccionada con la proporcionalidad de magnitudes. Criticae. Revista Científica para el Fomento del Pensamiento Crítico, 3(1), 28-45.

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Mathematics is a "deductive science that studies the properties of abstract entities, such as numbers, geometric figures or symbols, and their relationships", a definition given in the fifth meaning of the Dictionary of the Royal Spanish Academy. These properties are precisely the soul of mathematical science, as it requires progressing step by step, with logic, moving from the universal to the particular, demonstrating at each step the properties on which it is based, leaving no room for mere intuition.

Royal Decree 217/2022, of March 29, which establishes the organization and minimum teachings of Compulsory Secondary Education in Spain, states in Annex II regarding mathematics:

> *Reasoning, argumentation, modeling, knowledge of space and time, decision-making, prediction and control of uncertainty or the correct use of digital technology are characteristics of mathematics [...] it is important to develop in students the basic tools and knowledge of mathematics that enable them to navigate successfully in personal, academic and scientific contexts as well as in social and professional environments.*
>
> *[...]*
>
> *Research in didactics has shown that performance in mathematics can be improved if prejudices are challenged and positive emotions towards mathematics are developed. [...] Solving problems is not only a goal of learning mathematics, but is also one of the main ways to learn mathematics. Problem-solving involves processes such as interpretation, translation into mathematical language, application of mathematical strategies, evaluation of the process, and verification of the validity of solutions.*
>
> *[...]*
>
> *The connection between mathematics and other subjects should not be limited to concepts, but should extend to procedures and attitudes, so that basic mathematical knowledge can be transferred and applied to other subjects and contexts.*

As we will see in the following sections, this has been precisely the shortcoming observed with the so-called "compound rule of three", which cannot be referred to as "proportionality of magnitudes" or "direct and inverse proportion", at least not with the approach given in the current bibliography. This is because the rule of three with magnitudes is not simply mathematical; it goes beyond that, belonging to the field of applied mathematics from the very moment the magnitudes or dimensional elements are added, thus encroaching into another discipline such as physics.

At this point, it should be noted that the confusion surrounding the rule of three with magnitudes is not due to a mathematical error, as it is not purely numerical, but rather pertains to physics. Mochón (2012) has identified a problem with the rule of three, blaming it on its mechanical teaching in place of proportion, proposing very interesting didactic methods; however, all of them are characterized by the absence of the magnitudes themselves.

The observed problem is that operations involving physical magnitudes often omit the magnitudes themselves, leading to a significant issue: the arithmetization of physics. This has been identified by the author Arnaiz (2017, p. 25):

> *There is thus a gap yet to be resolved in the operations with physical magnitudes, which causes the proliferation of diverse and contradictory opinions regarding their nature and formulation, discussions that could be settled simply by defining the appropriate composition laws. A group of authors, such as Tolman, attribute to the symbols of dimensional expressions a certain impenetrable or mystical character and consider that "the true essence of magnitudes, from a physical point of view, is represented by their dimensional formula" (Physics Review, 1917, p. 25). This hypothesis does not seem to be true, because it would imply that such disparate magnitudes as the moment of a force and its work, both of which can be expressed as "newton×meter", are essentially manifestations of the same magnitude, energy, which clearly seems unreasonable, as we justify in section XXVI of the First Algebra of Magnitudes.*
>
> *Great authors such as Planck indicate that "it is as meaningless to speak of the 'real' dimension of a magnitude as of the 'real' name of an object", which would imply that physical magnitudes should be concealed from understanding.*

He continues stating (p. 27):

> *All authors versed in dimensional analysis take for granted that unit abbreviations operate with the same algebra as abstract numbers, and based on this implicit and unjustified assumption, they develop their respective theories, which completely omit any specific algebra for magnitudes.* ***The same occurs in the educational field, where the philosophical problems related to magnitudes and their composition laws are overlooked as if they did***

> ***not exist, teaching concrete operations in an intuitive, subjective and arbitrary manner, leaving students, even without knowing it, with a residue of uncertainty*** *that taints all the knowledge acquired, due to this unresolved gap.*

This problem of the arithmetization of magnitudes, which can be summarized as performing operations with concrete numbers by simply omitting the magnitudes ─thereby ignoring the influence of the dimensional element on the result and turning the physical operation into a merely arithmetic question (hence the term "arithmetization")─ is extensively researched by Arnaiz (2017), who has not only identified the problem and studied its historical origins, but also proposed a solution that has proven to be the only applicable way to demonstrate something as basic as the rule of three with magnitudes, also known as "proportional magnitudes" or "direct and inverse proportion" ─an inaccurate expression according to the didactic methodology employed since proportional applied is on the numbers, not to dimensional elements─.

By applying the *First Algebra of Magnitudes* to the proportional magnitudes, the reader will see that the current approach is meaningless, reducing everything to something much simpler and deductive, eliminating any hint of mere intuition referenced by Martínez, Muñoz and Oller (2015, p. 104).

The problem of the arithmetization of magnitudes with the current approach leads to issues such as the problems of estimating unattainable magnitudes posed by Albarracín and Gorgorio (2013).

To guide the reader toward a correct understanding of the practical didactic issues currently present in teaching, a practical problem will first be presented, outlining the different methodologies taught for its solution. Subsequently, the necessary concepts of the *First Algebra of Magnitudes* applicable to this topic will be introduced with the corresponding demonstrations, to finally apply them to the same problem statement.

Finally, the results of a statistical study will be presented, measuring the effectiveness in teaching each of the four methods (the three traditional ones and the proposed fourth method), evaluating the proportion of students who correctly reasoned a problem using each of the didactic methods. This statistical study was conducted among university students to first determine how many remember how to reason a problem correctly before teaching the different methods. This is without prejudice to the proposed subsequent research among secondary students outlined at the end of this article.

Literature review

Problems specific to the compound rule of three are those that involve more than two magnitudes. Problems with two magnitudes are simple.

The problem taken as a reference to contrast the main didactic methods with the fourth one proposed in this research involves four magnitudes, and its statement is as follows: If 5 men dig a 40-meters trench in 8 days working 6 hours a day, how many days will it take for 9 men working 8 hours a day to dig a 60-meter trench?

A problem with this context has been chosen for the arguments given by Martínez, Muñoz and Oller (2015, p. 105): "the predominant contexts in the problems [presented in school books] are those of 'production or consumption in a cooperative work framework' and 'economic or temporal costs of an activity'" with only Anaya's textbooks presenting "problems that involve magnitudes from Physics". However, it is considered that it is precisely with the magnitudes from this science where the true nature of applied mathematics concerning the content of the compound rule of three is observed, without prejudice to the proposed method being equally applicable to any other type of context.

Different resolution methods have been considered in this study: reduction to unity, proportions and practical method. These methods are applied complementarily or substitutely, and even in combination, across all the didactic mathematical bibliography consulted (Zuasti, 2022; González, 2020; García and Ortega, 2016; Mira, 2016; Vallejo and Fuentes, 2016; Didáctica de las ciencias para docentes de educación primaria, 2016; García and Resano, 2015; Segovia and Rico, 2015; Volera and Gazxtelu, 2014; Matemáticas 7, 2011; Gómez, 2006; Galdós, 2000; Postigo, 1998; Matemáticas. Regla de tres, 1982). The proportion method has not been observed independently in any school bibliography. However, it was considered relevant to include it in this research as it has been studied in research articles such as Gómez (2006).

An interesting study is the one proposed by Silvestre and da Ponte (2012), who suggest that before teaching direct proportion or the direct rule of three, students make use of simultaneous calculations with additions and multiplications with "rudimentary strategies" such as unit counting. They conclude with an observation about the student's ability to understand the relationship between variables and the problem's context. This last observation is very important because the system proposed in this article, in replacement of the purely intuitive methods taught, aims precisely to facilitate the algebraic representation of reality.

Quite interesting is the observation made by Ibáñez and Martínez (2020, p. 53), who state that none of the textbooks teaching compound proportions (referring to the compound rule of three) distinguish between quantities of magnitudes and numbers, an issue that has been anticipated here with the problem of the arithmetization of physics.

Theorical Framework

Before applying each of them, we must refer to two premises without demonstration on which they are based:

1. It is said that two magnitudes are directly proportional when, by multiplying one of them by a number, the other is multiplied by the same number, and by dividing one of them by a number, the other is divided by the same number (Grence, 2023, p. 27; Llanos Vaca et al., 2022, p. 131; Galdós, 2000, p.197; Carrillo et al., 2016, p. 157).
2. On the contrary, it is said that two magnitudes are inversely proportional when, by multiplying one of them by a number, the other is divided by the same number, and by dividing one of them by a number, the other is multiplied by the same number (Grence, 2023, p. 29; Llanos Vaca et al., 2022, p. 131; Carrillo et al., 2016, p. 157; Galdós, 2000, p. 198).

Reduction to unity method.

Following the same bibliography, "it is about finding the days it would take for 1 man working 1 hour a day to dig a 1-meter trench and then calculating the time it would take for 9 men working 8 hours a day to dig the 60-meter trench" (Galdós, 2000, p. 199). According to Zuasti (2022, p. 47), there are four magnitudes: the days of work, the daily working hours, the number of men, and the size of the trench, considering that "the best method is to reduce it to a simple proportionality problem".

To this end, it is proposed to multiply 5×8=40 days, which is the time it would take for one man working 6 hours a day to dig a 40-meter trench. By multiplying 40×6=240, we obtain the days it would take for one man working 1 hour a day to dig a 40-meter trench. Finally, they propose that by dividing 240÷40=6, we obtain the days it would take for one man working 1 hour a day to dig a 1-meter trench. By performing the operation

$$\frac{6}{9}=\frac{2}{3}$$

we get the days it would take 9 men working 1 hour a day to dig a 1-meter trench. By performing the operation

$$\frac{\frac{2}{3}}{8}=\frac{1}{12}$$

we obtain the portion of the day it would take for 9 men working 8 hours a day to dig a 1-meter trench. Finally, multiplying

$$\frac{1}{12}\times 60=5$$

we get the days it would take for 9 men working 8 hours a day to dig a 60-meter trench, thus solving the presented problem.

However, only the necessary operations to be performed are explained in the given terms; no general reasoning is provided that could apply to any problem of the same type.

Proportions method

Although this method was completely lost in the LOGSE system by combining with other methods (Martínez et al., 2015, p. 104), which is why it has been included in this study in its independent version, striving to remain faithful to the original symbolism[3].

It method involves organizing the different numerical data based on their magnitudes. Thus, in the first row, the data of the initial assumption are placed, and in a second row the data of the question where the unknown number is located:

Assumption	5 men	6 hours a day	40-meter	8 days
Question	9 men	8 hours a day	60-meter	x

The first step of this method consists of the decomposition into simple rules of three, so that, firstly, we have:

Assumption	5 men	8 days
Question	9 men	x

According to the explanation given for this methodology, "the number of men and the time spent are inversely proportional magnitudes since more men means less time and fewer men means the more time" (Galdós, 2000, p. 201). The following proportion is reached:

$$9:5::8:y$$

It can be observed that the proportion has been reduced to an operation with only two magnitudes, although these have been eliminated from the equation.

Similarly, it is said, we have:

Assumption	6 hours a day	y days
Question	8 hours a day	z

Again, it is noted that "the number of daily hours and the time used are inversely proportional magnitudes since more hours mean less time and fewer hours mean more time". Thus, obtaining:

$$8:6::y:z$$

The same criteria apply to the last proportion:

Assumption	40-meters	z days
Question	60-meters	x

In this case, it is stated that "the length of the trench and the time taken are directly proportional magnitudes", leading of the following proportion:

$$40:60::z:x$$

Finally, the obtained proportions are multiplied term by term:

$$\frac{9 \times 8 \times 40}{5 \times 6 \times 60} = \frac{8 \times y \times z}{y \times z \times x}$$

Solving the mathematical operation, we get:

$$x = \frac{8 \times 5 \times 6 \times 60}{9 \times 8 \times 40} = 5$$

Thus, the calculated solution is "5 days". As in the previous method, no general reasoning is carried out to attempt to prove it.

Practical method.

This method is explained in the following words by Galdós (2000, p. 202):

> *The assumption and the question are written and then each of the magnitudes is compared with the unknown number, assuming that the other magnitudes remain constant, to see if these magnitudes are directly or inversely proportional to the unknown number. The magnitudes that are directly proportional to the unknown number are marked with a + sign at the bottom and a – sign at the top, and the magnitudes that are inversely proportional to the unknown number are marked with a – sign at the bottom and a + sign at the top. Once this is done, the value of the unknown number will*

[3] The symbolism used in this method is specific to the original bibliography when dealing with proportions. That is, the symbol ":" corresponds arithmetically to division, and the symbol "::" to equality. This symbolism has been used since Bails, B. (1805), Galdós (2000, pp. 199-201), and Gómez Alfonso (2006, p. 58). However, it has not been observed in any recent editions of teaching materials for ESO, although it was considered interesting to maintain it in the study.

be equal to the known data of its kind, which is assumed to always carry a + sign, multiplied by all the quantities with a + sign and divided by all the quantities with a − sign. This is the fastest method for solving all types of rule of three problems.

Following these steps, our problem would be set up as follows:

	+	+	−	+
Assumption	5 men	6 hours a day	40-meter	8 days
Question	9 men	8 hours a day	60-meter	x
	−	−	+	−

Thus, Galdós (2000, p. 203) states:

As can be seen, the number of men and the time taken are inversely proportional magnitudes and, therefore, we put the + sign above the men and the − sign below. Similarly, the number of daily hours and the time taken are inversely proportional magnitudes, and, therefore, we put the + sign above the daily hours and the − sign below. Similarly, the length of the trench and the time taken are directly proportional magnitudes, and therefore, we put the − sign above the length and the + sign below.

With all this, it must be inferred that the operation to be performed is, in days:

$$x = \frac{8 \times 60 \times 6 \times 5}{40 \times 8 \times 9} = 5$$

Proposed method of resolution based on the *First Algebra of Magnitudes*.

The topic of the rule of three with magnitudes is the first lesson in Applied Mathematics, as it is the first time that magnitudes are introduced to numbers. Therefore, it is the first moment we encounter the problem of the arithmetization of physics.

When the problem statement specifies quantities "5 men", "6 hours a day", "40 meters" and "8 days", each of the four measurements is what is termed a "dyad". Every dyad consists of two elements: the abstract number and the dimensional element or magnitude it accompanies, which gives a natural sense to the abstract number. As stated by Arnaiz (2017, p. 51): "In general, we have called measurement the quantity, extension or portion of a magnitude expressed in the form q U, as a symbol of the times q, a real number, that a unitary quantity U is present in a phenomenon, denoting q as the measure with the unit U of the magnitude included in the observed fact".

Thus, in our case U=day and q=8, because as he continues, "this newborn entity that refers to physical measurement could also receive the mathematical name of, for example, concrete entity or physical dyad, and its elements will be called primary or measure q or $\overline{q}$, and unit U or secondary. The primary is the mathematical part of the dyad. The secondary is the physical or dimensional part."

The problem arises when we want to operate with dyads, performing additions, subtractions, multiplications, divisions, etc., in such a way that we do not omit the dimensional element ─the magnitude─ of the dyad. The proposed solution is the one that gives its name to the book itself: the *First Algebra of Magnitudes*, where the first steps are taken for operations with magnitudes based on geometry, seeking a new symbology that differentiates it from the merely arithmetic.

The deduction of the proportionality of magnitudes based on the algebra of magnitudes on which the compound rule of three problems are based is carried out in the following sections.

Proportions with homogeneous dyads.

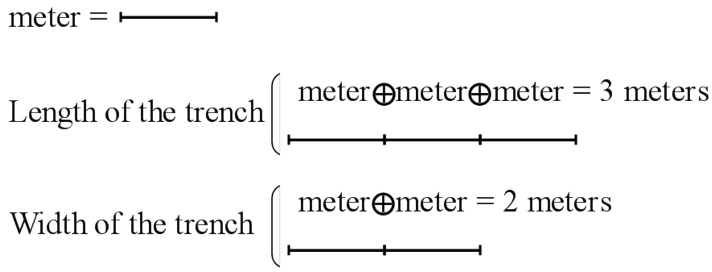
To be clear in the exposition, a simpler problem than the initial one will first be presented, based on the very definition of the rule of three according to Galdós (2000, p. 198): "the rule of three is the arithmetic operation that consists of determining the fourth term of a proportion knowing the other three".

The problem that is presented is the following: a trench 3 meters long by 2 meters wide has been dug. Another one is to be made with sides proportional to those of the previous trench, with the longer side being 6 meters. How wide should it be for both sides to be proportional to those of the smaller trench?

Figure 1 shows two dyads: the first refers to the length of the trench, “3 meters”; and the second to the width, “2 meters”. In *First Algebra of Magnitudes*, operations are performed geometrically by defining each magnitude according to a segment. Thus, “meter” is specified in a segment of a specific measurement. Thus, “meter” is represented by a segment of a specific measure. Since both dyads have “meter” as their dimensional element, the segments representing them will have the same measure.

Figure 1

Geometric representation of homogeneous dyads.



Note: First, the *meter* magnitude is defined by a fixed-length segment. Then, this segment is consecutively added for the length of the trench as many times as the number of meters it has, what is deffined by the abstract number. Since the trench is 3 *meters long*, the segment is added three times. The same applies to the 2 *meter width* of the trench. Since the length and width have different dimensions (a meter of length is not the same as a meter of width), the addition is performed separately.

If we analyze each dimension of the trench, the length is “3 meters,” which geometrically would mean taking the “meter” segment and adding it three times, as shown in Figure 1. Symbolically, we cannot use the “+” sign because we are in the algebra of magnitudes with geometry, not in arithmetic, and we must clearly distinguish between both kind of algebra. Thus, respecting the symbology of the *First Algebra of Magnitudes*[4], the sign we will use is ⊕, resulting in:

$$meter \oplus meter \oplus meter = 1meter \oplus 1meter \oplus 1meter = (1+1+1)\ meters = 3\ meters$$

The problem consists of adding 3 more “meter” segments of length and calculating how many “meter” segments we need to add to the 2 we already have in width to maintain proportionality. Thus, the geometric algebraic operation with respect to the length of the trench will be:

$$3meters \oplus 3meters = (3+3)\ meters = 6\ meters$$

To answer the question, we can observe that the length, composed of 3 “meter” segments, has been added by the same amount:

$$3meters \oplus 3meters = 2 \circ (3meters) = 6\ meters$$

Again, the multiplication symbol known to all for operations with abstract numbers cannot be used in the same way for geometric multiplications of an abstract number by a dyad, so the symbol ○ is chosen following the *First Algebra of Magnitudes*. The same applies to division, whose symbol used as a geometric operation is //.

Thus, if we compare the length of the trench before and after the addition, we find the multiplier of the width of the trench to answer the question:

$$\frac{6\ meters}{3\ meters} = \frac{6}{3} = 2$$

Therefore, 2 is the abstract multiplier of both sides of the trench to maintain proportionality between the two dyads. Note that dividing two homogeneous dyads results in an abstract number, as one magnitude is geometrically divided by itself. As Arnaiz (2017, p. 152) explains, algebraically:

$$a \circ (b\ magnitude) = c\ magnitude \rightarrow a = \ (c\ magnitude)//(b\ magnitude)$$

To facilitate the solution to the problem, given the measurements of the length and width of the trench before and after the addition, whose final width is unknown, we have:

[4] The symbology of the *First Algebra of Magnitudes* is developed throughout the entire volume as the various demonstrations are carried out. However, the author provides a synoptic diagram of all the signs used for the different operations, differentiating them according to whether they pertain to ordinary numerical algebra or dyadic algebra (Arnaiz, 2017, p. 220).

$$\frac{6\,meters}{3\,meters} = \frac{x\,meters}{2\,meters} \rightarrow \frac{6}{3} = \frac{x}{2} \rightarrow x = \frac{6}{3} \times 2 = 4$$

Let's look at it with another example: if 12 coats cost 360 euros, how much will 8 coats cost? We will pose the problem with magnitudes as:

$$\frac{12\,coats}{8\,coats} = \frac{360\,euros}{x\,euros} \rightarrow \frac{12}{8} = \frac{360}{x} \rightarrow x = \frac{360 \times 8}{12} = 240$$

But 240 for what exactly? To answer this question, we refer to the geometric proportion where we have the dyad with its magnitude, allowing us to complete the dyad of the consequent in the second geometric ratio, referring to 240 euros. Having explained the dyadic addition with two different examples, it is time to verify the generic property for its validity in any particular case.

Demonstration of proportions with homogeneous dyads

Given a numerical element "a" and its magnitude or dimensional element "$magnitude_a$" forming the dyad "a $magnitude_a$", if any abstract multiplier such as "x" is applied, the following geometric operation occurs:

$$a\ magnitude_a \rightarrow x\circ(a\ magnitude_a)$$

For the ratio of this addition to be proportional to the addition of another ratio of a different dyad such as "b $magnitude_b$", the multiplier must be the same:

$$b\ magnitude_b \rightarrow x\circ(b\ magnitude_b)$$

such that it will always be hold that:

$$\frac{x \circ (a\,magnitude_a)}{a\,magnitude_a} = \frac{x \circ (b\,magnitude_b)}{b\,magnitude_b} \rightarrow \frac{x \times a}{a} = \frac{x \times b}{b}$$

thus proving the property of proportions with homogeneous dyads, which can be stated as follows: for two ratios between homogeneous or additive dyads to be proportional, they must have the same abstract multiplier or quotient.

The reader will observe that this property nullifies one of the premises on which the rule of three with magnitudes is based, specifically the one that refers to the "direct proportionality between magnitudes".

Proportions with heterogeneous dyads

Continuing with the same type of approach, let's start with a simple example: if 6 workers complete a job in 20 days, how long will it take 8 workers to complete the same job?

In this case, the dyads we encounter have "workers" as one dimensional element and "days" as another. This means that the nature of the magnitudes is different, so their combination results in a third magnitude that can be defined as "work necessary to finish the job", meaning the relationship between both dyads is no longer additive, as in the previous problems.

This is what the *First Algebra of Magnitudes* refers to as "heterogeneous dyads", whose geometric treatment must be different in terms of the relationship between the two magnitudes.

Thus, like any other magnitude, "worker" and "day" will be geometrically defined by respective segments of equal or different lengths. How should both magnitudes be related? Through geometric multiplication.

In this way, defining the magnitude "worker" as a segment of a certain fixed length and the magnitude "day" as another segment of equal or different fixed length, multiplying both segments geometrically results an area that expresses the third magnitude directly related to the two previous ones: the composite unit in which the work necessary to complete the job will be measured. This is shown graphically in Figure 2.

To perform the geometric multiplication operation between two dyads, we cannot use the × or • signs used for abstract numbers, nor the ○ symbol defined in the *First Algebra of Magnitudes* for the product between an abstract number and a dyad. Thus, following the same bibliography, the chosen sign is ∗.

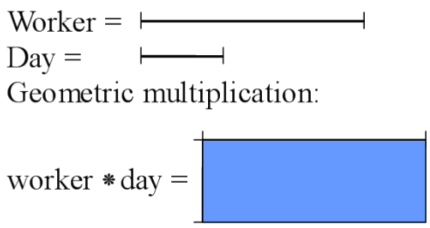


Figure 2. Graphic representation of heterogeneous dyads

Note: First, we define the fixed length of a segment for each magnitude. In this case, the *worker* magnitude will be a segment of equal or different length than the *day* magnitude. Since we are combining two magnitudes of different dimensions, the geometric operation to be performed is multiplication, which is symbolized according to the *First Algebra of Magnitudes* with the symbol $*$. The combination of these two magnitudes through geometric multiplication results in a new dyad, geometrically defined by the resulting surface. This is why they are heterogeneous dyads.

Therefore, completing the job in question requires 6 workers and 20 days of work. The geometric area obtained by multiplying both dyads is the "work needed to complete the job" (Figure 3), which symbolically will be:

$$6\ workers * 20\ days = work\ needed$$

The question posed is: to finish that same job, or in other words, the work needed to complete the job with 8 workers, how many days will it take?

This implies that we are being presented with another geometric operation, but the result is the same:

$$8\ workers * x\ days = work\ needed$$

Therefore:

$$6\ workers * 20\ days = 8\ workers * x\ days$$

If we operate separately on the numerical elements from the dimensional ones on the first side of the equation:

$$(6 \bullet 20) \circ workers * days = 8\ workers * x\ days$$

Solving geometrically for the dyad that is the unknown in the problem, we get:

$$\frac{(6 \bullet 20) \circ workers * days}{8\ workers} = x\ days$$

As explained earlier, "dividing two homogeneous dyads results in an abstract number, as a magnitude is geometrically divided by itself".

$$\frac{(6 \bullet 20) \circ days}{8} = x\ days$$

Separating on the first side of the equation and operating on the abstract numbers of the magnitude "day", we have:

$$\frac{6 \bullet 20}{8} \circ days = x\ days$$

Thus, it results in:

$$15\ days = x\ days$$

This answers the question, concluding that to complete the same work with 8 workers, 15 days are needed.

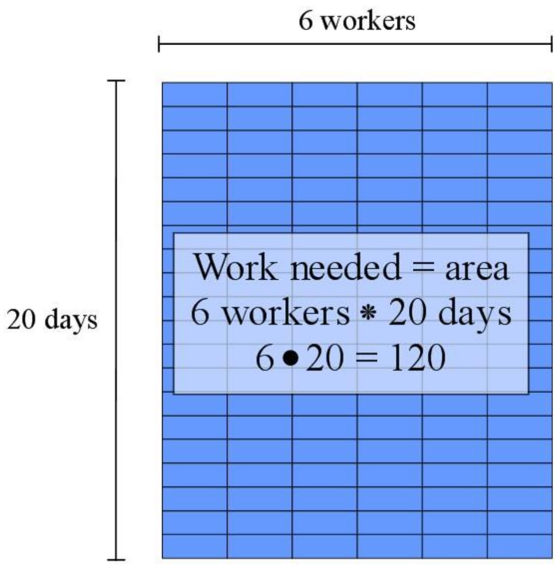


Figure 3. Graphical representation of the heterogeneous dyads in the example

Note: Once we have defined the segments of the two magnitudes separately and combined them as shown in Figure 2, the same operation can be performed after adding to each magnitude. In this way, if we add the *day* magnitude 20 times and the *worker* magnitude 6 times, resulting in two segments of size 6 *workers* and 20 *days*, and then combine the two resulting dyads through geometric multiplication, we obtain the third resulting heterogeneous dyad, defined by the expression 6 *workers* ∗ 20 *days*, which in this case can be defined as w*ork needed*. Geometrically, this new dyad is the surface.

As we have done in the case of proportions with homogeneous dyads, let's consider a second example of proportions with heterogeneous dyads: a car traveling at 140 kilometers per hour takes 4 hours to complete a trip. How long would it have taken if it had traveled at 112 kilometers per hour?

We observe that the given magnitudes are "speed" and "time", so both combined give rise to a third magnitude: "total distance traveled". Time is measured in this case in "hours", and speed in "kilometers per hour". By geometrically defining each of the two given magnitudes with a segment of a certain length, whether equal or different, their combination will result in an area that represents the distance traveled in kilometers. Thus, considering the data of the problem, the geometric operation will be:

$$Distance = 140\ kilometers\ per\ hour * 4\ hours$$

For the question posed, the distance (area of the geometric square obtained) being the same:

$$Distance = 112\ kilometers\ per\ hour * x\ hours$$

Since both geometric products are equal to the same area, we know their equality:

$$140\ kilometers\ per\ hour * 4\ hours = 112\ kilometers\ per\ hour * x\ hours$$

Operating with the established geometric symbolism:

$$\frac{140\ kilometers\ per\ hour}{\overline{112\ kilometers\ per\ hour}} * 4\ hours\ =\ x\ hours$$

Having both dyads in the geometric division with the same dimensional elements "kilometers per hour" in numerator and denominator and separating the numerical elements from the resulting dimensional:

$$\left(\frac{140}{112} \times 4\right) \circ\ hours =\ x\ hours$$

Thus, it resolves that:

$$5\ hours = x\ hours$$

Answering the question.

However, it is convenient to make a note here, as algebraic operations were performed with a magnitude (speed), which is itself part of another combination of dyads. Speed is measured in kilometers per hour as has been discussed in the problem; but this approach can be different if we consider that the speed is expressed as the distance traveled in a specific unit time period, so the assumption will be:

$$distance = \frac{140\ kilometres}{1\ hour} * 4\ hours$$

while the question will be:

$$distance = \frac{112\ kilometres}{1\ hour} * x\ hours$$

This leads to the following equality, which, when operating geometrically:

$$\frac{140\ km}{1\ hour} * 4\ hours = \frac{112\ km}{1\ hour} * x\ hours$$

$$(140 \bullet 4) \circ \frac{km * hours}{hour} = (112 \bullet x) \circ \frac{km * hours}{hour}$$

Since the combination of equal magnitudes results in an arithmetic equality, weobtain the same solution as the previous approach:

$$x = \frac{140 \bullet 4}{112} = 5$$

Demonstration of proportions with heterogeneous dyads

Continuing with the same methodology as with homogeneous dyads, let's demonstrate that proportions are met with heterogeneous dyads, establishing a new general property applicable to any particular case, as a deductive system that characterizes applied mathematics, in this case, physics.

Given two heterogeneous dyads such as "$a\ magnitude_a$" and "$b\ magnitude_b$", these give rise to a third "$c\ magnitude_c$" which is a combination of the two through geometric multiplication:

$$a\ magnitude_a * b\ magnitude_b = (a \bullet b) \circ magnitude_a * magnitude_b = c\ magnitude_c$$

such that the value of the combined dyad

$$(a \bullet b) \circ magnitude_a * magnitude_b$$

which is the area of the abstract rectangle formed, must remain invariant. When multiplying any of the two original dyads, which are the sides of this rectangle, the other will be divided, maintaining the geometric equality of the area.

Thus, in the previous general approach, let's suppose that the first dyad increases its value by an abstract multiplier such as "x". If the combined dyad (the area of the rectangle) remains invariant, the equality would be lost:

$$[x \circ (a\ magnitude_a)] * b\ magnitude_b \neq (a \bullet b) \circ magnitude_a * magnitude_b$$

To maintain this equality with the combined dyad being invariant, the second dyad will be altered, being divided by the same multiplier, which is symbolically expressed as:

$$[x \circ (a\ magnitude_a)] * [(1/x) \circ b\ magnitude_b] = (a \bullet b) \circ magnitude_a * magnitude_b$$

This is so because, if we operate the first member of the geometric equality:

$$[x \circ (a\ magnitude_a)] * [(1/x) \circ b\ magnitude_b] = (x \bullet a \bullet b/x) \circ magnitude_a * magnitude_b$$

It is not necessary to recall that a number divided by itself is the unit.

Thus, the proportion between heterogeneous dyads can be formulated as follows: given the combination between two heterogeneous dyads that result in a third combined dyad, when the first dyad is multiplied or divided by an abstract number, to maintain this equality with the combined dyad being invariant, the second dyad will be divided or multiplied, respectively, by the same abstract number.

Application of the proposed *First Algebra of Magnitudes* method to the given problem.

Let us recall the statement of the initial problem: if 5 men working 6 hours a day have dug a 40-meter trench in 8 days, how many days will it take for 9 men working 8 hours a day to dig a 60-meter trench?

In the given assumption, the following dyads are observed: "5 men", "6 hours a day", "8 days" and "40 meters". It is immediately observed that there is a homogeneous relationship between magnitudes "hours a day" and "days", which is "time", thus the dimensional element "days" is redundant of the "hours a day", making the numerical element the multiplier or abstract number. Therefore, the symbolic statement of this fact can be:

$$8 \circ (6\ hours\ a\ day) = 6 \circ (8\ days) = (6 \bullet 8) \circ hours$$

However, from the combination of "time" and the "men" working, a new magnitude arises, which is the "length of the trench" in question, representing the work done. Thus, geometrically, "time" and the "men" working will be defined by respective segments of certain lengths added as many times as the abstract multiplier establishes, and each resulting dyad will form one side of the area formed by geometric multiplication. Thus:

$$(6\bullet 8) \circ \textit{hours} * 5 \textit{ men} = 40 \textit{ meters}$$

Similarly, for the symbolic representation of the question:

$$(8\bullet x) \circ \textit{hours} * 9 \textit{ men} = 60 \textit{ meters}$$

It is observed that in this case, the area formed by the combination of the two dyads is not the same; however, they do have the same dimensional element, so if we reduce the combined dyad to the dimensional unit in each of the geometric equalities, we have in each case:

$$\frac{(6 \bullet 8) \circ hours * 5\, men}{40} = meters$$

$$\frac{(8 \bullet x) \circ hours * 9\, men}{60} = meters$$

Thus, we obtain the following equality:

$$\frac{(6 \bullet 8) \circ hours * 5\, men}{40} = \frac{(8 \bullet x) \circ hours * 9\, men}{60}$$

Operating geometrically distinguishing the abstract numbers from the magnitudes:

$$\left(\frac{6 \bullet 8 \bullet 5}{40}\right) \circ hours * men = \left(\frac{8 \bullet x \bullet 9}{60}\right) \circ hours * men$$

from which it results:

$$\frac{6 \bullet 8 \bullet 5}{40} = \frac{8 \bullet x \bullet 9}{60}$$

Therefore, we obtain the value of the unknown number:

$$x = \frac{6 \bullet 8 \bullet 5 \bullet 60}{40 \bullet 8 \bullet 9} = 5$$

Recovering the dimensional element of the unknown, the answer to the question is that 5 days of work are needed.

Application of the method to a more complex problem

Let's suppose the following problem: In a factory, 8 men and 12 women work. The men work 9 hours a day, and the women work 7 hours a day. Together, they produce 360 units of a product in 6 days. The factory decides to hire 4 more men and 6 more women, and also adjust the work schedule so that both men and women work 8 hours a day, but they just work 5 *days*.

To solve this problem, we must first identify the different magnitudes presented: men, women, hours per day, days, and units of product manufactured. It is necessary to distinguish between men and women because their working hours are different. Additionally, according to the problem statement, the contribution of men and women to the work may vary, meaning that the geometric segment defining these two magnitudes may also be different.

On the other hand, it is observed that the magnitudes *hours per day* and *days* share the same dimensional nature, which is *time*. Likewise, the relationship between *working time* and the number of *men* or *women* working determines the *total hours of work* based on gender, making them homogeneous. The total working time of men and women is then added together, and this total work results in the production of 360 units.

Symbolically, it would be represented as follows:

$$8 \textit{ men} * 9 \textit{ hours/day} * 6 \textit{ days} + 12 \textit{ women} * 7 \textit{ hours/day} * 6 \textit{ days} = 360 \textit{ units}$$

The question follows the same symbolic structure in its formulation, with the number of units produced as the unknown variable:

$$12 \textit{ men} * 8 \textit{ hours/day} * 5 \textit{ days} + 18 \textit{ women} * 8 \textit{ hours/day} * 5 \textit{ days} = \mathrm{X} \textit{ units}$$

Dividing both equations term by term, we obtain the following expression, easily reaching the result:

$$\frac{8\, men * 9\, hours/day * 6\, days \oplus 12\, women * 7\, hours/day * 6\, days}{12\, men * 8\, hours/day * 5\, days \oplus 18\, women * 8\, hours/day * 5\, days} = \frac{360\, units}{\mathrm{X}\, units}$$

$$\frac{8 \times 9 \times 6 + 12 \times 7 \times 6}{12 \times 8 \times 5 + 18 \times 8 \times 5} = \frac{360}{\mathrm{X}}$$
$$\mathrm{X} \approx 461{,}54 \textit{ units}$$

Proposition of the new method based on the *First Algebra of Magnitudes*.

With all the above reasoning, the method based on the *First Algebra of Magnitudes* is proposed in the following terms. The *First Algebra of Magnitudes* proposes not eliminating magnitudes in physical operations by the algebra of magnitudes as a solution of the problem of the arithmetization of physics. Thus, every measurement consists of two elements: a numerical one and a dimensional one. For example, “5 meters long” is composed of the numerical element “5” and the dimensional “meter long”, and algebraically they are multiplied. If more “meters long” are added, the abstract number 5 will increase, being a homogeneous proportion, as the dimensional element is the same: “meter long”. If measurements of a different nature are added, that is, with a different dimensional element such as “2 meters wide”, we will have a heterogeneous proportion, which is the geometric multiplication of the two measurements “5 meters long” and “2 meters wide”, resulting a third measurement: “surface”.

As with arithmetic operations, magnitudes can be operated on (although with different symbols as it is geometric algebra and not just arithmetic, symbolism that we are not going to introduce here). This is geometric algebra since it treats dimensional elements as segments of a certain length. Thus, “meter long” is a segment of a certain length. If more segments of this same dimension are added, they result in a longer segment determined by the abstract number, in our example “5” segments, which has been called homogeneous proportion. If segments of another dimension are added, such as “meter wide”, they will have equal or different length, but they can no longer be added after the “meters long” because they are of a different nature, and they must be multiplied together, geometrically resulting in an area that represents the third resulting dimensional element, in this case the “surface”. This is the heterogeneous proportion.

This method applied to problems with magnitudes consists of obtaining an equality, how much work is needed (a combination of measurements of equal or different dimensions) to obtain a result, that is:

Amount of work = Work result

Let's see it with an example: if 5 men working 6 hours a day have dug a 40-meter trench in 8 days, how many days will it take 9 men working 8 hours a day to dig a 60-meter trench?

For the given assumption, how much work is needed? 5 men, 6 hours a day and 8 days are needed. What is the result of the work? A 40-meter trench. Thus:

$$5 \textit{ men} \times 6 \textit{ hours a day} \times 8 \textit{ days} = 40 \textit{ meters}$$

To solve the unknown number, the question can be posed similarly, knowing that the work needed is: 9 men and 8 hours a day for x days (the dimensional elements are the same). The result of the work is a 60-meter trench:

$$9 \textit{ men} \times 8 \textit{ hours a day} \times x \textit{ days} = 60 \textit{ meters}$$

From here, it is sufficient to move everything to one side of the equation to isolate the variable (remember that "40 meters" and "60 meters" are algebraic multiplications). Thus, we obtain:

$$\frac{5\, men\, \times\, 6\, hours\ a\ day\, \times\, 8\, days}{40\, meters} = 1$$

Likewise:

$$\frac{9\, men\, \times\, 8\, hours\ a\ day\, \times\, X\, days}{60\, meters} = 1$$

Now, since both expressions equal 1, they are equal to each other. If we separate the numbers from the magnitudes, we get for the first case:

$$\frac{5 \times 6 \times 8}{40} \times \frac{men\, \times hours\ a\ day \times days}{meters} = 1$$

And for the second case:

$$\frac{9 \times 8 \times X}{60} \times \frac{men\, \times hours\ a\ day \times days}{meters} = 1$$

We can observe that we have the same magnitudes in both expressions, so all the magnitudes cancel out, leaving only the numerical part:

$$\frac{5 \times 6 \times 8}{40} = \frac{9 \times 8 \times X}{60}$$

Operating:

$$6 = 1{,}2 \times X$$

Solving for the unknown X, we get the result in a mere arithmetic equation:

$$6/1{,}2 = X = 5$$

Method

Research Model

The objective of the questionnaire is to measure the degree of effectiveness of each of the didactic methods of the compound rule of three.

To achieve the objective, the questionnaire was structured as follows:

- Firstly, the following problem was presented: if 5 men working 6 hours a day have dug a 40-meter trench in 8 days, how many days will it take for 9 men working 8 hours a day to dig a 60-meter trench?

The objective of this first question is to determine if they remember how to solve this type of problem.

- The second part of the questionnaire aims to measure the didactic effectiveness of each of the methods in this research. To this end, the steps followed were:
- First, an explanation of the reduction to unity method with an example.
- Second, presentation of a problem to be solved by the student using the reduction to unity method.
- Third, an explanation of the proportion method with an example.
- Fourth, presentation of a problem to be solved by the student using the proportion method
- Fifth, an explanation of the practical method with an example.
- Sixth, presentation of a problem to be solved by the student using the practical method.
- Seventh, an explanation of the method based on the *First Algebra of Magnitudes* with an example.
- Eighth, presentation of a problem to be solved by the student using the method of the First Algebra of Magnitudes.

Each didactic method was explained in writng according to the bibliography studied in this research in the same terms presented, having given a time of ten minutes for the study of each one. The method of the *First Algebra of Magnitudes* was conducted with the same degree of demonstration and extension as the three traditional methods in the terms of section III of this article.

The problem explained as an example in each didactic method in steps one, three, five and seven was always the same: if 5 men working 6 hours a day have dug a 40-meter trench in 8 days, how many days will it take for 9 men working 8 hours a day to dig a 60-meter trench?

The problem presented to be solved according to each method in steps two, four, six and eight was always the same to maintain the level of difficulty: if 60 printers working 40 days at 8 hours a day print 1,000 books, how many days will it take 30 printers working 6 hours a day to print 750 books?

Since the objective is to measure the didactic efficiency of each method, they were not allowed to take notes that they could consult during the subsequent problem-solving, and the written explanation was removed.

Additionally, they were never told whether the result they obtained in the problems they had to solve was correct or not, to avoid conditioning them for subsequent solutions.

A final problem was presented, different from all the previous ones, so that they could solve it by the method they chose. The problem in questionwas the following: leaving 9 taps open for 8 hours at 20 degrees results in an expense of 48 euros. At what temperature should the water be if 15 taps are left open for 5 hours to produce an expense of 60 euros?

Participants

The subjects who participated in the study were 97 students of the Business Administration and Management Degree in the 1st, 2nd and 3rd years. The age of the students was 18-21 years old.

Results and Discussion

The first problem presented was solved correctly by 26 students (26.80% of the sample) out of the 97 who participated. Of these, 19 solved it by intuition, 1 by the reduction to unity method, 3 by the proportion method and 3 by the practical method.

The results obtained regarding point 2 of the previous section can be seen in Table 1. To present these results more simply, Illustration 1 is provided showing whether it was correctly solved or, on the contrary, it was incorrectly solved or not solved at all.

Table 1. Results of the solution to the presented problems according to each didactic method (point 2 of section VI.1):

Method	No reasoning	%	Incorrectly reasoning	%	Correctly reasoning	%
Reduction to unity	39	40,63	53	55,21	4	4,17
Proportions	33	34,02	57	58,76	7	7,22
Practical	21	21,65	44	45,36	32	32,99
First Algebra of Magnitudes	8	8,25	12	12,37	77	79,38

Note: The results obtained in solving the proposed problem are compared according to each method. It has been specified based on whether the solution was not formulated, leaving the answer blank, whether the solution was set up using the requested method, and whether the approach was correct.

It can be observed that the best result was obtained with the *First Algebra of Magnitudes* method, with which 79.38% have reasoned it correctly, while with the practical method this figure rises to 32.99%. The methods of reduction to unity and proportion had a very low success rate, remaining at 4.17% and 7.22% respectively.

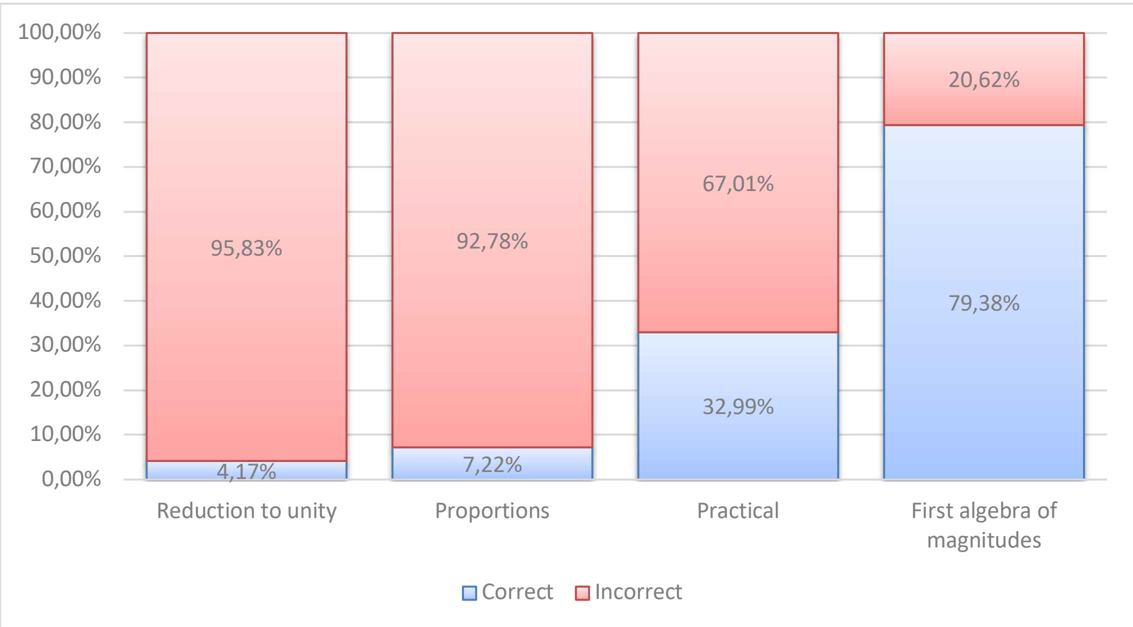


Figure 1. Results of the solutions to the presented problems according to each didactic method (point 2 of section IV.1)

Note: This illustration shows graphically the same results ot table 1 comparing if the problem was correctly solved according to each method.

With all this, a notable improvement in the understanding of this type of problem can be observed with the method of the *First Algebra of Magnitudes* compared to the others, considering that this method is completely new to the students since they had never encountered it before completing this questionnaire, which is not the case with at least one of the other three methods.

It is also observed that the practical method is easier to understand compared to the other two (reduction to unity and proportions), although the success rate in solving problems remains low.

Note that the number of students in the reduction to unity method is 96 (one less than the rest of the methods). This is because, once they began solving the problem according to this method, it was observed that one student had copied the explanation previously made having solved the problem with that material, so it has been removed from the results. Finally, Table 2 shows the results obtained regarding point 3 of section IV. 1, that is, the resolution of a new problem by the method freely chosen by each student. Additionally, Illustration 2 shows the graphical representation of these results.

Table 2. *Results of the last problem proposed according to the method chosen by each student (point 3 of section 3.2.).*

Method	Correct	%	Incorrect	%
Reduction to unity	0	0	3	3,09
Proportions	0	0	3	3,09
Practical	3	3,09	6	6,19
First Algebra of Magnitudes	72	74,23	1	1,03

Note: This table shows the results obtained in solving the last problem of the questionary. The students freely chose the method they preferred to solve it. The absolute and relative distribution of correct or incorrect solutions for each method is indicated.

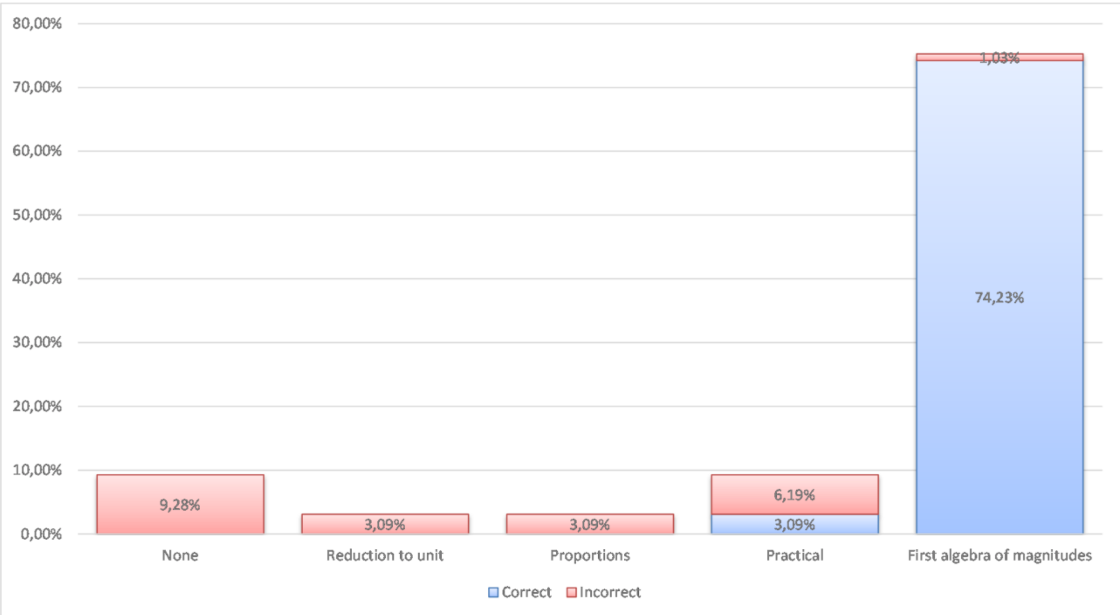


Figure 2. Results of the last problem proposed according to the method chosen by each student (point 3 of section 3.2.)

Note: Graphical representation of the results obtained in solving the last problem of the questionary, based on the method chosen by each student, indicating the relative distribution of correct and incorrect solutions for each method in relation to the total number of students.

It can be seen that 9.28% of the students did not choose any method, and were unable to reason the solution correctly. In most of these cases, there is an attempt to solve the problem by intuition. 75.26% (73 students) preferred the method of the *First Algebra of Magnitudes*, with only one student erring in the reasoning. 9 students preferred the practical method, with 66.67% of them erring, a proportion very similar to that in Illustration 1 for this same method. Finally, the 3 students who chose the proportion method reasoned it incorrectly. The same occurs with the 3 students who chose the reduction to unity method.

Conclusions

The primary conclusion is that the rule of three, as a didactic method, becomes obsolete due to its lack of foundation, relying solely on intuition, and is replaced by a true proportionality of magnitudes through the geometric algebra of dyads that are traditionally known as concrete numbers.As has been seen, the traditional rule of three relies on two theories that are displaced by the demonstrations of homogeneous and heterogeneous dyads based on the geometry of the algebra of magnitudes.

The theory of direct proportionality of magnitudes —"it is said that two magnitudes are directly proportional when, by multiplying one of them by a number, the other is multiplied by the same number, and by dividing one of them by a number, the other is divided by the same number"— loses rigor with the arithmetization of physics and is replaced by the property of dyadic addition or "proportion with homogeneous dyads" which can be summarized as: for two ratios between homogeneous or additive dyads to be proportional they must have the same abstract multiplier or quotient.
The theory of inverse proportionality of magnitudes —"it is said that two magnitudes are inversely proportional when, multiplying one of them by a number, the other is divided by the same number, and by dividing one of them by a number, the other is multiplied by the same number"— suffers from the same problem as the previous one, being replaced by the property of "proportion with heterogeneous dyads", that is, "given the combination between two heterogeneous dyads that result in a third combined dyad, when the first dyad is multiplied or divided by an abstract number, to maintain this equality with the combined dyad being invariant, the second dyad will be divided or multiplied respectively by that same abstract number".

Thus, the limitations of the intuitive principles on which traditional methods are based are overcome, and the problem of the arithmetization of physics is resolved, where the traditional solution relies on eliminating the magnitudes from operations, turning them into mere arithmetic, which, as has been justified in this article, is not the best idea. Instead, magnitudes with their respective numerical elements should be maintained, forming dyads.
Thus, the problems of the simple "rule of three" —with two magnitudes— and the compound "rule of three" —with more than two magnitudes— are reduced to a mere proportionality of magnitudes, whose only difficulty lies in the correct reasoning according to the relationship between the different dyads, which is determined by the nature of their dimensional elements.

If they are of the same nature, they will be homogeneous dyads consisting of addition, which is directly reflected in the numerical element of the dyad, being increased or decreased while maintaining the dimensional element. Conversely, if they are heterogeneous dyads, their respective dimensional elements or magnitudes are of a different nature and the operation between them is based on geometric multiplication and division, resulting in another dyad with a different dimensional element resulting from the combination of the previous ones.

In the problems of the so-called "compound rule of three", a series of homogeneous and heterogeneous dyads are always presented, which combination results in another dyad, and two assumptions are provided where only the numerical elements vary, leaving one as the unknown to be calculated. Thus, the problems of the "compound rule of three" are reduced to a mere algebraic proportional reasoning of magnitudes.

Finally, a didactic method based on the *First Algebra of Magnitudes* has been proposed —without prejudice to the fact that the method can be improved by including even the real symbolism of the algebra of magnitudes— and its didactic effectiveness has been verified with a questionnaire which results are summarized below.
The method of the *First Algebra of Magnitudes* has greater ease of understanding than the other three traditional methods, reaching a 79.38% correct reasoning of the problem proposed. The students who correctly reasoned the problem according to the other methods were quite inferior (32.99% the practical method; 7.22% the proportion method; 4.17% the reduction to unity method).

It has also been concluded that students prefer the method based on the *First Algebra of Magnitudes* (75.26% compared to 9.28% of the practical method, 3.09% of the proportion method, 3.09% of the reduction to unity method and 9.28% for none).

This new method, based on the *First Algebra of Magnitudes*, is important as a logical reasoning approach for solving typical problems of the classic rule of three. It gives meaning to the chosen title for this topic, "direct and inverse proportion of magnitudes", since, although this name is commonly used, there is no actual proportional calculation of magnitudes. Instead, it falls into the problem of the arithmetization of physics, which is simply resolved by removing the magnitudes from the operations.

Being the first topic in applied mathematics to physics, it is crucial to resolve the problem of arithmetization, preventing adolescents from developing an aversion to something so simple at an early age. Additionally, based on the *First Algebra of Magnitudes*, it is concluded that the possible existence of dismetric magnitudes could be considered—that is, the idea that the same magnitude may not always have the same measure depending on the surrounding circumstances. A meter in the vacuum of space may not be the same as a meter in a black hole.

This is precisely the case observed with the last proposed and reasoned problem in section 4.6. Although the problem was solved using isometric magnitudes—meaning that all men and women are considered equal—the reality is that each man is different from the others, just as each woman is, and no two people are exactly alike. Additionally, the same person is not always the same from one day to the next, so his contribution to work is not always the same.

This leads us to the concept of dismetric magnitudes, and these types of problems can be solved by taking this factor into account. However, operations involving such magnitudes become more complex, even requiring the use of tensors. In conclusion, to begin with, a review of the didactic methods of the classic rule of three is necessary, incorporating a true proportionality of magnitudes that corrects what has so far been referred to as direct and inverse proportionality.

Recomendations for Researchers

Based on this study, the following lines of research are suggested:

First: repeat the same statistical study to students who are learning the different methods of the compound rule of three for the first time, that is, to 12-year-old students. However, the explanation of the method based on the *First Algebra of Magnitudes* can be improved (even considering the possibility of teaching the method with the symbolism of the *First Algebra of Magnitudes* and all the reasoning presented in this research) as follows:

The *First Algebra of Magnitudes* proposes not eliminating magnitudes in physical operations by the algebra of magnitudes as a solution of the problem of the arithmetization of physics. Thus, every measurement consists of two elements: a numerical one and a dimensional one. For example, "5 meters long" is composed of the numerical element "5" and the dimensional "meter long", and algebraically they are multiplied. If more "meters long" are added, the abstract number 5 will increase, being a homogeneous proportion, as the dimensional element is the same: "meter long". If measurements of a different nature are added, that is, with a different dimensional element such as "2 meters wide", we will have a heterogeneous proportion, which is the geometric multiplication of the two measurements "5 meters long" and "2 meters wide", resulting a third measurement: "surface".

As with arithmetic operations, magnitudes can be operated on (although with different symbols as it is geometric algebra and not just arithmetic, symbolism that we are not going to introduce here). This is geometric algebra since it treats dimensional elements as segments of a certain length. Thus, "meter long" is a segment of a certain length. If more segments of this same dimension are added, they result in a longer segment determined by the abstract number, in our example "5" segments, which has been called homogeneous proportion. If segments of another dimension are added, such as "meter wide", they will have equal or different length, but they can no longer be added after the "meters long" because they are of a different nature, and they must be multiplied together, geometrically resulting in an area that represents the third resulting dimensional element, in this case the "surface". This is the heterogeneous proportion.

Here its is observed that dividing two homogeneous dyads results in an abstract number. This is because if we add 2 segments to the same dimension three times, for example, "meter", it results in:

$$2meters + 2meters + 2meters = 3 \times (2meters) = 6meters \rightarrow \frac{6meters}{2meters} = \frac{6}{2} = 3$$

This method applied to problems with magnitudes consists of obtaining an equality, how much work is needed (a combination of measurements of equal or different dimensions) to obtain a result, that is:

Amount of work = Work result

Let's see it with an example: if 5 men working 6 hours a day have dug a 40-meter trench in 8 days, how many days will it take 9 men working 8 hours a day to dig a 60-meter trench?

For the given assumption, how much work is needed? 5 men, 6 hours a day and 8 days are needed. What is the result of the work? A 40-meter trench. Thus:

$$5 \; men \times 6 \; hours \; a \; day \times 8 \; days = 40 \; meters$$

To solve the unknown number, the question can be posed similarly, knowing that the work needed is: 9 men and 8 hours a day for x days (the dimensional elements are the same). The result of the work is a 60-meter trench:

$$9 \; men \times 8 \; hours \; a \; day \times X \; days = 60 \; meters$$

From here, it is enough to divide both equalities term by term, obtaining the following homogeneous operation:

$$\frac{5 \; men \times 6 \; hours \; a \; day \times 8 \; days}{9 \; men \times 8 \; hours \; a \; day \times X \; days} = \frac{40 \; meters}{60 \; meters}$$

Since the magnitudes are divided by themselves, the operation becomes merely arithmetic:

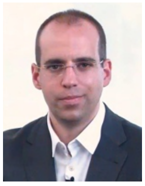
$$\frac{5 \times 6 \times 8}{9 \times 8 \times X} = \frac{40}{60}$$

Solving for the unknown X, we get the value of X:

$$X = \frac{5 \times 6 \times 8 \times 60}{9 \times 8 \times 40} = 5$$

Second: apply the *First Algebra of Magnitudes* to different topics in Physics with the corresponding didactic method, also conducting a statistical study to compare the teaching effectiveness and student acceptance.

Biodata of the Author



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