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A Comprehensive Study on Restricted and Extended Intersection Operations of Soft Sets

Research Article

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Abstract

Soft set theory has gained prominence as a revolutionary approach for handling and modeling uncertainty since it was proposed by Molodtsov. The concept of soft set operations, which is the major notion for the theory, has served as the foundation for theoretical and practical advances in the theory, therefore deriving the algebraic properties of the soft set operations and studying the algebraic structure of soft sets associated with these operations have attracted the researchers' interest continuously. In the theory of soft set, many soft intersection operations have been defined up to now among which there are some differences, and some of which are no longer preferred for use as they are essentially not useful and functional. While the restricted intersection definition is widely accepted and used in literature, it remains incomplete, as it ignores certain cases where the parameter sets of soft sets may be disjoint, thus not all conditions in the theorems are considered in the related proofs, leading to inaccuracies or deficiencies in the studies where this operation is used or its properties are investigated. There is a critical lack of comprehensive research in the existing literature on the correctly defined restricted intersection operation, along with the extended intersection, including their proper properties and distributions and the correct algebraic structures associated with these soft set operations. In this study, we primarily intend to fill this crucial gap by first correcting the deficiencies in the presentation of the definition of restricted intersection and revising it. Moreover, in many papers related to this operation, several theorems were presented without their proofs, or there were some incorrect parts in the proofs. In this study, all the proofs based on the function-equality are regularly provided. Besides, the relationships between the concept of soft subset and restricted and extended intersection operations are presented for the first time with their detailed proofs. Furthermore, we obtain many new properties of these operations as analogies and counterparts of the intersection operation in classical set theory. Moreover, the operations' full properties and distributions over other soft set operations are thoroughly investigated to determine the correct algebraic structures the operations form individually and in combination with other soft set operations both in the set of soft sets over the universe and with a fixed parameter set. This study demonstrates that the restricted/extended intersection operations, when combined with other kinds of soft set operations, form several significant algebraic structures, such as monoid, bounded semi-lattice, semiring, hemiring, bounded distributive lattice, Bool algebra, De Morgan Algebra, Kleene Algebra, Stone algebra, and MV-algebra but with detailed explanations. Accordingly, this study offers the most comprehensive analysis of restricted and extended intersection operations to date. It corrects earlier theorems and proofs, thereby advancing the theory and addressing a significant gap in the literature. Furthermore, it serves as a guide for beginners and sheds light on future research directions in soft set theory.

Keywords: Soft sets, Soft set operations, Restricted intersection operation, Extended intersection operation.

1. INTRODUCTION

It is difficult to explain and precisely describe many events in our daily lives, including uncertainty. Modeling situations involving uncertainty using classical mathematics or Aristotelian reasoning is extremely challenging. Set theory is considered a fundamental tool in mathematics, as it forms the basis for nearly all mathematical disciplines. To address and overcome uncertainty, many scientists from various fields have conducted research and proposed new theories. Among them, fuzzy set theory, introduced by Zadeh (1965), is one of the most widely used methods for dealing with uncertainty. In Aristotelian logic, the truth value of a proposition is either 0 or 1, whereas in fuzzy logic, it can be any real number within the range [0,1]. However, despite its popularity, fuzzy set theory faces certain limitations. The construction of the membership function is very subjective, which can lead to varying outcomes for the same problem. These challenges created a need for a new theory to address both uncertainty and cases of certainty. As an alternative, Molodtsov (1999) established "Soft Set (SS) Theory" as a mathematical technique to deal with uncertainty. SS theory's lack of a membership function construction issue makes it more practical. This advantage has led to its rapid application in fields such as mathematics, engineering, medicine, social sciences, and daily life situations like information systems and decision-making problems. Additionally, Molodtsov (1999) effectively used SS theory in domains like game theory, operations research, continuous differentiable functions, probability, measurement theory, Riemann integration, and Perron integration.

The fundamental notions of the SS were initially presented by Molodtsov (1999) in his pioneer study, and they were further expanded upon by a theoretical study of Maji et al. (2003) which introduces intersection and union operation with AND and OR operations, as well as the concepts of soft subset, soft equality, soft complement, NULL SSs, and absolute SSs. In contrast to Maji et al. (2003), Pei and Miao (2005) introduced a new intersection operation and proposed a new soft subset concept in their study on soft-based information systems. For the intersection operation, Feng et al. (2008) proposed an alternative concept known as "bi-intersection" (double intersection). To better understand the evolution and variations in soft set operations, it is essential to review key contributions in literature. For instance, introducing new definitions such as restricted union, intersection, difference, and extended intersection of SSs, Ali et al. (2009) aimed to resolve certain limitations in earlier operations defined by Maji et al. (2003). These foundational modifications laid the groundwork for exploring algebraic properties. Building on this, the concept of "relative complement" was developed, and it was shown that De Morgan's laws hold in soft set theory under these refined operations. Subsequently, Qin and Hong (2010) introduced a new form of "soft equality" and explored the algebraic structures of SSs, applying absorption laws to investigate whether these structures form lattices. In their extensive study, Ali et al. (2011) showed that some operations in the fixed-parameter SSs form MV-algebras and BCK-algebras, and they also showed that the SS operations defined by Ali et al. (2009) form a variety of algebraic structures, including monoids, hemirings, and lattices in the collection of SSs over the universe as well as in the fixed-parameter SSs. Sezgin and Atagün (2011) introduced restricted symmetric difference for SSs and investigated its characteristics. They also further explored the fundamental properties of restricted and extended intersection and union operation defined by Maji et al. (2003) and Ali et al. (2009). Redefining the notion of an SS's complement, Singh and Onyeozili (2012a, 2012b, 2012c, 2012d) published research on SS operations, the distributive and absorption laws of SS operations. Sen (2014) showed that restricted and extended intersection and union operations constitute a Boolean algebra in the set of the SSs with a fixed parameter set. A new SS operation known as "extended difference" was added to the list of extended operations in SSs by Sezgin et al. (2019) Additionally, Sezgin et al. (2019) investigated the characteristics of the operation and its connections to other SS operations. By proposing and examining the operation of extended symmetric difference, Stojanovic (2021) addressed a gap in the literature about the extended operation in SS theory. Some papers, such as Neog and Sut, 2011; Fu, 2011; Ge and Yang, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018 contain incorrect assertions that must be corrected.

As we see, "restricted" and "extended" SS operations are two primary categories under which the advancements in SS operations may be divided after a study of the research done up to this point. In contrast to the restricted and extended operation forms, the "soft binary piecewise difference operation" was an innovative SS operation that Eren and Çalışıcı (2019) described and investigated the characteristics of. A thorough investigation of the properties of the soft binary piecewise difference operation was explored by Sezgin and Çalışıcı (2024). Sezgin et al. (2023a) studied several novel binary set operations, motivated by Çağman (2021) work on conditional complements of sets. These binary set operations were transferred to SSs by Aybek (2024), who also defined novel restricted and extended SS operations, investigated their characteristics, and explored how they related to other SS operations. Additionally, Akbulut (2024), Demirci (2024), and Sarıalioğlu (2024) investigated a new type of SS operations known as "complementary extended SS operations". Yavuz (2024), Sezgin and Yavuz (2023a), and Sezgin and Yavuz (2024) defined and thoroughly examined a number of new soft binary piecewise operations, all of which were defined within the framework of the soft binary piecewise operation that was first presented in the study of Eren and Çalışıcı (2019). Besides, several authors (Sezgin et al., 2023b, Sezgin et al. 2023c; Sezgin and Dagtoros, 2023; Sezgin and Demirci, 2023; Sezgin and Yavuz, 2023b; Sezgin and Sarıalioğlu; 2024a; Sezgin and Sarıalioğlu; 2024b; Sezgin and Çağman, 2024; Sezgin and Şenyiğit, 2025)

An algebraic structure is made up of a set that has one or more binary operations defined on it along with those binary operations. Classifying algebraic structures and finding, showing, and deriving results from their common features are the goals of abstract algebra. It conducts this regardless of the sets and binary operations that make up these structures. This is the reason abstract algebra is the name given to this area of mathematics. Fundamentally, algebraic structures are involved in many branches of mathematics. Mathematicians have studied algebraic structures for millennia as they offer a universal and abstract approach to understanding and comprehending mathematical subjects. Understanding the properties of algebraic structures enables mathematicians to solve challenging problems, create new theories, and apply ideas to a variety of mathematical, scientific, and engineering domains. Furthermore, applications frequently provide special examples of algebraic structures, which help to clarify specific circumstances and make it easier to examine more general scenarios. When a particular set S is recognized as an illustration of a well-known algebraic structure, all of the well-known results regarding this algebraic structure also inherently hold for S. Abstraction is primarily motivated by this advantage. As a result, algebraic structures play a significant role in abstract algebra and mathematics.

One of the most well-known binary algebraic structures, which is a generalization of rings, is the notion of semirings which has been a subject of study and fascination for scholars from the past to the present. Vandiver (1934) introduced the concept of semirings. Several researchers have also studied semirings with additive inverses (Karvellas,1974; Goodearl, 1976; Petrich, 1973). While semirings are especially important in geometry, they are also important in pure mathematics and are critical for resolving problems in many practical mathematics and information science applications. Hemiring is a special class of semirings with commutative addition and a zero element. Additionally, there are several algebras related to logic. MV algebras are suited for multi-valued logic, while Boolean algebras are connected to traditional two-valued Aristotelean logic.

Just as the basic operations such as addition, subtraction, multiplication, and division in the set of integers and intersection, union, difference, complement, and symmetric difference in the set of sets are fundamental for the related theories, operations on SSs are equally vital in SS theory. SS operations serve as the theoretical basis for several soft computing and decision-making approaches. Furthermore, a thorough understanding of the algebraic structure of SSs may be obtained by looking at the algebraic structures formed by SSs and operations. This improves the comprehension of applications and makes it possible to see how SS algebra can be used in both classical and non-classical logic, which paves the way for a number of uses, such as the development of new SS-based cryptography techniques and decision-making processes. We refer to the study by Alcantud et al. (2024), where a comprehensive survey of SS theory, encompassing its foundational concepts, developments, and applications are presented. As regards the studies on soft algebraic structures for all of which SS operations have been the basis, we refer to Aktaş and Çağman, 2007; Jun, 2008; Jun and Park, 2008; Park et al., 2008; Feng et al., 2008; Sun et al., 2008; Acar et al., 2010; Zhan and Jun, 2010; Sezer et al., 2013, Sezer et al., 2014; Atagün and Sezgin, 2015; Sezer et al, 2015; Muştuoğlu et al., 2016; Mahmood et al., 2015; Sezer and Atagün, 2016; Tunçay and Sezgin, 2016; Sezer et al., 2017; Khan et al., 2017; Atagün and Sezgin, 2017; Sezgin et al., 2017; Atagün and Sezer, 2018; Ullah et al., 2018; Iftikhar and Mahmood; 2018; Gulistan et al., 2018; Sezgin, 2018; Atagün et al., 2019; Jana et al., 2019; Karaaslan, 2019; Özlü and Sezgin, 2020; Karaaslan et al., 2021; Sezgin et al., 2022, Atagün and Sezgin, 2022; Sezgin and Orbay, 2022, Riaz et al., 2023; Manikantan et al., 2023; Sezgin and İlgin, 2024; Sezgin and Onur, 2024; Sezgin et al., 2024).

In the theory of SS, many soft intersection operations have been defined up to now. There are some differences among them, and some definitions are no longer preferred for use as they are essentially not very useful. The intersection of SSs was first defined by Maji et al. (2003), however, it is problematic as it is obvious from the nature of the definition of SS that the condition put in the definition is not necessarily the case. This problematic nature of the definition was detailed by Ali et al. (2009) and Pei and Miao (2005). Pei and Miao (2005) defined a new intersection operation for SS, which they believed would be more functional, however in this definition it was not addressed what the result of the operation would be in the case where the parameter sets of the SSs are disjoint. Feng et al. (2008) defined an alternative intersection operation for SS, called the "bi-intersection" of SSs. This definition is problematic as well, as it is not addressed what the result of the operation would be in the case where the parameter sets of the SSs are disjoint. Ali et al. (2009) defined a new intersection operation for SSs called the "restricted intersection operation". Unlike the definition by Feng et al. (2008), this definition starts with the condition that the parameter sets of the SSs whose intersection is calculated should be disjoint. Moreover, it did not address what the result of the operation would be in the case where the parameter sets of the SSs are disjoint as well. Ali et al. (2011) evaluated the case of the intersection of the parameter sets of two SSs being empty, which was not considered in the restricted soft intersection operation defined by Ali et al. (2009), and updated the definition by adding the note that if the parameter sets of the SSs whose restricted intersection is calculated are disjoint, then the result of the operation is the empty SS. This is the first study to provide information on the result of the restricted intersection operation when the intersection of the parameter sets is empty set. Although the most current and useful definition for the restricted intersection operation is the one provided by Ali et al. (2011), in this definition, the condition that the parameter sets of the SSs should not be disjoint to calculate their restricted intersection was included as a necessary condition; however this is not the case, because whether the intersection of the parameter sets of the two SSs is an empty set or not, the restricted intersection of these two SSs can be calculated in any case. The intersection of the parameter sets of these two SSs being non-empty is never a necessary condition for their restricted intersection to be calculated. In this sense, from a chronological perspective, although the idea of the restricted intersection operation in SSs was first proposed by Pei and Miao (2005), as in their definition, the case where the intersection of the parameter sets of the SSs is empty was not considered, and this was addressed for the first time by Ali et al. (2011), the study by Ali et al. (2011) is of great importance. Besides, inspired by the union definition of SSs by Maji et al. (2013), a similar type operation defined as the "extended intersection operation" of SSs was proposed by Ali et al. (2009), and its properties and distributive rules were studied by various authors (Ali et al. 2009; Ali et al. 2011; Qin and Hong 2010); Sezgin and Atagün, 2011; Singh and Onyeozili, 2012c).

As restricted and extended intersection operations are anyway existing concepts in the literature, the properties of them were already studied in many studies, thus it may seem that some properties included in this paper were already presented in the previous studies (Ali et al., 2009; Ali et al., 2011; Feng et al. 2008; Maji et al., 2003; Pei and Miao, 2005; Qin and Hong, 2010; Sezgin and Atagün, 2011). However, we find it beneficial to note that the properties of the operations together with their distribution rules were not handled in the mentioned papers by considering the important point that the parameter sets of the SSs may be disjoint. This is due to the incomplete definition of restricted intersection operation and thus, ignoring some of the cases in the theorems and proofs. From this perspective, there is a significant gap in the literature for providing a comprehensive study of restricted and extended intersection operations by taking into account these ignored cases. Moreover, as SS operations serve as both the theoretical and practical basis for the theory, and in many studies as regards the soft algebraic structures, these basic two SS operations are always used while exploring the properties of the soft structures, this critical gap needs to be filled immediately beginning with a precise exposition of the definition of restricted intersection. This study aims to encompass all the prior studies regarding these operations. Moreover, in the above-mentioned papers together with (Neog and Sut, 2011; Fu, 2011; Ge and Yang S, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018) several theorems and propositions were presented without their proofs, or there were some incorrect or missing parts in the proofs due to the incomplete definition of restricted intersection; however, in this study, the proofs based on the function equality are regularly provided, and thus all the incorrect parts are corrected. Additionally, as the definition of subset by Pei and Miao (2005) is more functional and rational, and thus has a wide-spread usage than that of Maji et al. (2003), and since in the existing studies, (especially in the study of Sezgin and Atagün, 2011), the relationships between restricted and extended intersection operations and soft subset were handled with regard to the definition of subset proposed by Maji et al. (2003), in the literature there is a wide gap needs to be filled in this regard as well. In this study, the relationships in this regard which were not addressed in previous studies, are presented for the first time with detailed proofs and with their classical set counterparts as well. We do not only correct the problematics parts in the existing papers, but also we obtain many new properties of restricted and extended intersection operations together with their relationships with the SS operations defined by Aybek (2024) and Yavuz (2024). By looking at the distribution rules, the algebraic structures formed by these operations in the set of SSs with a fixed parameter set and in the set of sets over the universe are examined and presented thoroughly with their detailed proofs. Furthermore, when a distribution rule does not hold, unlike the studies by Ali et al. (2011) and Qin and Hong (2010), where the distribution rules are investigated to obtain the algebraic structures of SSs associated with the operations, we also explore and put forward the condition(s) under which the assertions hold. In this sense, we obtain many new algebraic structures related to SSs and restricted and extended intersection operations. Thus, this study presents a detailed and complete examination of all the properties of restricted intersection and extended intersection operation, which are the basic SS operations. As the intersection operation exists in classical set theory, all of the properties of the operations together with their counterparts in classical sets have been thoroughly investigated without omission. Additionally, the properties that were previously handled with incorrect/lengthy proofs or without proof in earlier studies by Ali et al. (2009), Ali et al. (2011), Sezgin and Atagün, 2011, Singh and Onyeozili (2012c) are handled again by presenting them in their correct forms. In order to find out whether the collection of SSs and restricted and extended intersection operations form lattice structures in the collection of SSs over the universe and in the collection of SS with a fixed parameter set, the so-called absorption laws are examined with detailed their proofs. Although the absorption laws were presented in previous works by Ali et al. (2011), Qin and Hong (2010), Singh and Onyeozili (2012c) presented the results only with a table without proofs, and since the proofs in other studies are element-based and relatively long proofs, they are presented in this study with their more rational proofs. Furthermore, in this study, the absorption laws for the SSs with a fixed parameter set are given in detail for the newly-defined operations by Yavuz (2024) and Aybek (2024) as well. Additionally, the distributive rules are presented collectively in a table. Finally, we systematically, in detail, and collectively present the unary and binary algebraic structures, and lattice structures formed by the restricted intersection and extended intersection together with other SS operations both in the collection of SS over the universe and in the collection of SSs with a fixed parameter set together with their detailed explanations and with the corrected ones.

The stream of the paper is as follows: In Section 2, we review the fundamental concepts regarding SSs and certain algebraic structures which are obtained to be associated with the SSs throughout the paper. In Section 3, first of all, we give the original definitions of intersection operations of SSs proposed up now together with the historical improvements of these operations in chronological order to indicate what deficiencies these definitions have and to contribute to the comprehensibility of the study. Then, the revised and updated definition of restricted intersection and all the properties of the restricted and extended intersection operations are presented and examined in detail and demonstrated with their complete proofs. When investigating the properties and distributive rules, the case where the intersection of the parameter sets of the SSs is empty is always considered in the assertions and the proofs. As intersection operation also exists in classical sets, special attention is given to show how the properties of intersection operation in classical sets reflect these operations to obtain their counterparts and analogies in SS theory. Thus, numerous new properties have also been added to those previously presented in this field. Besides, previously presented properties that were either unproven or had lengthy or erroneous proofs are presented with simplified proofs. Incorrect parts in previous studies, as regards these operations, are corrected with detailed explanations. Additionally, in order to see which algebraic structures these basic SS operations form, the distributions of restricted and extended intersection operations over other types of SS operations are examined in Section 3, and the absorption laws are investigated in detail in Section 4, and it is observed that these SS operations individually and together with other types of SSs form a wide variety of algebraic structures in the set of SSs over the universe and in the set of SSs with a fixed parameter set, such as monoid, bounded semi-lattice, semiring, hemiring, bounded distributive lattice, Bool algebra, De Morgan Algebra, Kleene Algebra, Stone algebra and MV-algebra, which are given collectively and with their detailed explanations in Section 4. We also, by providing a methodical study, correct some algebraic structures associated with the restricted and extended intersection operations obtained by Ali et al. (2009) presenting the corrected new ones. In the conclusion section, we highlight the significance of the study's results and their possible impact on both the SS theory, classical algebra, and real-world scenario. Taking all of these into account, this paper is the most comprehensive study in the existing literature of SSs as regards the restricted and extended intersection operations which encompass all the previous studies on this subject (Maji et al., 2003; Pei and Miao, 2005; Ali et al., 2009; Qin and Hong, 2010; Sezgin and Atagün, 2011; Ali et al., 2011; Singh and Onyeozili, 2012c, Sen, 2014) and (Neog and Sut, 2011; Fu, 2011; Ge and Yang, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018) serving as a handbook for those who start to study SS theory and advancing the theory by closing the big gap in the literature in this regard, as such an inclusive study does currently not exist in the literature and it is quite necessary in terms of shedding light on the future studies and preventing possible errors in the theory.

2. PRELIMINARIES

In this section, several algebraic structures and several fundamental concepts in SS theory are provided. Soft set first proposed by Molodtsov (1999); however its definition was revised by Maji et al. (2003).

Definition 1. Let E be the parameter set, Ɲ⊆ E, U be the universal set, P(U) be the power set of U, A pair (Ꞙ,Ɲ) is called an SS over U, where Ꞙ is a function given by Ꞙ : Ɲ → P(U) (Maji et al., 2003).

While the SS (Ꞙ,Ɲ) is denoted as $F_Ɲ$ in some papers, we prefer the commonly-held representation "(Ꞙ,Ɲ)" in this study. Besides, the definition of SS, proposed by Maji et al. (1999), has been reorganized by Çağman and Enginoğlu (2010) however, we use the definition of Maji et al. (2003) to be faithful to the original definition of SS throughout this paper.

More than one SS can be defined with a subset Ɲ of the set of parameters E. In this case, these SSs are denoted as (Ꞙ,Ɲ) (ℭ, Ɲ), (Ƕ, Ɲ), etc. Also, more than one SS can be defined with different subsets Ɲ,Y, Ᵽ etc. of the set of parameters E. In this case, the SSs are denoted as (Ꞙ,Ɲ), (Ꞙ,Y), (Ꞙ,Ᵽ), etc. (Maji et al, 2003). The collection of all SSs over U is denoted by S_E(U), and $S_Ɲ$(U) indicates the collection of all SSs over U with a fixed parameter set Ɲ, where Ɲ is a subset of E.

The definitions of "NULL SS" and "absolute SS" were first introduced by Maji et al. (2003), where a NULL SS (Ꞙ,A) was represented by Φ, and an absolute SS (Ꞙ,A) by $\tilde{A}$. However, it was extensively shown by Ali et al. (2009), Sezgin and Atagün (2011), and Yang (2008) that these definitions and notations, unfortunately, pose certain mathematical problems. Specifically, these definitions create many problematic situations in theorems and propositions, as the parameter set of the SS need not be a fixed set changing from SS to SS.

To address these problematic situations, Ali et al. (2009) updated these definitions, introducing the definitions of "relative null SS with respect to parameter set Ɲ" and "relative whole SS with respect to parameter set Ɲ" which are all determined by the parameter set of the SS. Consequently, in several significant studies (Ali et al. (2009), Ali et al. (2011) Sezgin and Atagün (2011)), the mathematically correct versions of all problematic theorems and propositions related to NULL SS and absolute SS operations were provided. Throughout this paper, in order to avoid confusion, we use the definitions of Ali et al. (2011) for absolute SS, null SS, and whole SS.

A function whose domain is the empty set is known as the empty function. Since the empty function is also a function, it is evident that by taking the domain as $\emptyset$, an SS can be defined as F: $\emptyset \longrightarrow$ P(U). This type of SS is referred to as an empty SS and is represented by $\emptyset_\emptyset$. As stated by Ali et al. (2011), the only SS with an empty parameter set is $\emptyset_\emptyset$. In this study, unless otherwise stated, all the SSs over U are different from $\emptyset_\emptyset$.

Definition 2. Let (Ꞙ,Ɲ)∈ $S_E(U)$. If Ꞙ(ʑ)=∅ for all ʑ∈ Ɲ, then (Ꞙ,Ɲ) is referred to as a null SS with respect to Ɲ, signified by $\emptyset_Ɲ$ (Ali et al, 2009). An SS with an empty parameter set is denoted as $\emptyset_\emptyset$ and is called an empty SS (Ali et al., 2011).

Definition 3. Let (Ꞙ,Ɲ)∈ $S_E(U)$. If Ꞙ(ʑ)=U for all ʑ∈ Ɲ, then (Ꞙ,Ɲ) is referred to as a relative whole SS with respect to Ɲ, signified by $U_Ɲ$. The relative whole SS U_E with respect to the universe set of parameters E is called the absolute SS over U (Ali et al., 2009).

Soft subsets and soft equal relations, in the framework of SS theory, are core concepts as well. Maji et al. (2003) were the pioneers in using a very strict definition of soft subsets. The definition is as follows:

Definition 4. Let (Ꞙ,Ɲ), (ℭ,Ɏ)∈ $S_E(U)$. If

i. Ɲ⊆ Ɏ and

ii. For all w ∈ Ɲ, Ꞙ(w)= ℭ(w)

then (Ꞙ,Ɲ) is called a soft subset of (ℭ, Ɏ), denoted as (Ꞙ,Ɲ) $\tilde{\subseteq}$ (ℭ, Ɏ).

If (ℭ, Ɏ) is a soft subset of (Ꞙ,Ɲ), then (Ꞙ,W) is a soft superset of (ℭ, Ɏ), denoted as (Ꞙ,Ɲ) $\tilde{\supseteq}$ (ℭ, Ɏ). If (Ꞙ,Ɲ) $\tilde{\subseteq}$ (ℭ, Ɏ) and (ℭ, Ɏ) $\tilde{\subseteq}$ (Ꞙ,Ɲ), then (Ꞙ,Ɲ) and (ℭ, Ɏ) are said to be (soft) equal sets (Maji et al., 2003).

In the study by Sezgin and Atagün (2011), the properties of restricted intersection and restricted union, and extended intersection and extended union operations were examined according to the soft subset definition by Maji et al. (2003). However, since the soft subset definition in the study by Pei and Miao (2005) is more related to classical sets, and thus is more useful and functional, the soft subset definition by Pei and Miao (2005) is used throughout this study.

Definition 5. Let (Ꞙ,Ɲ), (ℭ, Ɏ)∈ $S_E(U)$. If Ɲ ⊆ Ɏ and Ꞙ(ʑ) ⊆ ℭ(ʑ), for all ʑ∈ Ɲ, then (Ꞙ,Ɲ) is a soft subset of (ℭ,Ɏ), indicated by (Ꞙ,Ɲ)$\tilde{\subseteq}$(ℭ,Ɏ). If (ℭ,Ɏ) is a soft subset of (Ꞙ,Ɲ), then (Ꞙ,Ɲ) is a soft superset of (ℭ,Ɏ), indicated by (Ꞙ,Ɲ)$\tilde{\supseteq}$(ℭ, Ɏ). If (Ꞙ,Ɲ)$\tilde{\subseteq}$(ℭ,Ɏ) and (ℭ,Ɏ)$\tilde{\subseteq}$(Ꞙ,W), then (Ꞙ,Ɲ) and (ℭ,Ɏ) are called soft equal sets (Pei and Mio, 2005).

In the literature, various and updated definitions of soft subset and soft equal set have been introduced. For these definitions and the relationships between them, we refer to the studies by Qin and Hong, 2010); Jun and Yang, 2011; Liu et al. 2012; Feng and Li, 2013; Abbas et al., 2014; Mujahid et al., 2017; Abbas et al., 2017; Al-Shami, 2019; Al-Shasi and El-Shafei, 2020; and Ali et al., 2022.

The concept of the complement of an SS was first introduced by Maji et al. (2003). In this definition, when the complement of an SS (Ꞙ,Ɲ) is calculated, the complement of Ɲ is also conducted, thus the parameter set of the SS changes. It was shown by Ali et al. (2009) that this causes problematic situations in important aspects such as De Morgan's laws. To overcome this confusion, Ali et al. (2009) introduced the concept of "relative complement" of an SS which is more rational. In this definition, when the complement of an SS is conducted, the parameter set remains unchanged, that is, it is preserved. This definition became preferred, as it is more functional than the complement defined by Maji et al. (2003). To avoid confusion, in the study by Ali et al. (2009), the complement defined by Maji et al. (2003) is called the "neg-complement", and the updated complement concept for SSs is called the "relative complement" (briefly soft complement). Below, the concept of (relative) complement introduced by Ali et al. (2011) is presented and it used throughout this study is provided.

Definition 6. Let (Ꞙ,Ɲ)∈ $S_E(U)$. The relative complement of (Ꞙ,Ɲ), indicated by $(Ꞙ,Ɲ)^r$ =($Ꞙ^r$,Ɲ), is defined as follows: $Ꞙ^r$(ʑ)=U-Ꞙ(ʑ), for all ʑ∈ Ɲ (Ali et al, 2011).

From here, $(\emptyset_A)^r = U_A$, $(U_A)^r = \emptyset_A$, $(\emptyset_E)^r = U_E$, $(U_E)^r = \emptyset_E$, $(\emptyset_\emptyset)^r = \emptyset_\emptyset$ (Since the parameter set remains unchanged and $\emptyset_\emptyset$ is the only SS that has an empty parameter set), and $((Ғ, Ɲ)^r)^r = (Ғ, Ɲ)$.

Moreover, it is clear that $\emptyset_A \,\tilde{\subseteq}\, (Ғ,A) \,\tilde{\subseteq}\, U_A \,\tilde{\subseteq}\, U_E$ (Ali et al., 2011). Here we want to point out one issue: In Ali et al. (2011) it is stated that $\emptyset_\emptyset \,\tilde{\subseteq}\, \emptyset_A \,\tilde{\subseteq}\, (Ғ,A) \,\tilde{\subseteq}\, U_A \,\tilde{\subseteq}\, U_E$; however it is unknown whether $\emptyset_\emptyset$ is a subset of $\emptyset_A$ or not due to the definition of empty SS.

Çağman (2021) introduced two new complements as the inclusive complement (Here, we denote by +) and the exclusive complement (Here, we denote by θ). For two sets Ɲ and $\mathfrak{I}$, these binary operations are defined as $Ɲ + \mathfrak{I} = Ɲ' \cup \mathfrak{I}$ and $Ɲ\, \theta\, \mathfrak{I} = Ɲ' \cap \mathfrak{I}'$. Sezgin et al. (2023a) investigated the relationship between these two operations and also introduced new binary operations: For the sets Ɲ and $\mathfrak{I}$, $Ɲ * \mathfrak{I} = Ɲ' \cup \mathfrak{I}'$, $Ɲ\, \gamma\, \mathfrak{I} = Ɲ' \cap \mathfrak{I}$, $Ɲ\, \lambda\, \mathfrak{I} = Ɲ \cup \mathfrak{I}'$. Let the set operations be denoted by "⊛" (that is, ⊛ can be ∩, ∪,\, Δ, +,θ, *, λ,γ), then the following definitions are applied to all forms of SS operations:

Definition 7. Let (Ғ, Ɲ), $(\mathfrak{C}, Ұ) \in S_E(U)$ such that $Ɲ \cap Ұ \neq \emptyset$. The restricted ⊛ operation of (Ғ, Ɲ) and $(\mathfrak{C}, Ұ)$ is the SS (Ҥ,Ᵽ), denoted by $(Ғ,Ɲ) \circledast_R (\mathfrak{C}, Ұ) = (Ҥ, Ᵽ)$, where $Ᵽ = Ɲ \cap Ұ$ and for all ̡ϩ∈Ᵽ, $Ҥ(ϩ) = Ғ(ϩ) \circledast \mathfrak{C}(ϩ)$. Here, if $Ᵽ = Ɲ \cap Ұ = \emptyset$, then $(Ғ,Ɲ) \circledast_R (\mathfrak{C}, Ұ) = \emptyset_\emptyset$ (Ali et al., 2011; Pei and Mia, 2005; Sezgin and Atagün, 2011).

Definition 8. Let (Ғ,Ɲ), $(\mathfrak{C},Ұ) \in S_E(U)$. The extended ⊛ operation (Ғ,Ɲ) and $(\mathfrak{C},Ұ)$ is the SS (Ҥ,Ᵽ), denoted by $(Ғ, Ɲ) \circledast_\varepsilon (\mathfrak{C},Ұ) = (Ҥ, Ᵽ)$, where $Ᵽ = Ɲ \cup Ұ$, and for all ϩ∈Ᵽ,

$$Ҥ(ϩ) = \begin{cases} Ғ(ϩ), & ϩ \in Ɲ - Ұ \\ \mathfrak{C}(ϩ), & ϩ \in Ұ - Ɲ \\ Ғ(ϩ) \circledast \mathfrak{C}(ϩ), & ϩ \in Ɲ \cap Ұ \end{cases}$$

(Maji et al., 2003; Ali et al., 2009; Ali et al., 2011; Sezgin et al, 2019; Stojanavic, 2021; Aybek, 2024).

Definition 9. Let (Ғ,Ɲ), $(\mathfrak{C}, Ұ) \in S_E(U)$. The complementary extended ⊛ operation (Ғ,Ɲ) and $(\mathfrak{C}$,Y) is the SS (Ҥ,Ᵽ), denoted by $(Ғ,Ɲ) \overset{*}{\circledast}_\varepsilon (\mathfrak{C}, Ұ) = (Ҥ, Ᵽ)$, where $Ᵽ = Ɲ \cup Ұ$, and for all ϩ∈Ᵽ,

$$Ҥ(ϩ) = \begin{cases} Ғ'(ϩ), & ϩ \in Ɲ - Ұ \\ \mathfrak{C}'(ϩ), & ϩ \in Ұ - Ɲ \\ Ғ(ϩ) \circledast \mathfrak{C}(ϩ), & ϩ \in Ɲ \cap Ұ \end{cases}$$

(Akbulut, 2024; Demirci, 2024; Sarıalioğlu, 2024; Sezgin and Sarıalioğlu, 2024b)

Definition 10. Let (Ғ,Ɲ), $(\mathfrak{C}, Ұ) \in S_E(U)$. The soft binary piecewise ⊛ operation of (Ғ,Ɲ) and $(\mathfrak{C}, Ұ)$ is the SS (Ҥ, Ɲ), denoted by $(Ғ,Ɲ) \underset{\circledast}{\sim} (\mathfrak{C},Ұ) = (Ҥ,Ɲ)$, where for all ϩ∈ Ɲ,

$$Ҥ(ϩ) = \begin{cases} Ғ(ϩ), & ϩ \in Ɲ - Ұ \\ Ғ(ϩ) \circledast \mathfrak{C}(ϩ), & ϩ \in Ɲ \cap Ұ \end{cases}$$

(Eren and Çalışıcı, 2019; Sezgin and Yavuz, 2023a; Sezgin and Çalışıcı, 2024; Yavuz, 2024)

Definition 11. Let (Ғ,Ɲ), $(\mathfrak{C}, Ұ) \in S_E(U)$. The complementary soft binary piecewise ⊛ operation of (Ғ,Ɲ) and $(\mathfrak{C}, Ұ)$ is the SS (Ҥ, Ɲ), denoted by $(Ғ,Ɲ) \overset{*}{\underset{\circledast}{\sim}} (\mathfrak{C},Ұ) = (Ҥ,Ɲ)$, where for all ϩ∈ Ɲ,

$$Ҥ(ϩ) = \begin{cases} Ғ'(ϩ), & ϩ \in Ɲ - Ұ \\ Ғ(ϩ) \circledast \mathfrak{C}(ϩ), & ϩ \in Ɲ \cap Ұ \end{cases}$$

(Sezgin and Demirci, 2023; Sezgin et al. 2023a, 2023b; Sezgin and Yavuz, 2023b; Sezgin and Dagtoros, 2023; Sezgin and Çağman, 2024; Sezgin and Sarıalioğlu, 2024a)

Definition 12. An algebraic structure (Ś,⋆) is said to be idempotent if $s^2=s$ for all s∈ Ś, then. An idempotent semigroup is said to be a band, a commutative band is called a semi-lattice, and a semi-lattice with an identity is called a bounded semi-lattice (Clifford, 1954).

Definition 13. Let Ś be a non-empty set and "+" and "⋆" be two binary operations defined on Ś. If the algebraic structure (Ś, +, ⋆) satisfies the following properties, then it is called a semiring:

i. (Ś, +) is a semigroup.
ii. (Ś, ⋆) is a semigroup,
iii. For all ƀ, đ, z ∈ Ś, ƀ⋆(đ + z) = ƀ⋆ đ + ƀ⋆z and (ƀ + đ) ⋆z = ƀ⋆z + đ⋆z

If ƀ + đ = đ + ƀ for all ƀ,đ∈ Ś, then Ś is called an additive commutative semiring. If ƀ⋆ đ =đ⋆ ƀ for all ƀ,đ∈ Ś, then Ś is called a multiplicative commutative semiring. If there exists an element 1∈ Ś such that ƀ⋆1=1⋆ ƀ = ƀ for all ƀ∈ Ś (multiplicative identity), then Ś is called semiring with unity. If there exists 0∈ Ś such that for all ƀ ∈ Ś, 0⋆ ƀ =ƀ⋆0=0 and 0+ ƀ = ƀ +0= ƀ, then 0 is called the zero of Ś. A semiring with commutative addition and a zero element, is called a hemiring (Vandiver, 1934).

Definition 14. Let ζ be a non-empty set, and let "∨" and "∧" be two binary operations defined on ζ. If the algebraic structure. (ζ,∨,∧) satisfies the following properties, then it is called a lattice:

i. (ζ, ∨) is a semi-lattice
ii. (ζ, ∧) is a semi-lattice
iii. For all Ջ,Ց ∈ ζ , Ջ ∨(Ջ ∧ Ց) =Ջ ∧(Ջ ∨ Ց) (absorption law)

A lattice with an identity element according to both operations is called a bounded lattice. In a bounded lattice, the identity element of ζ with respect to the ∧ operation is usually denoted by 1, while the identity element with respect to the ∨ operation is denoted by 0. If the bounded lattice ζ has an element Ջ′ such that Ջ ∧ Ջ′ = 0 and Ջ∨ Ջ′=1 for all Ջ ∈ ζ, then ζ is called a complemented lattice. A lattice holding distribution law is called a distributive lattice. A lattice that is bounded, distributive, and at the same time complemented is called Boolean algebra. The lattice with De Morgan's law, i.e. (Ջ ∨ Ց)′ = Ջ ′∧ Ց ′ and (Ջ ∧ Ց)′ = Ջ ′ ∨ Ց ′ for all Ջ, Ց ∈ ζ is called De Morgan algebra. If De Morgan algebra satisfies the condition Ջ ∧ Ջ ′≤ Ց∨ Ց′ for all Ջ, Ց ∈ ζ, then it is called a Kleene algebra. For Ջ ∈ ζ, Ջ* is called the pseudo-complete of Ջ if Ջ ∧ Ջ*= 0 and Ց ≤ Ջ* whenever Ջ ∧Ց =0. The equality Ջ*∨ Ջ**=1 is called the Stone's identity. A pseudo-complemented distributive lattice satisfying the Stone's identity is called a Stone algebra (Ali et al., 2011)

Definition 15. Let Ƨ be a non-empty set with binary operation "⊕" and a unary operation "*" defined on Ƨ. If 0 is a constant that fulfills the following axioms for each ȥ and y in Ƨ, then the structure (Ƨ,⊕,∗ ,0) is called an MV-algebra:

i. (Ƨ, ⊕, 0) commutative monoid.
ii. (ȥ*)*= ȥ
iii. 0*⊕ ȥ =0*
iv. (ȥ* ⊕ y)* ⊕ y=(y* ⊕ ȥ)* ⊕ ȥ

The concept of MV-algebras was introduced by Chang (1959) with the aim of providing an algebraic proof for Lukasiewicz logic, a many-valued logic introduced by Lukasiewicz in the 1920s. We refer to Pant et al. (2024) regarding the possible applications of network analysis and graph applications on SSs, and to Ali et al. (2015), Jan et al. (2020), Irfan Siddique et al. (2021), and Mahmood (2020) for bipolar soft sets, double framed soft sets and lattice ordered soft sets. For more about soft AG-groupoids, soft KU-algebras, and picture soft sets, see Khan et al. (2015), Gulistan and Shahzad (2014), Memiş (2022), and Naeem and Memiş (2023), respectively.

3. MORE ON RESTRICTED AND EXTENDED INTERSECTION OPERATIONS

In the theory of SS, which was proposed in 1999, many soft intersection operations have been defined. There are some differences among them, and some definitions are no longer preferred for use because they are essentially not very useful. We find it beneficial to start this section by recalling these points. From this perspective, we first aim to present the soft intersection operations existing in the literature in chronological order, and to indicate what deficiencies these definitions have, as we believe this will contribute to the comprehensibility of the study.

In 2003, the intersection of SSs was first defined by Maji et al. (2003) as follows: Let (Ӻ,A) and (ℭ,B) be two SSs over U. The intersection operation of these SSs, denoted by (Ӻ,A)$\tilde{\cap}$(ℭ,B), is defined as (Ӻ,A)$\tilde{\cap}$(ℭ,B)=(Ӈ, C), where C=A∩B and Ӈ(α)=Ӻ(α) or Ӈ(α)=ℭ(α), for all α ∈C (as both are the same sets) (Maji et al, 2003). Although it was claimed in this definition that Ӻ(α) = ℭ(α), for α ∈ C, it is clear from the definition and nature of the SS that such a situation is not necessarily the case. Therefore, the problematic nature of this definition has been detailed in the studies by Ali et al. (2009) and Pei and Miao (2005).

Pei and Miao (2005) defined a new soft intersection operation, which they believed would be more functional, as follows: Let (Ӻ, A) and (ℭ, B) be two SSs over U. The restricted intersection of these SSs, denoted by (Ӻ,A)∩(ℭ,B), is defined as (Ӻ,A)∩(ℭ,B)=(Ӈ,C), where C=A∩B and for all α ∈C, Ӈ(α) = Ӻ(α)∩ℭ(α) (Pei and Mia, 2005). In this definition, however, it is not addressed what the result of the operation (Ӻ,A)∩(ℭ,B) would be in the case where the parameter sets of the SSs are disjoint, and the notation for the restricted soft intersection operation is chosen to be similar to the intersection operation in classical sets.

Feng et al. (2008) defined an intersection operation, called the bi-intersection operation, as follows: Let (Ӻ, A) and (ℭ, B) be two SSs over U. The bi-intersection of these SSs, where C=A∩B and for all ϛ ∈ C, Ӈ: C → P(U) is defined as Ӈ(ϛ)=Ӻ(ϛ)∩ℭ(ϛ), and is denoted by (Ӻ, A) $\tilde{\sqcap}$ (ℭ, B) = (Ӈ, C). In this definition, as well, it is not addressed what the result of the operation (Ӻ, A) $\tilde{\sqcap}$ (ℭ, B) would be in the case where A∩B=∅.

Ali et al. (2009) defined an SS operation called the restricted intersection operation as follows: Let (Ӻ, A) and (ℭ, B) be two SSs over U such that A∩B≠ ∅. The restricted intersection operation of (Ӻ, A) and (ℭ, B) denoted by (Ӻ,A)⋒(ℭ,B), is defined as (Ӻ,A)⋒(ℭ,B)=(Ӈ,C), where C=A∩B, and for all α ∈C, Ӈ(α)= Ӻ(α)∩ℭ(α). Unlike the definition by Feng et al. (2008), this definition starts with the condition "Let (Ӻ,A) and (ℭ,B) be two SSs such that A∩B ≠ ∅," treating this condition as a necessary condition. Moreover, it does not address what the result of the operation (Ӻ,A)⋒(ℭ,B) would be in the case where A∩B = ∅.

Ali et al. (2011) evaluated the case of the intersection of the parameter sets of two SSs being empty, which was not considered in the restricted soft intersection operation defined by Ali et al. (2009), and updated the definition of the restricted intersection operation as follows: "Let (Ӻ, A) and (ℭ, B) be two SSs over U such that A∩B=∅. The restricted intersection operation (Ӻ,A) and (ℭ,B) is denoted by (Ӻ,A)$\cap_R$(ℭ,B), and is defined as (Ӻ,A) $\cap_R$(ℭ,B)=(Ӈ,C), where C = A∩B, and for all α ∈ C, Ӈ(α)= Ӻ(α)∩ℭ(α). If A∩B=∅, then (Ӻ,A) $\cap_R$(ℭ,B)= $\emptyset_{\emptyset}$. "This was the first study to provide information on the result of the restricted intersection operation when the intersection of the parameter sets is empty. Additionally, this study preferred to use the symbol "$\cap_R$" which is the most useful notation for the restricted intersection operation. The letter "R" under the intersection symbol is in harmony with and meaningful because it stands for "restricted" in English. Indeed, in subsequent studies on SS operations, this notation form was preferred for restricted SS operations. The most current and useful definition of the restricted intersection definition is the one provided by Ali et al. (2011). However, in this definition, for two SSs (Ғ,A) and ($\mathfrak{C}$, B) over the same universe, the condition $A \cap B \neq \emptyset$ is again included as a necessary condition for the restricted intersection of these two SSs to be calculated by adding the expression "Let (Ғ,A) and ($\mathfrak{C}$, B) be two SSs over U such that $A \cap B \neq \emptyset$." However, even if $A \cap B = \emptyset$, the restricted intersection operation of these SSs is still defined, and in this case, $(\text{Ғ},A) \cap_R (\mathfrak{C},B) = \emptyset_\emptyset$. That is, whether the intersection of the parameter sets of the two SSs to be intersected is an empty set or not, the restricted intersection of the two SSs can be calculated in any case. The parameter sets of the two SSs being not disjoint is never a necessary condition for their restricted intersection to be calculated.

In this sense, from a chronological perspective, although the idea of the restricted intersection operation in SSs was first proposed by Pei and Miao (2005), as in their definition, the case where the intersection of the parameter sets of the SSs to be intersected is empty was not considered and it is addressed firstly in the study of Ali et al. (2011), the updated definition for the restricted intersection provided in this section as Definition 3.1.1 will be given and used.

In this section, the algebraic properties of the restricted intersection and extended intersection operations of SSs, updated considering the above-mentioned points, are examined in comparison with the properties of the intersection operation in classical sets. We investigate the distribution rules so as to obtain the algebraic structures formed by these operations in the collection of SSs with a fixed parameter set, and in the collection of softs over the universe in the following section.

Here, we find it beneficial to note the following: Although the restricted intersection operation exists in the literature as the restricted and extended intersection operations are pre-defined operations, many properties included in this section have been given in previous works (Ali et al. 2009; Ali et al., 2011; Feng et al. 2008; Pei and Mia, 2005; Maji et al, 2003; Qin and Hong, 2010; Sezgin and Atagün, 2011), but without considering the important points detailed in this section. Moreover, in most studies, many of these properties are provided without proofs, or with insufficiently detailed proofs. In this study, all proofs are systematically provided based on function equality. Additionally, the relationships between Pei and Miao's (2005) definition of soft subsets and restricted/extended intersection operations, which were not addressed in previous studies, are presented for the first time in this study with detailed proofs and their classical set counterparts.

Moreover, since all properties of the fundamental SS operations, namely the restricted and extended intersection operations, are provided together with their proofs, and since this study takes care the case where the intersection of the parameter sets of the SSs may be empty set for each property-previously overlooked in other studies-this work is comprehensive of all previous works. From this perspective, we consider this study to be of significant importance, and hope it serves as a handbook for beginners in SS theory.

3.1. More on Restricted Intersection Operation

In this subsection, the updated and revised presentation of the definition of restricted intersection defined by Ali et al. (2011), its example, and all its properties are provided with their detailed proofs.

Definition 16 Let (Ꞙ, Ʝ) and (ℭ, Ᵽ) be SSs over U. The restricted intersection of (Ꞙ, Ʝ) and (ℭ, Ᵽ), denoted by (Ꞙ, Ʝ) $\cap_R$ (ℭ, Ᵽ), is defined as (Ꞙ, Ʝ) $\cap_R$ (ℭ, Ᵽ) = (Ή, C), where C= Ʝ ∩Ᵽ. Here, if C= Ʝ ∩Ᵽ≠∅, then Ή(α)= Ꞙ(α) ∩ℭ(α) for all α∈C, and if C= Ʝ ∩Ᵽ=∅, then (Ꞙ,Ʝ) $\cap_R$ (ℭ,Ᵽ)=(Ή, C)= $\emptyset_{\emptyset}$.

Since the only SS with an empty parameter set is $\emptyset_{\emptyset}$, it is clear from the definition that if C=Ʝ∩Ᵽ=∅, then (Ꞙ,Ʝ) $\cap_R$ (ℭ,Ᵽ)= $\emptyset_{\emptyset}$. Therefore, it can be seen that there is no requirement for Ʝ∩Ᵽ≠ ∅ for the restricted intersection operation to be defined, where (Ꞙ,Ʝ) and (ℭ, Ᵽ) are two SSs over U.

The reason we have not cited Ali et al. (2009), Ali et al. (2011), Feng et al. (2008), and Pei and Miao (2005) in Definition 16 is due to the detailed explanations we provided in the introduction of Section 3, where we updated the definition by considering the case of the intersection of parameter sets being empty. In many studies (Ali et al., 2009; Ali et al., 2011; Sezgin and Atagün, 2011; Singh and Onyeozili, 2012c), properties related to the restricted intersection operation were stated overlooking this case, but throughout this study, it has been shown that this condition is not necessary.

The symbol "$\cap_R$" used to denote the restricted intersection operation aligns well with the English word "restricted," forming a meaningful whole. This notation form has been preferred for restricted SS operations in other studies on SS operations as well.

Example 1 Let E={e_1,e_2,e_3,e_4} be the parameter set Ʝ ={e_1, e_3} and Ᵽ ={e_2, e_3, e_4} be the subsets of E and U={h_1,h_2,h_3,h_4,h_5} be the initial universe set. Assume that (Ꞙ,Ʝ) and (ℭ,Ᵽ) are the SSs over U defined as follows:

(Ꞙ,Ʝ)={(e_1,{h_2,h_5}),(e_3,{h_1,h_2,h_5})}, (ℭ,Ᵽ)={(e_2,{h_1,h_4,h_5}), (e_3,{h_2,h_3,h_4}),(e_4,{h_3,h_5})}.

Let (Ꞙ,Ʝ)$\cap_R$(ℭ,Ᵽ)=(Ή,Ʝ∩Ᵽ), where for all α ∈ Ʝ ∩ Ᵽ = {e_3},

Ή(e_3)=Ꞙ(e_3)∩ℭ(e_3)={h_1,h_2,h_5}∩{h_2,h_3,h_4}={h_2}.

Thus, (Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)={(e_3,{h_2})}.

Proposition 1 The set $S_E(U)$ is closed under the operation $\cap_R$. That is, when (Ꞙ,Ʝ) and (ℭ,Ᵽ) are two SSs over U, then so is (Ꞙ, Ʝ) $\cap_R$ (ℭ, Ᵽ).

Proof: It is clear that $\cap_R$ is a binary operation in $S_E(U)$. That is,

$\cap_R$: $S_E(U)$x $S_E(U)$→ $S_E(U)$

((Ꞙ,Ʝ), (ℭ,Ᵽ)) → (Ꞙ, Ʝ) $\cap_R$ (ℭ, Ᵽ)=(Ή,Ʝ∩Ᵽ)

Hence, the set $S_E(U)$ is closed under the operation $\cap_R$. Similarly,

$\cap_R$: $S_Ʝ(U)$ x $S_Ʝ(U)$→ $S_Ʝ(U)$

((Ꞙ,Ʝ), (ℭ,Ʝ)) → (Ꞙ, Ʝ) $\cap_R$ (ℭ, Ʝ)=(K,Ʝ∩Ʝ)=(K,Ʝ)

That is, $\cap_R$ is also closed in $S_Ʝ(U)$, where Ʝ is a fixed subset of E.

Proposition 2 Let (Ꞙ,Ʝ), (ℭ,Ᵽ), and (Ή,Ꝣ) be SSs over U. Then, [(Ꞙ,Ʝ) $\cap_R$ (ℭ,Ᵽ)] $\cap_R$ (Ή,Ꝣ) = (Ꞙ,Ʝ) $\cap_R$ [(ℭ,Ᵽ) $\cap_R$(Ή,Ꝣ)] (Pei and Miao, 2005).

Proof: Pei and Miao (2005) presented this property without its proof; however, we provide it in detail with its rigorous proof. Let (Ꞙ,Ʝ)$\cap_R$(ℭ,Ᵽ)=(S,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, S(α)=Ꞙ(α)∩ℭ(α). Let (S,Ʝ∩Ᵽ)$\cap_R$ (Ⱨ,Ȝ) =(R,(Ʝ∩Ᵽ)∩Ȝ)), where for all α∈(Ʝ∩Ᵽ)∩Ȝ, R(α)=S(α)∩Ⱨ(α). Thus,

R(α)=[Ꞙ(α)∩ℭ(α)]∩Ⱨ(α)

Let (ℭ,Ᵽ)$\cap_R$(Ⱨ,Ȝ)=(K,Ᵽ∩Ȝ), where for all α∈Ᵽ∩Ȝ, K(α)=ℭ(α)∩Ⱨ(α). Let (Ꞙ,Ʝ)$\cap_R$ (K,Ᵽ ∩ Ȝ) =(L,Ʝ∩(Ᵽ∩Ȝ)), where for all α∈Ʝ∩(Ᵽ∩Ȝ), L(α)=Ꞙ(α)∩K(α). Thus,

L(α)=Ꞙ(α)∩[ℭ(α)∩Ⱨ(α)]

Here it is seen that (R,(Ʝ ∩ Ᵽ) ∩ Ȝ) =(L, Ʝ∩(Ᵽ∩Ȝ)). That is, $\cap_R$ is associative in the S_E(U).

Here, it is obvious that if Ʝ∩Ᵽ=∅ or Ᵽ∩Ȝ=∅ or Ʝ∩Ȝ=∅, then (R,(Ʝ ∩ Ᵽ) ∩ Ȝ) =(L, Ʝ∩(Ᵽ∩Ȝ))= $\emptyset_{\emptyset}$, thus $\cap_R$ is associative under these conditions as well.

Proposition 3 Let (Ꞙ,Ʝ), (ℭ,Ʝ), and (Ⱨ,Ʝ) be SSs over U. Then, [(Ꞙ,Ʝ) $\cap_R$(ℭ,Ʝ)] $\cap_R$(Ⱨ,Ʝ)=(Ꞙ,Ʝ) $\cap_R$[(ℭ,Ʝ) $\cap_R$(Ⱨ,Ʝ)].

Proof: Let (Ꞙ,Ʝ)$\cap_R$(ℭ,Ʝ)=(K,Ʝ), where for all α∈Ʝ∩Ʝ=Ʝ, K(α)=Ꞙ(α) ∩ℭ(α). Let (K,Ʝ)$\cap_R$ (Ⱨ, Ʝ) =(R,Ʝ), where for all α∈Ʝ∩Ʝ=Ʝ, R(α)=K(α)∩Ⱨ(α). Thus,

R(α)=[Ꞙ(α)∩ℭ(α)]∩Ⱨ(α)

Let (ℭ,Ʝ)$\cap_R$(Ⱨ,Ʝ)=(L,Ʝ), where for all α∈Ʝ∩Ʝ, L(α)=ℭ(α)∩Ⱨ(α). Let (Ꞙ,Ʝ) $\cap_R$(L,Ʝ)= (Ṅ,Ʝ), where for all α∈Ʝ∩Ʝ, Ṅ(α)= Ꞙ(α)∩L(α). Thus,

Ṅ(α)=Ꞙ(α) ∩ [ℭ(α)∩Ⱨ(α)]

It is seen that (R,Ʝ)=(Ṅ,Ʝ). That is, $\cap_R$ is associative in $S_{Ʝ}$(U).

Proposition 4 Let (Ꞙ,Ʝ), and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)=(ℭ,Ᵽ) $\cap_R$(Ꞙ,Ʝ) (Qin and Hong, 2010).

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it in detail with its rigorous proof. Let (Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)=(Ⱨ,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, Ⱨ(α)=Ꞙ(α)∩ℭ(α). Let (ℭ,Ᵽ)$\cap_R$(Ꞙ,Ʝ)=(S,Ᵽ∩Ʝ), where for all α∈Ᵽ∩Ʝ, S(α)=ℭ(α)∩Ꞙ(α). Thus,

(Ꞙ,Ʝ)$\cap_R$(ℭ,Ᵽ)=(ℭ,Ᵽ) $\cap_R$ (Ꞙ, Ʝ).

Hence, $\cap_R$ is commutative in S_E(U). It is obvious that (Ꞙ,Ʝ)$\cap_R$(ℭ,Ʝ)=(ℭ,Ʝ)$\cap_R$(Ꞙ,Ʝ). That is, $\cap_R$ is commutative in $S_{Ʝ}$(U) as well. Here, it is also obvious that if Ʝ∩Ᵽ=∅, then (Ⱨ, ∩ Ᵽ) =(S, Ᵽ∩ Ʝ)= $\emptyset_{\emptyset}$, thus $\cap_R$ is commutative under this condition as well.

Proposition 5 Let (Ꞙ,Ʝ) be an SS over U. Then, (Ꞙ,Ʝ) $\cap_R$(Ꞙ,Ʝ)=(Ꞙ,Ʝ) (Pei and Miao, 2005).

Proof: Pei and Miao (2005) presented this property without proof in their study; however, we provide it in detail with its rigorous proof. Let (Ꞙ,Ʝ)$\cap_R$(Ꞙ,Ʝ)=(Ⱨ,Ʝ∩Ʝ), where for all α∈Ʝ, Ⱨ(α)=Ꞙ(α)∩Ꞙ(α)=Ꞙ(α). Thus. (Ⱨ,Ʝ)= (Ꞙ, Ʝ). That is, $\cap_R$ is idempotent in S_E(U).

Proposition 6 Let (Ꞙ,Ʝ) be an SS over U. Then, (Ꞙ,Ʝ)$\cap_R$ $U_{Ʝ}$= $U_{Ʝ}$ $\cap_R$(Ꞙ,Ʝ)=(Ꞙ,Ʝ) (Ali et al. 2011).

Proof: Ali et al. (2011) presented this property with the relative whole SS with respect to Ʝ only on the right side, and without its proof; however, we provide the property with the relative whole SS with respect to Ꭻ on both the right and left sides, along with its detailed proof. Let $U_{Ꭻ}$=(K,Ꭻ), where for all α∈Ꭻ, K(α)=U. Let (Ꞙ,Ꭻ) $\cap_R$ (K, Ꭻ)=(Ⱨ,Ꭻ∩Ꭻ), where for all α∈ Ꭻ, Ⱨ(α)= Ꞙ(α)∩ K(α)=Ꞙ(α)∩ U=Ꞙ(α). Thus, (Ⱨ,Ꭻ)= (Ꞙ,Ꭻ) implying that (Ꞙ,Ꭻ)$\cap_R$ U_T=(Ꞙ,Ꭻ), and by Proposition 4, $U_{Ꭻ}$ $\cap_R$(Ꞙ,Ꭻ)=(Ꞙ,Ꭻ) as well. That is, $U_{Ꭻ}$ is the identity element of $\cap_R$ in $S_{Ꭻ}$(U).

Here, it is obvious that there is no inverse element for the operation $\cap_R$ other than $U_{Ꭻ}$ in $S_{Ꭻ}$(U). Naturally, $U_{Ꭻ}$ itself is the identity element for the operation $\cap_R$ in $S_{Ꭻ}$(U).

Proposition 7 Let (Ꞙ,Ꭻ) be an SS over U. Then, (Ꞙ,Ꭻ) $\cap_R$ $\emptyset_{Ꭻ}$=$\emptyset_{Ꭻ}$ $\cap_R$(Ꞙ,Ꭻ)=$\emptyset_{Ꭻ}$ (Ali et al. 2011).

Proof: Ali et al. (2011) presented this property with the relative null SS with respect to Ꭻ only on the right side, and without its proof; however, we provide the property with the relative null SS with respect to Ꭻ on both the right and left sides, along with its detailed proof. Let $\emptyset_{Ꭻ}$=(S,Ꭻ), where for all α∈ Ꭻ, S(α)= ∅. Let (Ꞙ,Ꭻ) $\cap_R$(S,Ꭻ)=(Ⱨ,Ꭻ∩Ꭻ), where for all α∈Ꭻ, Ⱨ(α)=Ꞙ(α)∩S(α)=Ꞙ(α)∩ ∅=∅. Thus, (Ⱨ,Ꭻ)=$\emptyset_{Ꭻ}$, implying that (Ꞙ,Ꭻ) $\cap_R$ $\emptyset_{Ꭻ}$=$\emptyset_{Ꭻ}$, and by Proposition 4, $\emptyset_{Ꭻ}$ $\cap_R$(Ꞙ,Ꭻ)=$\emptyset_{Ꭻ}$ as well. That is, $\emptyset_{Ꭻ}$ is the absorbing element of $\cap_R$ in $S_{Ꭻ}$(U).

Theorem 1 ($S_{Ꭻ}$(U),$\cap_R$) is a bounded semi-lattice, whose identity is $U_{Ꭻ}$ and the absorbing element is $\emptyset_{Ꭻ}$.

Proof: By Proposition 1, Proposition 3, Proposition 4, Proposition 5, Proposition 6, and Proposition 7 (S_T(U),$\cap_R$) is a commutative, idempotent monoid whose identity is $U_{Ꭻ}$ and absorbing element $\emptyset_{Ꭻ}$, that is, a bounded semi-lattice.

Proposition 8 Let (Ꞙ,Ꭻ) be an SS over U. Then, (Ꞙ,Ꭻ)$\cap_R$ U_E=U_E $\cap_R$(Ꞙ,Ꭻ)=(Ꞙ,Ꭻ).

Proof: Let U_E=(K,E), where for all α∈E, K(α)=U. Let (Ꞙ, Ꭻ) $\cap_R$ (K, E) = (Ⱨ, Ꭻ ∩ E), where for all α∈Ꭻ∩E=Ꭻ, Ⱨ(α)=Ꞙ(α)∩K(α)=Ꞙ(α)∩ U=Ꞙ(α). Thus, (Ⱨ,Ꭻ)=(Ꞙ,Ꭻ), implying that (Ꞙ,Ꭻ)$\cap_R$ U_E=(Ꞙ,Ꭻ), and by Proposition 4, U_E $\cap_R$(Ꞙ,Ꭻ)=(Ꞙ,Ꭻ) as well. That is, U_E is the identity element of $\cap_R$ in S_E(U).

From this, we can conclude that in S_E(U), no element has an inverse element for the operation $\cap_R$ other than U_E, which is the identity element. Naturally, the SS U_E itself is the identity element of the operation $\cap_R$ in S_E(U).

Proposition 9 Let (Ꞙ,Ꭻ) be an SS over U. Then, (Ꞙ,Ꭻ)$\cap_R$ $\emptyset_\emptyset$=$\emptyset_\emptyset$ $\cap_R$(Ꞙ,Ꭻ)= $\emptyset_\emptyset$.

Proof: Let $\emptyset_\emptyset$ =(S,∅). (Ꞙ,Ꭻ)$\cap_R$ (S,∅)=(Ⱨ,Ꭻ ∩ ∅)=(Ⱨ, ∅), $\emptyset_\emptyset$ since the parameter set is the only SS that is the empty set, (Ⱨ, ∅)=$\emptyset_\emptyset$, implying that (Ꞙ,Ꭻ)$\cap_R$ $\emptyset_\emptyset$= $\emptyset_\emptyset$, and by Proposition 4, $\emptyset_\emptyset$ $\cap_R$(Ꞙ,Ꭻ)= $\emptyset_\emptyset$ as well. That is, $\emptyset_\emptyset$ is the absorbing element of $\cap_R$ in S_E(U).

Theorem 2 (S_E(U),$\cap_R$) is a bounded semi-lattice, whose identity is U_E and the absorbing element is $\emptyset_\emptyset$.

Proof: By Proposition 1, Proposition 2, Proposition 4, Proposition 5, Proposition 8, and Proposition 9, (S_E(U),$\cap_R$) is a commutative, idempotent monoid whose identity is U_E, that is, a bounded semi-lattice.

Proposition 10 Let (Ꞙ,Ꞩ) be an SS over U. Then, (Ꞙ,Ꞩ)$\cap_R$ $\emptyset_E$=$\emptyset_E$ $\cap_R$ (Ꞙ,Ꞩ)= $\emptyset_Ꞩ$.

Proof: Let $\emptyset_E$=(S,E), where for all α∈E, S(α)=∅. Let (Ꞙ,Ꞩ)$\cap_R$(S,E)=(Ꞡ,Ꞩ∩E), where for all α∈Ꞩ∩E=Ꞩ, Ꞡ(α)=Ꞙ(α)∩S(α)=Ꞙ(α)∩ ∅=∅. Thus, (Ꞡ,Ꞩ)= $\emptyset_T$, implying that (Ꞙ,Ꞩ)$\cap_R$ $\emptyset_E$=$\emptyset_Ꞩ$, and by Proposition 4, $\emptyset_E$ $\cap_R$ (Ꞙ,Ꞩ)= $\emptyset_Ꞩ$ as well.

Proposition 11 Let (Ꞙ,Ꞩ) be an SS over U. Then, (Ꞙ,Ꞩ) $\cap_R$(Ꞙ,Ꞩ)^r=(Ꞙ,Ꞩ)^r $\cap_R$(Ꞙ,Ꞩ)= $\emptyset_Ꞩ$ (Sezgin and Atagün, 2011; Ali et al., 2011).

Proof: In the studies by Sezgin and Atagün (2011) and Ali et al. (2011), this property was presented with the relative complement of the SS (Ꞙ,Ꞩ) only on the right side, and without its proof. However, we provide the property with the relative complement of the SS (Ꞙ,Ꞩ) on both the right and left sides, along with its detailed proof. Let (Ꞙ,Ꞩ)^r=(Ꞡ,Ꞩ), where for all α∈ Ꞩ, Ꞡ(α)=Ꞙ'(α). Let (Ꞙ,Ꞩ)$\cap_R$ (Ꞡ, Ꞩ)=(L,Ꞩ∩Ꞩ), where for all α∈Ꞩ, L(α)=Ꞙ(α)∩Ꞡ(α)=Ꞙ(α)∩Ꞙ'(α)=∅. Thus, (L,Ꞩ)= $\emptyset_Ꞩ$, implying that (Ꞙ,Ꞩ)$\cap_R$(Ꞙ,Ꞩ)^r=$\emptyset_Ꞩ$, and by Proposition 4, (Ꞙ,Ꞩ)^r $\cap_R$(Ꞙ,Ꞩ)= $\emptyset_Ꞩ$ as well.

Proposition 12 Let (Ꞙ,Ꞩ) and (ℭ,Ᵽ) be SSs over U. Then, [(Ꞙ,Ꞩ) $\cap_R$(ℭ,Ᵽ)]^r=(Ꞙ,Ꞩ)^r $\cup_R$(ℭ,Ᵽ)^r (De Morgan Law) (Ali et al., 2009).

Proof: In the study by Ali et al. (2009), the De Morgan property was presented with the condition Ꞩ∩Ᵽ≠∅ and the proof was relatively lengthy. In this study, we state that the condition Ꞩ∩Ᵽ≠∅ is not a necessary condition for the proposition and provide a simpler proof. Let (Ꞙ,Ꞩ)$\cap_R$(ℭ,Ᵽ)=(Ꞡ,Ꞩ∩Ᵽ), where for all α∈Ꞩ∩Ᵽ, Ꞡ(α)=Ꞙ(α)∩ℭ(α). Let (Ꞡ,Ꞩ∩Ᵽ)^r=(K,Ꞩ∩Ᵽ), where for all α∈Ꞩ∩Ᵽ, K(α)=H'(α)=Ꞙ'(α)∪G'(α). Thus, (K,Ꞩ∩Ᵽ)=(Ꞙ,Ꞩ)^r $\cup_R$(ℭ,Ᵽ)^r. Here, if Ꞩ∩Ᵽ=∅, then the equality is again satisfied since the right and left sides will be $\emptyset_\emptyset$. So, Ꞩ∩Ᵽ≠∅ is not a necessary condition for this proposition.

Proposition 13 Let (Ꞙ,Ꞩ) and (ℭ,Ꞩ) be SSs over U. Then, (Ꞙ,Ꞩ) $\cap_R$ (ℭ, Ꞩ)= $U_Ꞩ \Leftrightarrow$ (Ꞙ, Ꞩ) $= U_Ꞩ$ and (ℭ, Ꞩ) $= U_Ꞩ$.

Proof: Let (Ꞙ, Ꞩ) $\cap_R$ (ℭ, Ꞩ) = (K,Ꞩ∩Ꞩ), where for all α∈Ꞩ, K(α)= Ꞙ(α)∩ℭ(α). Since (K,Ꞩ)= U_T, K(α)=U, for all α∈Ꞩ. Thus, K(α)=Ꞙ(α)∩ℭ(α)=U, for all α∈Ꞩ ⇔Ꞙ(α)= U and ℭ(α)= U, for all α ∈ Ꞩ ⇔ (Ꞙ, Ꞩ) $= U_Ꞩ$ and (ℭ,Ꞩ)= $U_Ꞩ$.

In the study by Sezgin and Atagün (2011), the properties related to restricted intersection and soft subsets were examined according to the definition of soft subsets given in the study by Maji et al. (2003). In this study, we examine the properties related to soft subsets according to the definition given in the study by Pei and Miao (2005), which is widely accepted. Therefore, the following properties related to soft subsets have not been included in previous studies.

Proposition 14 Let (Ꞙ,Ꞩ) and (ℭ,Ᵽ) be SSs over U. Then, $\emptyset_{Ꞩ\capᵽ}$ $\tilde{\subseteq}$(Ꞙ,Ꞩ) $\cap_R$(ℭ,Ᵽ). Moreover, (Ꞙ,Ꞩ) $\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$ $U_Ꞩ$, and (Ꞙ,Ꞩ) $\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$ $U_Ᵽ$.

Proof: The proof is evident from the fact that the empty set is a subset of every set and the universal set includes every set.

Proposition 15 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$ (Ꞙ,Ɉ) and (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$(ℭ,Ᵽ), where Ɉ∩Ᵽ≠∅.

Proof: Let (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)=(Ἡ,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, Ἡ(α)=Ꞙ(α) ∩ℭ(α). Thus, Ἡ(α)=Ꞙ(α) ∩ ℭ(α) ⊆Ꞙ(α), for all α∈Ɉ∩Ᵽ. Hence, (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)$\tilde{\subseteq}$(Ꞙ,Ɉ). Furthermore, since Ꞙ(α) ∩ ℭ(α) ⊆ℭ(α), (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) $\tilde{\subseteq}$ (ℭ,Ᵽ) is obvious.

Proposition 16 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, (Ꞙ,Ɉ)$\tilde{\subseteq}$ (ℭ,Ᵽ) if and only if (Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ)=(Ꞙ,Ɉ)**.**

Proof: Let (Ꞙ,Ɉ)$\tilde{\subseteq}$ (ℭ,Ᵽ) and (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)=(K,Ɉ∩Ᵽ=Ɉ). Thus, Ɉ⊆Ᵽ and Ꞙ(α)⊆ ℭ(α), for all α∈Ɉ, and so K(α)=Ꞙ(α)∩ℭ(α)=Ꞙ(α), for all α∈Ɉ. Therefore, (K,Ɉ)= (Ꞙ,Ɉ) $\cap_R$(ℭ,Ᵽ) =(Ꞙ,Ɉ)**.** Conversely, let (Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ) =(Ꞙ,Ɉ). Hence, Ɉ∩Ᵽ=Ɉ, implying that Ɉ⊆Ᵽ**.** Moreover, since Ꞙ(α)∩ℭ(α)=Ꞙ(α), for all α∈Ɉ, this implies that Ꞙ(α)⊆ℭ(α). Thereby, (Ꞙ,Ɉ) $\tilde{\subseteq}$ (ℭ,Ᵽ).

Proposition 17 Let (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ἡ,Ꙁ) be SSs over U such that Ɉ∩Ꙁ≠∅. If (Ꞙ,Ɉ) $\tilde{\subseteq}$ (ℭ,Ᵽ), then (Ꞙ,Ɉ) $\cap_R$ (Ἡ,Ꙁ) $\tilde{\subseteq}$ (ℭ,Ᵽ) $\cap_R$ (Ἡ,Ꙁ).

Proof: Let (Ꞙ,Ɉ)$\tilde{\subseteq}$ (ℭ,Ᵽ). Then, Ɉ⊆Ᵽ, and Ꞙ(α)⊆ ℭ(α), for all α∈Ɉ. Let (Ꞙ,Ɉ)$\cap_R$(Ἡ,Ꙁ)=(K,Ɉ∩Ꙁ). Thus, K(α)=Ꞙ(α)∩Ἡ(α), for all α∈Ɉ∩Ꙁ. Let (ℭ,Ᵽ)$\cap_R$(Ἡ,Ꙁ)=(L,Ᵽ∩Ꙁ). Hence, L(α)=ℭ(α)∩Ἡ(α), for all α∈Ᵽ∩Ꙁ. Hence, Ɉ∩Ꙁ ⊆ Ᵽ∩Ꙁ, and K(α)=Ꞙ(α)∩Ἡ(α)⊆ℭ(α)∩Ἡ(α)=L(α), for all α∈Ɉ∩Ꙁ. Thereby, (Ꞙ,Ɉ)$\cap_R$(Ἡ,Ꙁ) $\tilde{\subseteq}$(ℭ,Ᵽ) $\cap_R$(Ἡ,Ꙁ).

Here, if Ᵽ∩Ꙁ=∅, this would require Ɉ∩Ꙁ=∅ (as Ɉ∩Ꙁ⊆Ᵽ∩Ꙁ) making the proof evident once again.

Proposition 18 Let (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ἡ,Ꙁ) be SSs over U such that Ɉ∩Ꙁ≠∅. If (Ꞙ,Ɉ) $\cap_R$ (Ἡ,Ꙁ) $\tilde{\subseteq}$ (ℭ,Ᵽ) $\cap_R$ (Ἡ,Ꙁ), thn (Ꞙ,Ɉ) $\tilde{\subseteq}$ (ℭ,Ᵽ) needs not be true. That is, the converse of Proposition 17 is not true.

Proof: Let us give an example to show that the converse of Proposition 17 is not true. Let E={e_1,e_2,e_3,e_4,e_5} be the parameter set Ɉ={e_1,e_3}, Ᵽ={e_1,e_3,e_5}, Ꙁ={e_1,e_3,e_5, e_6} be the subsets of E and U={h_1,h_2,h_3,h_4,h_5} be the universel set. Assume that (Ꞙ,Ɉ), (ℭ,Ᵽ), and (Ἡ,Ꙁ) are the SSs over U defined as follows:

(Ꞙ,Ɉ)={(e_1,{h_2, h_5}),(e_3,{h_1,h_2,h_5})},

(ℭ,Ᵽ)={(e_1,{h_2}),(e_3,{h_1,h_2}),(e_5,{h_3})},

(Ἡ,Ꙁ)={ e_1,∅),(e_3,∅),(e_5, ∅),(e_6,{h_1, h_5})}.

Let (Ꞙ,Ɉ)$\cap_R$(Ἡ,Ꙁ)=(L,Ɉ∩Ꙁ), where for all α∈Ɉ ∩ Ꙁ = {e_1, e_3}, L(α)=Ꞙ(α) ∩Ἡ(α). Thus, L(e_1)=Ꞙ(e_1) ∩ Ἡ(e_1)=∅, L(e_3)=Ꞙ(e_3) ∩ Ἡ(e_3)=∅. Hence, (Ꞙ,Ꭻ)$\cap_R$(Ή,Ꝣ)={(e_1,∅),(e_3,∅)}.

Now let (ℭ,Ᵽ)$\cap_R$(Ή,Ꝣ)=(K,Ᵽ∩Ꝣ), where for all α∈Ᵽ ∩ Ꝣ = {e_1, e_3, e_5}, K(α)=ℭ(α) ∩Ή(α). Hence, K(e_1)=ℭ(e_1) ∩ Ή(e_1)=∅, K(e_3)=ℭ(e_3) ∩ Ή(e_3)=∅, and K(e_5)=ℭ(e_5) ∩ Ή(e_5)=∅. Thus,

(ℭ,Ᵽ) $\cap_R$(Ή,Ꝣ)={(e_1,∅), (e_3,∅),(e_5,∅)}.

Hence, (Ꞙ,Ꭻ) $\cap_R$(Ή,Ꝣ) $\tilde{\subseteq}$(ℭ,Ᵽ) $\cap_R$(Ή,Ꝣ), but (Ꞙ,Ꭻ) isn't a subset of ℭ,Ᵽ).

Proposition 19 Let (Ꞙ,Ꭻ), (ℭ,Ᵽ), (K,V) and (L,W) be SSs over U such that Ꭻ∩V≠∅. If (Ꞙ,Ꭻ) $\tilde{\subseteq}$ (ℭ,Ᵽ) and (K,V) $\tilde{\subseteq}$ (L,W), then (Ꞙ,Ꭻ) $\cap_R$ (K,V) $\tilde{\subseteq}$ (ℭ,Ᵽ) $\cap_R$ (L,W)

Proof: Let (Ꞙ,Ꭻ) $\tilde{\subseteq}$ (ℭ,Ᵽ) and (K,V) $\tilde{\subseteq}$ (L,W). Hence, for all α∈Ꭻ, Ꞙ(α)⊆ ℭ(α) and for all α∈V, K(α)⊆ L(α). Let (Ꞙ,Ꭻ)$\cap_R$(K,V)=(M,Ꭻ∩V). Thus, for all α∈Ꭻ∩V, M(α)=Ꞙ(α)∩K(α). Let (ℭ,Ᵽ)$\cap_R$(L,W)=(Ǹ,Ᵽ∩W). Thus, for all α∈Ᵽ∩W, Ǹ(α)=ℭ(α)∩L(α). Hence, for all α∈ Ꭻ∩V, M(α)=Ꞙ(α)∩K(α)⊆ℭ(α)∩L(α)=Ǹ(α). Hence, (Ꞙ,Ꭻ)$\cap_R$(K,V)$\tilde{\subseteq}$(ℭ,Ᵽ) $\cap_R$(L,W). Here, if Ᵽ∩W=∅, this would require Ꭻ∩V=∅ (since Ꭻ∩V⊆Ᵽ∩W) making the proof clear once again.

Proposition 20 Let (Ꞙ,Ꭻ) and (ℭ,Ᵽ) be SSs over U. If (Ꞙ,Ꭻ) $\tilde{\subseteq}$ (ℭ,Ᵽ)^r, then (Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ) = $\emptyset_{Ꭻ}$. Moreover, (Ꞙ,Ꭻ) $\tilde{\subseteq}$ (ℭ,Ꭻ)^r if and only if (Ꞙ,Ꭻ) $\cap_R$(ℭ,Ꭻ) = $\emptyset_{Ꭻ}$.

Proof: Let (Ꞙ,Ꭻ) $\tilde{\subseteq}$ (ℭ,Ᵽ)^r. Hence, Ꭻ ⊆ Ᵽ and Ꞙ(α) ⊆ ℭ'(α), for all α∈Ꭻ. Let (Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ) =(Ή, Ꭻ∩Ᵽ=Ꭻ). Hence, Ή(α)= Ꞙ(α)∩ℭ(α)=∅, for all α∈Ꭻ. Thus, (Ή,Ꭻ) = $\emptyset_{Ꭻ}$. Similarly, it can be shown that (Ꞙ,Ꭻ) $\tilde{\subseteq}$(ℭ,Ꭻ)^r ⇔ (Ꞙ,Ꭻ) $\cap_R$(ℭ,Ꭻ)= $\emptyset_{Ꭻ}$. In other words, for the converse of the theorem to be true, the parameter sets of the SSs must be the same. It was given without proof in Ali et al. (2011) that if (Ꞙ,Ꭻ) $\cap_R$ (ℭ,Ꭻ)= $\emptyset_{Ꭻ}$, then (Ꞙ,Ꭻ) $\tilde{\subseteq}$(ℭ,Ꭻ)^r . However, it is also evident that if (Ꞙ,Ꭻ) $\tilde{\subseteq}$(ℭ,Ꭻ)^r, then (Ꞙ,Ꭻ) $\cap_R$ (ℭ,Ꭻ) = $\emptyset_{Ꭻ}$.

3.1.1. The distributions of the restricted intersection operation over other SS operations:

In this subsection, the distributions of restricted intersection operation over other SS operations such as restricted SS operations, extended SS operations, and soft binary piecewise operations are examined in detail and many interesting results are obtained.

3.1.1.1. The distributions of the restricted intersection operation over other restricted SS operations:

Here, the distributions of the restricted intersection operation over other restricted operations have been examined. First, the left distributions, and then the right distributions were investigated. It is worth mentioning an important point here. Although Sezgin and Atagün (2011) showed that the restricted intersection operation distributes over the restricted union and restricted difference from both the right and the left, their proofs repeatedly emphasized that the intersections of the parameter sets of the SSs involved in the restricted operations should be non-empty. However, even if the intersections of the parameter sets of the SSs involved in the restricted operations are empty, these distributions still hold. Besides, in the study by Ali et al. (2011), only the left distributions were presented in a table and without proofs. Therefore, in this subsection, considering all these conditions, detailed proof is provided for the distributions.

a) LHS distributions of restricted intersection over other restricted SS operations:

Let (Ӻ,Ʝ), (ℭ,Ᵽ), and (Ή,Ƨ) be SSs over U. Then, we have the following distributions:

i) $(Ӻ,Ʝ)\cap_R[(ℭ,Ᵽ)\cup_R(Ή,Ƨ)]=[(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)]\cup_R[(Ӻ,Ʝ)\cap_R(Ή,Ƨ)]$ (Sezgin and Atagün, 2011).

Proof: In their study, Sezgin and Atagün (2011) provided the proof of this property under the condition that the intersection of the parameter sets of the SSs involved in the restricted operations should be non-empty. However, this proof will specifically indicate that this property holds even if the intersection of the parameter sets of the SSs involved in the restricted operations is empty.

First, let's consider the left-hand side (LHS), and let $(ℭ,Ᵽ)\cup_R(Ή,Ƨ)=(R,Ᵽ\cap Ƨ)$, where for all $\alpha\in Ᵽ\cap Ƨ$, $R(\alpha)=ℭ(\alpha)\cup Ή(\alpha)$. Let $(Ӻ,Ʝ)\cap_R(R,Ᵽ\cap Ƨ)=(Ǹ,Ʝ\cap(Ᵽ\cap Ƨ))$, where for all $\alpha\in Ʝ\cap(Ᵽ\cap Ƨ)$, $Ǹ(\alpha)=Ӻ(\alpha)\cap R(\alpha)$. Hence, for all $\alpha\in Ʝ\cap Ᵽ\cap Ƨ$,

$$Ǹ(\alpha)= Ӻ(\alpha)\cap[(ℭ(\alpha)\cup Ή(\alpha)].$$

Now let's handle the right hand side (RHS). Let $(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)=(V,Ʝ\cap Ᵽ)$, where for all $\alpha\in Ʝ\cap Ᵽ$, $V(\alpha)=Ӻ(\alpha)\cap ℭ(\alpha)$, and let $(Ӻ,Ʝ)\cap_R(Ή,Ƨ)=(W,Ʝ\cap Ƨ)$, where for all $\alpha\in Ʝ\cap Ƨ$, $W(\alpha)=Ӻ(\alpha)\cap Ή(\alpha)$. Let $(V,Ʝ\cap Ᵽ)\cup_R(W,Ʝ\cap Ƨ)=(S,(Ʝ\cap Ᵽ)\cap(Ʝ\cap Ƨ))$, where for all $\alpha\in Ʝ\cap Ᵽ\cap Ƨ$, $S(\alpha)=V(\alpha)\cup W(\alpha)$. Thereby,

$$S(\alpha)= [Ӻ(\alpha)\cap ℭ(\alpha)]\cup[Ӻ(\alpha)\cap Ή(\alpha)]$$

Thus, it is seen that $(Ǹ,Ʝ\cap Ᵽ\cap Ƨ)=(S,Ʝ\cap Ᵽ\cap Ƨ)$. Here, if $Ᵽ\cap Ƨ=\emptyset$ or $Ʝ\cap Ᵽ=\emptyset$ or $Ʝ\cap Ƨ=\emptyset$, then in every case, both the left side and the right side will be $\emptyset_\emptyset$. Thus, the equality holds in this case as well. Therefore, there is no need to impose the condition that these sets are non-empty.

ii) $(Ӻ,Ʝ)\cap_R[(ℭ,Ᵽ)\cap_R(Ή,Ƨ)] = [(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)]\cap_R[(Ӻ,Ʝ)\cap_R(Ή,Ƨ)]$.

iii) $(Ӻ,Ʝ)\cap_R[(ℭ,Ᵽ)\setminus_R(Ή,Ƨ)] = [(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)]\setminus_R[(Ӻ,Ʝ)\cap_R(Ή,Ƨ)]$ (Sezgin and Atagün, 2011).

iv) $(Ӻ,Ʝ)\cap_R[(ℭ,Ᵽ)\gamma_R(Ή,Ƨ)] = [(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)]\gamma_R[(Ӻ,Ʝ)\cap_R(Ή,Ƨ)]$.

v) $(Ӻ,Ʝ)\cap_R[(ℭ,Ᵽ)\Delta_R(Ή,Ƨ)]=[(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)]\Delta_R[(Ӻ,Ʝ)\cap_R(Ή,Ƨ)]$ (Singh and Onyeozili, 2012c).
Although Singh and Onyeozili (2012c) provided this property, their proof contains numerous mathematical errors. Therefore, we are presenting the proof again in a structured and corrected manner. First, let's consider the LHS, and let $(ℭ,Ᵽ)\Delta_R(Ή,Ƨ)=(R,Ᵽ\cap Ƨ)$, where for all $\alpha\in Ᵽ\cap Ƨ$, $R(\alpha)=ℭ(\alpha)\Delta Ή(\alpha)$. Let $(Ӻ,Ʝ)\cap_R(R,Ᵽ\cap Ƨ)=(Ǹ,Ʝ\cap(Ᵽ\cap Ƨ))$, for all $\alpha\in Ʝ\cap(Ᵽ\cap Ƨ)$, $Ǹ(\alpha)= Ӻ(\alpha)\cap R(\alpha)$. Hence, for all $\alpha\in Ʝ\cap Ᵽ\cap Ƨ$,

$$Ǹ(\alpha)= Ӻ(\alpha)\cap[(ℭ(\alpha)\Delta Ή(\alpha)].$$

Now let's handle the RHS. Let $(Ӻ,Ʝ)\cap_R(ℭ,Ᵽ)=(V,Ʝ\cap Ᵽ)$, where for all $\alpha\in Ʝ\cap Ᵽ$, $V(\alpha)=Ӻ(\alpha)\cap ℭ(\alpha)$. Now let $(Ӻ,Ʝ)\cap_R(Ή,Ƨ)=(W,Ʝ\cap Ƨ)$, where for all $\alpha\in Ʝ\cap Ƨ$, $W(\alpha)=Ӻ(\alpha)\cap Ή(\alpha)$. Let $(V,Ʝ\cap Ᵽ)\Delta_R(W,Ʝ\cap Ƨ)=(S,(Ʝ\cap Ᵽ)\cap(Ʝ\cap Ƨ))$. Hence, for all $\alpha\in Ʝ\cap Ᵽ\cap Ƨ$, $S(\alpha)=V(\alpha)\Delta W(\alpha)$,

$$S(\alpha)= [Ӻ(\alpha)\cap ℭ(\alpha)]\Delta[Ӻ(\alpha)\cap Ή(\alpha)]$$

Thus, it is seen that $(Ǹ,Ʝ\cap Ᵽ\cap Ƨ)=(S,Ʝ\cap Ᵽ\cap Ƨ)$. Here, if $Ᵽ\cap Ƨ=\emptyset$ or $Ʝ\cap Ᵽ=\emptyset$ or $Ʝ\cap Ƨ=\emptyset$, then in every case, both the left side and the right side will be $\emptyset_\emptyset$. Thus, the equality holds in this case as well. Therefore, there is no need to impose the condition that these sets are non-empty.

b) RHS distributions of restricted intersection over other restricted SS operations:
Let (Ӻ,Ʝ), (ℭ,Ᵽ), and (Ή,Ƹ) be SSs over U. Then, we have the following distributions:

i) [(Ꞙ,Ꭻ)$\cup_R$(ℭ,Ꝑ)] $\cap_R$(Ή,Ȝ)=[(Ꞙ,Ꭻ) $\cap_R$(Ή,Ȝ)] $\cup_R$[(ℭ,Ꝑ) $\cap_R$ (Ή,Ȝ)] (Sezgin and Atagün, 2011).

Proof: Sezgin and Atagün (2011) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's handle the LHS. Let (Ꞙ,Ꭻ) $\cup_R$(ℭ,Ꝑ)=(R,Ꭻ∩Ꝑ), where for all α∈Ꭻ∩Ꝑ, R(α)=Ꞙ(α)∪ℭ(α). Let (R,Ꭻ∩Ꝑ) $\cap_R$(Ή,Ƨ)=(Ǹ,(Ꭻ∩Ꝑ)∩Ƨ)), where for all α∈(Ꭻ∩Ꝑ)∩Ƨ, Ǹ(α)=R(α)∩Ή(α). Hence,

Ǹ(α)= [Ꞙ(α)∪ℭ(α)]∩Ή(α)

Now let's handle the RHS. Let (Ꞙ,Ꭻ)$\cap_R$(Ή,Ƨ)=(S,Ꭻ∩Ƨ), where for all α∈Ꭻ∩Ƨ, S(α)=Ꞙ(α)∩Ή(α). Let (ℭ,Ꝑ) $\cap_R$(Ή,Ƨ)=(K,Ꝑ∩Ƨ), where K(α)=ℭ(α)∩Ή(α), for all α∈Ꝑ∩Ƨ. Let (S,Ꭻ∩Ƨ)$\cap_R$(K,Ꝑ∩Ƨ)=(L,(Ꭻ∩Ꝑ∩Ƨ)), where for all α∈(Ꭻ∩Ƨ)∩(Ꝑ∩Ƨ) , L(α)=S(α)∩K(α). Hence,

L(α)= ([Ꞙ(α)∩Ή(α)] ∪[ℭ(α)∩Ή(α)]

Thus, it is seen that (Ǹ,Ꭻ∩Ꝑ∩Ƨ)=(L,Ꭻ∩Ꝑ∩Ƨ). Here, if Ꝑ∩Ƨ=∅ or Ꭻ∩Ꝑ=∅ or Ꭻ∩Ƨ=∅, =∅, then in every case, both the left side and the right side will be $\emptyset_\emptyset$. Thus, the equality holds in this case as well. Therefore, there is no need to impose the condition that these sets are non-empty.

ii) [(Ꞙ,Ꭻ) $\cap_R$ (ℭ,Ꝑ)] $\cap_R$ (Ή,Ƨ)=[(Ꞙ,Ꭻ) $\cap_R$(Ή,Ƨ)] $\cap_R$[(ℭ,Ꝑ) $\cap_R$(Ή,Ƨ)].

iii) [(Ꞙ,Ꭻ) $\setminus_R$ (ℭ,Ꝑ)] $\cap_R$ (Ή,Ƨ)= [(Ꞙ,Ꭻ)$\cap_R$(Ή,Ƨ)] $\setminus_R$ [(ℭ,Ꝑ)$\cap_R$(Ή,Ƨ)] (Sezgin and Atagün, 2011).

iv) [(Ꞙ,Ꭻ)γ_R(ℭ,Ꝑ)] $\cap_R$(Ή,Ƨ)= [(Ꞙ,Ꭻ) $\cap_R$ (Ή,Ƨ)] γ_R [(ℭ,Ꝑ) $\cap_R$ (Ή,Ƨ)].

v) [(Ꞙ,Ꭻ) Δ_R (ℭ,Ꝑ)] $\cap_R$ (Ή,Ƨ)= [(Ꞙ,Ꭻ)$\cap_R$(Ή,Ƨ)] Δ_R [(ℭ,Ꝑ)$\cap_R$(Ή,Ƨ)].

3.1.1.2. The distributions of the restricted intersection operation over extended SS operations:

Here, the distributions of the restricted intersection operation over extended operations have been examined. First, left distributions were investigated, followed by right distributions. It is important to note a significant point here. In some studies, although it has been stated that the restricted intersection operation distributes over extended union, intersection, and difference operations from the left side, no attention has been given to the right distributions, and all right distributions have been provided without proofs in the study by Ali et al. (2011). In other studies where proofs are provided, emphasis has been placed on the requirement that the intersection of the parameter sets of the SSs involved in the restricted operations must be non-empty. However, even if the intersection of the parameter sets of the SSs involved in the restricted operations is the empty set, these distributions are still valid. Therefore, detailed proofs are provided taking into account these considerations, especially in the distributions under this subsection.

a) LHS distributions of restricted intersection operation over extended SS operations:

Let (Ꞙ,Ꭻ), (ℭ,Ꝑ), and (Ή,Ƨ) be SSs over U. Then, we have the following distributions:

i) (Ꞙ,Ꭻ) $\cap_R$[(ℭ,Ꝑ) $\cup_\varepsilon$ (Ή,Ƨ)] = [(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ꝑ)] $\cup_\varepsilon$ [(Ꞙ,Ꭻ) $\cap_R$ (Ή,Ƨ)] (Pei and Miao, 2005)

Proof: Pei and Miao (2005) presented this property without proof in their study; however, we provide it with its detailed proof here. We also state and prove that the property holds even when the intersection of the parameter sets of the SSs involved in the restricted operations is empty.

First, let's consider the LHS, and let (ℭ,Ᵽ) $\cup_{\varepsilon}$ (Ⱨ,Ƨ)=(R,Ᵽ∪Ƨ), where for all α∈Ᵽ∪Ƨ,

R(α)= { ℭ(α)   α∈Ᵽ\Ƨ
Ⱨ(α)   α∈Ƨ\Ᵽ
ℭ(α)∪Ⱨ(α)   α∈Ᵽ∩Ƨ

Let (Ꞙ,Ʝ) $\cap_R$ (R,Ᵽ∪Ƨ) =(Ṅ,(Ʝ∩(Ᵽ∪Ƨ)), where for all α∈Ʝ∩(Ᵽ∪Ƨ), Ṅ(α)=Ꞙ(α)∩R(α). Thus,

Ṅ(α)= { Ꞙ(α)∩ℭ(α)   α∈Ʝ∩(Ᵽ\Ƨ)=Ʝ∩Ᵽ∩Ƨ'
Ꞙ(α)∩Ⱨ(α)   α∈Ʝ∩(Ƨ\Ᵽ)=Ʝ∩Ᵽ'∩Ƨ
Ꞙ(α)∩[ℭ(α)∪Ⱨ(α)]   α∈Ʝ∩(Ᵽ∩Ƨ)=Ʝ∩Ᵽ∩Ƨ

Now let's handle the RHS. Let (Ꞙ,Ʝ)$\cap_R$(ℭ,Ᵽ)=(K,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, K(α)=Ꞙ(α)∩ℭ(α). Let (Ꞙ,Ʝ) $\cap_R$(Ⱨ,Ƨ)=(S,Ʝ∩Ƨ), where for all α∈Ʝ∩Ƨ, S(α)=Ꞙ(α)∩Ⱨ(α). Let (K,Ʝ∩Ᵽ) $\cup_{\varepsilon}$(S,Ʝ∩Ƨ)=(L,(Ʝ∩Ᵽ)∪(Ʝ∩Ƨ)), where for all α∈(Ʝ∩Ᵽ)∪(Ʝ∩Ƨ),

L(α)= { K(α)   α∈(Ʝ∩Ᵽ)\(Ʝ∩Ƨ)=Ʝ∩(Ᵽ\Ƨ)
S(α)   α∈(Ʝ∩Ƨ)\(Ʝ∩Ᵽ)=Ʝ∩(Ƨ\Ᵽ)
K(α)∪S(α)   α∈(Ʝ∩Ᵽ)∩(Ʝ∩Ƨ)=Ʝ∩(Ᵽ∩Ƨ)

Hence,

L(α)= { Ꞙ(α)∩ℭ(α)   α∈Ʝ∩Ᵽ∩Ƨ'
Ꞙ(α)∩Ⱨ(α)   α∈Ʝ∩Ᵽ'∩Ƨ
[Ꞙ(α)∩ℭ(α)] ∪[Ꞙ(α)∩Ⱨ(α)]   α∈Ʝ∩Ᵽ∩Ƨ

It is seen that (Ṅ,Ʝ∩(Ᵽ∪Ƨ))=(L,(Ʝ∩Ᵽ)∪(Ʝ∩Ƨ)). Here, if Ʝ∩Ᵽ=∅, then Ṅ(α)=L(α)=Ꞙ(α)∩Ⱨ(α), and if Ʝ∩Ƨ=∅, then Ṅ(α)=L(α)=Ꞙ(α)∩ℭ(α). That is, in these cases, the left-hand side and the right-hand side are still equal. Therefore, there is no need to impose the condition that these sets are non-empty.

ii) (Ꞙ,Ʝ) $\cap_R$[(ℭ,Ᵽ) $\cap_{\varepsilon}$(Ⱨ,Ƨ)]=[(Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)] $\cap_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_R$ (Ⱨ,Ƨ)] (Singh and Onyeozili, 2012c).

iii) (Ꞙ,Ʝ) $\cap_R$[(ℭ,Ᵽ) $\setminus_{\varepsilon}$(Ⱨ,Ƨ)]=[(Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)]$\setminus_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_R$ (Ⱨ,Ƨ)] (Sezgin et al., 2019).

iv) (Ꞙ,Ʝ) $\cap_R$[(ℭ,Ᵽ) γ_{ε}(Ⱨ,Ƨ)]=[(Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)] γ_{ε}[(Ꞙ,Ʝ) $\cap_R$ (Ⱨ,Ƨ)].

v) (Ꞙ,Ʝ) $\cap_R$[(ℭ,Ᵽ) Δ_{ε}(Ⱨ,Ƨ)]=[(Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)] Δ_{ε}[(Ꞙ,Ʝ) $\cap_R$ (Ⱨ,Ƨ)].

vi) (Ꞙ,Ʝ) $\cap_R$[(ℭ,Ᵽ) $+_{\varepsilon}$ (Ⱨ,Ƨ)] = [(Ꞙ,Ʝ) $\cap_R$(ℭ,Ᵽ)] $+_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_R$ (Ⱨ,Ƨ)], where Ʝ∩Ᵽ∩Ƨ=∅.

vii) $(Ꞙ,Ɉ) \cap_R[(ℭ,Ᵽ)\ \lambda_\varepsilon\ (Ҥ,Ƨ)] = [(Ꞙ,Ɉ) \cap_R(ℭ,Ᵽ)]\ \lambda_\varepsilon[(Ꞙ,Ɉ) \cap_R (Ҥ,Ƨ)]$, where $Ɉ\cap Ᵽ\cap Ƨ=\emptyset$.

viii) $(Ꞙ,Ɉ) \cap_R[(ℭ,Ᵽ)\ *_\varepsilon\ (Ҥ,Ƨ)] = [(Ꞙ,Ɉ) \cap_R(ℭ,Ᵽ)]\ *_\varepsilon[(Ꞙ,Ɉ) \cap_R (Ҥ,Ƨ)]$, where $Ɉ\cap Ᵽ\cap Ƨ=\emptyset$.

ix) $(Ꞙ,Ɉ) \cap_R[(ℭ,Ᵽ)\ \theta_\varepsilon\ (Ҥ,Ƨ)] = [(Ꞙ,Ɉ) \cap_R(ℭ,Ᵽ)]\ \theta_\varepsilon[(Ꞙ,Ɉ) \cap_R (Ҥ,Ƨ)]$, where $Ɉ\cap Ᵽ\cap Ƨ=\emptyset$.

b) RHS distributions of restricted intersection over extended SS operations:
Let $(Ꞙ,Ɉ)$, $(ℭ,Ᵽ)$, and $(Ҥ,Ƨ)$ be SSs over U. Then, we have the following distributions:

i) $[(Ꞙ,Ɉ) \cup_\varepsilon (ℭ,Ᵽ)] \cap_R (Ҥ,Ƨ) = [(Ꞙ,Ɉ) \cap_R (Ҥ,Ƨ)] \cup_\varepsilon [(ℭ,Ᵽ) \cap_R (Ҥ,Ƨ)]$.

Proof: First, let's consider the LHS, and let $(Ꞙ,Ɉ) \cup_\varepsilon(ℭ,Ᵽ)=(R,Ɉ\cup Ᵽ)$, where for all $\alpha\in Ɉ\cup Ᵽ$,

$$R(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\backslash Ᵽ \\ ℭ(\alpha) & \alpha\in Ᵽ\backslash Ɉ \\ Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Let $(R,Ɉ\cup Ᵽ) \cap_R(Ҥ,Ƨ) =(\grave{N},(Ɉ\cup Ᵽ)\cap Ƨ)$, where for all $\alpha\in(Ɉ\cup Ᵽ)\cap Ƨ$, $\grave{N}(\alpha)=R(\alpha)\cap Ҥ(\alpha)$. Thus,

$$\grave{N}(\alpha)=\begin{cases} Ꞙ(\alpha)\cap Ҥ(\alpha) & \alpha\in(Ɉ\backslash Ᵽ)\cap Ƨ=Ɉ\cap Ᵽ'\cap Ƨ \\ ℭ(\alpha)\cap Ҥ(\alpha) & \alpha\in(Ᵽ\backslash Ɉ)\cap Ƨ=Ɉ'\cap Ᵽ\cap Ƨ \\ [Ꞙ(\alpha)\cup ℭ(\alpha)]\cap Ҥ(\alpha) & \alpha\in(Ɉ\cap Ᵽ)\cap Ƨ=Ɉ\cap Ᵽ\cap Ƨ \end{cases}$$

Now let's handle the RHS. Let $(Ꞙ,Ɉ)\cap_R(Ҥ,Ƨ)=(K,Ɉ\cap Ƨ)$, where for all $\alpha\in Ɉ\cap Ƨ$, $K(\alpha)=Ꞙ(\alpha)\cap Ҥ(\alpha)$. Let $(ℭ,Ᵽ) \cap_R(Ҥ,Ƨ)=(S,Ᵽ\cap Ƨ)$, where for all $\alpha\in Ᵽ\cap Ƨ$, $S(\alpha)=ℭ(\alpha)\cap Ҥ(\alpha)$. Then, let $(K,Ɉ\cap Ƨ) \cup_\varepsilon(S,Ᵽ\cap Ƨ)=(L,(Ɉ\cap Ƨ)\cup(Ᵽ\cap Ƨ))$, where for all $\alpha\in(Ɉ\cap Ƨ)\cup(Ᵽ\cap Ƨ)$,

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(Ɉ\cap Ƨ)\backslash(Ᵽ\cap Ƨ)=(Ɉ\backslash Ᵽ)\cap Ƨ \\ S(\alpha) & \alpha\in(Ᵽ\cap Ƨ)\backslash(Ɉ\cap Ƨ)=(Ᵽ\backslash Ɉ)\cap Ƨ \\ K(\alpha)\cup S(\alpha) & \alpha\in(Ɉ\cap Ƨ)\cap(Ᵽ\cap Ƨ)=(Ɉ\cap Ᵽ)\cap Ƨ \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} Ꞙ(\alpha)\cap Ҥ(\alpha) & \alpha\in Ɉ\cap Ᵽ'\cap Ƨ \\ ℭ(\alpha)\cap Ҥ(\alpha) & \alpha\in Ɉ'\cap Ᵽ\cap Ƨ \\ [Ꞙ(\alpha)\cap Ҥ(\alpha)]\cup[ℭ(\alpha)\cap Ҥ(\alpha)] & \alpha\in Ɉ\cap Ᵽ\cap Ƨ \end{cases}$$

It is seen that $(\grave{N},(Ɉ\cup Ᵽ)\cap Ƨ)=(L,(Ɉ\cap Ƨ)\cup(Ᵽ\cap Ƨ))$. Here, if $Ɉ\cap Ᵽ=\emptyset$ and $\alpha\in Ɉ\cap Ᵽ'\cap Ƨ$, then $\grave{N}(\alpha)=L(\alpha)=Ꞙ(\alpha)\cap Ҥ(\alpha)$; if $Ɉ\cap Ƨ=\emptyset$ and $\alpha\in Ɉ'\cap Ᵽ\cap Ƨ$, then $\grave{N}(\alpha)=L(\alpha)=ℭ(\alpha)\cap Ҥ(\alpha)$. If $Ᵽ\cap Ƨ=\emptyset$, then

Ǹ(α)=L(α)=Ғ(α)∩Ƕ(α). Since the right and left sides are equal in these cases, it is not necessary to impose the condition that these sets must be non-empty.

ii) [(Ғ,Ɉ) $\cap_\varepsilon$(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] $\cap_\varepsilon$ [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)].

iii) [(Ғ,Ɉ) $\setminus_\varepsilon$(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)=[(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)]$\setminus_\varepsilon$[(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)] (Sezgin et al, 2019).

iv) [(Ғ,Ɉ) γ_ε(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] γ_ε [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)].

v) [(Ғ,Ɉ)Δ_ε(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] Δ_ε [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)] (Sezgin and Çağman, 2025).

vi) [(Ғ,Ɉ)$+_\varepsilon$(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] $+_\varepsilon$ [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)], where Ɉ∩Ᵽ∩Ꝣ=∅.

vii) [(Ғ,Ɉ)λ_ε(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] λ_ε [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)], where Ɉ∩Ᵽ∩Ꝣ=∅.

viii) [(Ғ,Ɉ)θ_ε(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] θ_ε [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)], where Ɉ∩Ᵽ∩Ꝣ=∅.

ix) [(Ғ,Ɉ)$*_\varepsilon$(ℭ,Ᵽ)] $\cap_R$(Ƕ,Ꝣ)= [(Ғ,Ɉ) $\cap_R$(Ƕ,Ꝣ)] $*_\varepsilon$ [(ℭ,Ᵽ) $\cap_R$ (Ƕ,Ꝣ)], where Ɉ∩Ᵽ∩Ꝣ=∅.

3.1.1.3. The distributions of the restricted intersection operation over soft binary piecewise operations:

In this subsection, the distributions of the restricted intersection operation to soft binary piecewise operations have been examined. First, left distributions were investigated, followed by right distributions. It is worth noting here that these distributions are satisfied even if the intersection of the parameter sets of the SSs involved in the restricted operations is the empty set.

a) LHS distributions of restricted intersection operation over soft binary piecewise operations:
Let (Ғ,Ɉ), (ℭ,Ᵽ), and (Ƕ,Ꝣ) be SSs over U. Then, we have the following distributions:

i)(Ғ,Ɉ) $\cap_R$ [(ℭ,Ᵽ) $\overset{\sim}{\cup}$ (Ƕ,Ꝣ)] = [(Ғ,Ɉ) $\cap_R$ (ℭ,Ᵽ)] $\overset{\sim}{\cup}$ [(Ғ,Ɉ) $\cap_R$ (Ƕ,Ꝣ)].

Proof: First, let's consider the LHS, and let (ℭ,Ᵽ)$\overset{\sim}{\cup}$ (Ƕ,Ꝣ)=(R,Ᵽ), where for all α∈Ᵽ,

$$R(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ᵽ\setminus Ꝣ \\ ℭ(\alpha)\cup Ƕ(\alpha) & \alpha\in Ᵽ\cap Ꝣ \end{cases}$$

Let (Ғ,Ɉ) $\cap_R$(R,Ᵽ) =(Ǹ,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, Ǹ(α)=Ғ(α)∩R(α). Hence,

$$\dot{N}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ɉ}\cap(\text{Ᵽ}\setminus\text{Ƨ}) \\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cup\text{Ⱨ}(\alpha)] & \alpha\in\text{Ɉ}\cap(\text{Ᵽ}\cap\text{Ƨ}) \end{cases}$$

for all α∈Ɉ∩Ᵽ. Now let's handle the RHS. Let (Ꞙ,Ɉ) $\cap_R$ (ℭ,Ᵽ)=(K,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ, K(α)=Ꞙ(α)∩ℭ(α). Let (Ꞙ,Ɉ) $\cap_R$(Ⱨ,Ƨ)=(S,Ɉ∩Ƨ), where for all α∈Ɉ∩Ƨ, S(α)=Ꞙ(α)∩Ⱨ(α). Assume that (K,Ɉ∩Ᵽ)$\tilde{\cup}$(S,Ɉ∩Ƨ)=(L,Ɉ∩Ᵽ), where for all α∈Ɉ∩Ᵽ,

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\setminus(\text{Ɉ}\cap\text{Ƨ}) \\ K(\alpha)\cup S(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ɉ}\cap\text{Ƨ}) \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\setminus(\text{Ɉ}\cap\text{Ƨ})=\text{Ɉ}\cap(\text{Ᵽ}\setminus\text{Ƨ}) \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup[\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha)] & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ɉ}\cap\text{Ƨ})=\text{Ɉ}\cap(\text{Ᵽ}\cap\text{Ƨ}) \end{cases}$$

It is seen that (Ṅ,Ɉ∩Ᵽ)=(L,Ɉ∩Ᵽ). Here, if Ɉ∩Ᵽ=∅, then (Ṅ,Ɉ∩Ᵽ)=(L,Ɉ∩Ᵽ)=$\emptyset_{\emptyset}$, and if Ɉ∩Ƨ=∅, then Ṅ(α)=L(α)=Ꞙ(α)∩ℭ(α). Since the right and left sides are equal in these cases, it is not necessary to impose the condition that these sets must be non-empty.

ii) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{\cap}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{\cap}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)].

iii) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{\setminus}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{\setminus}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)].

iv) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{\gamma}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{\gamma}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)].

v) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{\Delta}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{\Delta}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)].

vi) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{+}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{+}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

vii) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{\lambda}$(Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{\lambda}$ [(Ꞙ,Ɉ) $\cap_R$ (Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

viii) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{\theta}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{\theta}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

ix) (Ꞙ,Ɉ) $\cap_R$[(ℭ,Ᵽ) $\tilde{*}$ (Ⱨ,Ƨ)]=[(Ꞙ,Ɉ)$\cap_R$(ℭ,Ᵽ)] $\tilde{*}$ [(Ꞙ,Ɉ)$\cap_R$(Ⱨ,Ƨ)], where Ɉ∩Ᵽ∩Ƨ=∅.

b) RHS distributions of restricted intersection operation over soft binary piecewise operations:
Let (Ꞙ,Ʝ), (ℭ,Ᵽ), and (Ή,Ꝣ) be SSs over U. Then, we have the following distributions:

i) [(Ꞙ,Ʝ) $\tilde{\cup}$(ℭ,Ᵽ)] $\cap_R$(Ή,Ꝣ)= [(Ꞙ,Ʝ) $\cap_R$(Ή,Ꝣ)] $\tilde{\cup}$ [(ℭ,Ᵽ) $\cap_R$ (Ή,Ꝣ)].

Proof: First, let's consider the LHS, and let (Ꞙ,Ʝ) $\tilde{\cup}$ (ℭ,Ᵽ)=(R,Ʝ), where for all α∈Ʝ,

$$R(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ʝ\setminus Ᵽ \\ Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in Ʝ\cap Ᵽ \end{cases}$$

Let (R,Ʝ) $\cap_R$(Ή,Ꝣ) =(Ṅ,Ʝ∩Ꝣ), where for all α∈Ʝ∩Ꝣ, Ṅ(α)=R(α)∩Ή(α). Thus,

$$Ṅ(\alpha)=\begin{cases} Ꞙ(\alpha)\cap Ή(\alpha) & \alpha\in (Ʝ\setminus Ᵽ)\cap Ꝣ \\ [Ꞙ(\alpha)\cup ℭ(\alpha)]\cap Ή(\alpha) & \alpha\in (Ʝ\cap Ᵽ)\cap Ꝣ \end{cases}$$

Now let's handle the RHS. Let (Ꞙ,Ʝ)$\cap_R$(Ή,Ꝣ)=(K,Ʝ∩Ꝣ), where for all α∈Ʝ∩Ꝣ, K(α)=Ꞙ(α)∩Ή(α). Let (ℭ,Ᵽ) $\cap_R$ (Ή,Ꝣ)=(S,Ᵽ∩Ꝣ), where for all α∈Ᵽ∩Ꝣ, S(α)= ℭ(α)∩Ή(α). Let (K,Ʝ∩Ꝣ) $\tilde{\cup}$(S,Ᵽ∩Ꝣ)=(L,Ʝ∩Ꝣ), where for all α∈Ʝ∩Ꝣ,

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in (Ʝ\cap Ꝣ)\setminus(Ᵽ\cap Ꝣ) \\ K(\alpha)\cup S(\alpha) & \alpha\in (Ʝ\cap Ꝣ)\cap(Ᵽ\cap Ꝣ) \end{cases}$$

Thus,

$$L(\alpha)=\begin{cases} Ꞙ(\alpha)\cap Ή(\alpha) & \alpha\in (Ʝ\cap Ꝣ)\setminus(Ᵽ\cap Ꝣ)=(Ʝ\setminus Ᵽ)\cap Ꝣ \\ [Ꞙ(\alpha)\cap Ή(\alpha)]\cup[ℭ(\alpha)\cap Ή(\alpha)] & \alpha\in (Ʝ\cap Ꝣ)\cap(Ᵽ\cap Ꝣ)=(Ʝ\cap Ᵽ)\cap Ꝣ \end{cases}$$

It is seen that (Ṅ,Ʝ∩Ꝣ)=(L,Ʝ∩Ꝣ). Here, if Ʝ∩Ꝣ=∅, then (Ṅ,Ʝ∩Ꝣ)=(L,Ʝ∩Ꝣ)=$\emptyset_{\emptyset}$, and if Ᵽ∩Ꝣ=∅, then Ṅ(α)=L(α)=Ꞙ(α)∩Ή(α). Since the right and left sides are equal in these cases, it is not necessary to impose the condition that these sets must be non-empty.

ii) [(Ꞙ,Ʝ) $\tilde{\cap}$(ℭ,Ᵽ)]$\cap_R$(Ή,Ꝣ)= [(Ꞙ,Ʝ)$\cap_R$(Ή,Ꝣ)] $\tilde{\cap}$ [(ℭ,Ᵽ)$\cap_R$(Ή,Ꝣ)].

iii) [(Ꞙ,Ʝ) $\tilde{\setminus}$(ℭ,Ᵽ)]$\cap_R$(Ή,Ꝣ)= [(Ꞙ,Ʝ)$\cap_R$(Ή,Ꝣ)] $\tilde{\setminus}$[(ℭ,Ᵽ)$\cap_R$(Ή,Ꝣ)].

iv) $[(\text{Ꞙ},\text{Ɉ})\,\tilde{\gamma}\,(\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ͱ},\text{Ƨ})= [(\text{Ꞙ},\text{Ɉ})\cap_R(\text{Ͱ},\text{Ƨ})]\,\tilde{\gamma}\,[(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ͱ},\text{Ƨ})]$.

v) $[(\text{Ꞙ},\text{Ɉ})\,\tilde{\Delta}\,(\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ͱ},\text{Ƨ})= [(\text{Ꞙ},\text{Ɉ})\cap_R(\text{Ͱ},\text{Ƨ})]\,\tilde{\Delta}\,[(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ͱ},\text{Ƨ})]$.

vi) $[(\text{Ꞙ},\text{Ɉ})\,\tilde{+}\,(\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ͱ},\text{Ƨ})= [(\text{Ꞙ},\text{Ɉ})\cap_R(\text{Ͱ},\text{Ƨ})]\,\tilde{+}\,[(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ͱ},\text{Ƨ})]$, where $\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

vii) $[(\text{Ꞙ},\text{Ɉ})\,\tilde{\lambda}\,(\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ͱ},\text{Ƨ})= [(\text{Ꞙ},\text{Ɉ})\cap_R(\text{Ͱ},\text{Ƨ})]\,\tilde{\lambda}\,[(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ͱ},\text{Ƨ})]$, where $\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

viii) $[(\text{Ꞙ},\text{Ɉ})\,\tilde{\theta}\,(\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ͱ},\text{Ƨ})= [(\text{Ꞙ},\text{Ɉ})\cap_R(\text{Ͱ},\text{Ƨ})]\,\tilde{\theta}\,[(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ͱ},\text{Ƨ})]$, where $\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

ix) $[(\text{Ꞙ},\text{Ɉ})\,\tilde{*}\,(\mathfrak{C},\text{Ᵽ})]\cap_R(\text{Ͱ},\text{Ƨ})= [(\text{Ꞙ},\text{Ɉ})\cap_R(\text{Ͱ},\text{Ƨ})]\,\tilde{*}\,[(\mathfrak{C},\text{Ᵽ})\cap_R(\text{Ͱ},\text{Ƨ})]$, where $\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}=\emptyset$.

3.2. More on Extended Intersection Operation

To further clarify the conceptual expansion, this subsection is inspired by the extended union definition for SSs by Maji et al. (2003), a similar operation defined as the extended intersection operation of SSs by Ali et al. (2009) is examined in detail. Its properties similar to the intersection operation in classical sets, distributive rules, and relationships with other operations are thoroughly investigated. Since the extended intersection operation for SSs is not a new definition, some of its properties and its distributive rules have already been studied by various authors (Ali et al. (2009), Ali et al. (2011), Qin and Hong (2010), Sezgin and Atagün (2011)) However, in most studies, these properties have been presented without their proofs. From this perspective, we want to emphasize the importance of this study, as it includes all the properties of the extended intersection operation with their proofs, and provides detailed proofs of many new properties, especially those relating to their counterparts in classical set theory as regards intersection operation.

Definition 17 Let $(\text{Ꞙ},\text{Ɉ})$ and $(\mathfrak{C},\text{Ᵽ})$ be SSs over U. The extended intersection $(\text{Ꞙ},\text{Ɉ})$ and $(\mathfrak{C},\text{Ᵽ})$ is the SS (Ͱ,C) denoted by $(\text{Ꞙ},\text{Ɉ})\cap_\varepsilon(\mathfrak{C},\text{Ᵽ})=(\text{Ͱ},\mathrm{C})$, where $\mathrm{C}=\text{Ɉ}\cup\text{Ᵽ}$ and for all $\alpha\in\mathrm{C}$,

$$\text{Ͱ}(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\setminus\text{Ᵽ}\\ \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ɉ}\\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\end{cases}$$

(Ali et al., 2009).

Here note that the letter "ε" written below the symbol " ∩" which represents the extended intersection operation, forms a meaningful and consistent whole with its English meaning "extended". In other studies on SS operations, extended SS operations are also represented in this form. From the definition, it is obvious that if $Ɉ=\emptyset$, then $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=(\mathfrak{C},Ᵽ)$; if $Ᵽ=\emptyset$, then $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=(Ꞙ,Ɉ)$, and if $Ɉ=Ᵽ=\emptyset$, then $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=\emptyset_\emptyset$.

Example 2 Let $E=\{e_1,e_2,e_3,e_4\}$ be the parameter set $Ɉ=\{e_1, e_3\}$ and $Ᵽ=\{e_2, e_3, e_4\}$ be the subsets of E and $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the initial universe set. Assume that $(Ꞙ,Ɉ)$ and $(\mathfrak{C},Ᵽ)$ are the SSs over U defined as follows: $(Ꞙ,Ɉ)=\{(e_1,\{h_2,h_5\}), (e_3,\{h_1,h_2,h_5\})\}$, $(\mathfrak{C},Ᵽ)=\{(e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$. Let $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=(Ꞁ,Ɉ\cup Ᵽ)$, where for all $\alpha\in Ɉ\cup Ᵽ$,

$$Ꞁ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ \mathfrak{C}(\alpha) & \alpha\in Ᵽ\setminus Ɉ \\ Ꞙ(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Since $Ɉ\cup Ᵽ=\{e_1,e_2,e_3,e_4\}$, $Ɉ\setminus Ᵽ=\{e_1\}$, $Ᵽ\setminus Ɉ=\{e_2, e_4\}$, and $Ɉ\cap Ᵽ=\{e_3\}$, $Ꞁ(e_1)=Ꞙ(e_1)=\{h_2,h_5\}$, $Ꞁ(e_2)=\mathfrak{C}(e_2)=\{h_1,h_4,h_5\}$, $Ꞁ(e_4)=\mathfrak{C}(e_4)=\{h_3,h_5\}$, $Ꞁ(e_3)=Ꞙ(e_3)\cap\mathfrak{C}(e_3)=\{h_1,h_2,h_5\}\cap\{h_2,h_3,h_4\}=\{h_2\}$. Thus,

$$(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=\{(e_1,\{h_2,h_5\}), (e_2,\{h_1,h_4,h_5\}), (e_3,\{h_2\}), (e_4,\{h_3,h_5\})\}.$$

Note 1 The restricted intersection and extended intersection operations in $S_{Ꞩ}(U)$ are coincident, where Ꞩ is a fixed subset of E. That is, $(Ꞙ,Ꞩ)\cap_\varepsilon(\mathfrak{C},Ꞩ)=(Ꞙ,Ꞩ)\cap_R(\mathfrak{C},Ꞩ)$.

Proposition 21 The set $S_E(U)$ is closed under the operation $\cap_\varepsilon$. That is, when $(Ꞙ,Ɉ)$ and $(\mathfrak{C},Ᵽ)$ are two SSs over U, then so is $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)$.

Proof: It is clear that $\cap_\varepsilon$ is a binary operation in $S_E(U)$. That is,

$$\cap_\varepsilon : S_E(U) \times S_E(U)\rightarrow S_E(U)$$
$$((Ꞙ,Ɉ), (\mathfrak{C},Ᵽ)) \rightarrow (Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=(Ꞁ, Ɉ\cup Ᵽ)$$

Hence, the set $S_E(U)$ is closed under $\cap_\varepsilon$. Similarly,

$$\cap_\varepsilon: S_Ɉ(U) \times S_Ɉ(U)\rightarrow S_Ɉ(U)$$
$$((Ꞙ,Ɉ), (\mathfrak{C},Ɉ)) \rightarrow (Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ɉ)=(K,Ɉ\cup Ɉ)=(K,Ɉ)$$

Let $(Ꞙ,Ɉ)$ and $(\mathfrak{C},Ɉ)$ be elements of the set $S_T(U)$, where Ɉ is a fixed subset of the set E. Then, $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ɉ)$ is an element of the set $S_Ɉ(U)$. That is, the operation $\cap_\varepsilon$ is also closed in $S_T(U)$.

Proposition 22 Let $(Ꞙ,Ɉ)$, $(\mathfrak{C},Ᵽ)$ and $(Ꞁ,Ƨ)$ be SSs over U. Then, $[(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)]\cap_\varepsilon(Ꞁ,Ƨ)=(Ꞙ,Ɉ)\cap_\varepsilon[(\mathfrak{C},Ᵽ)\cap_\varepsilon(Ꞁ,Ƨ)]$ (Qin and Hong, 2010).

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's consider the LHS, and let $(Ꞙ,Ɉ)\cap_\varepsilon(\mathfrak{C},Ᵽ)=(S,Ɉ\cup Ᵽ)$, where for all $\alpha\in Ɉ\cup Ᵽ$,

$$S(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ᵽ \\ \mathfrak{C}(\alpha) & \alpha\in Ᵽ\setminus Ɉ \\ Ꞙ(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Let (S,ɈUᵽ) $\cap_\varepsilon$ (Ή,Ƨ) =(Ṅ,(ɈUᵽ)UƧ)), where for all α∈(ɈUᵽ)UƧ,

$$\text{Ṅ}(\alpha)=\begin{cases}\text{S}(\alpha) & \alpha\in(\text{Ɉ}\cup\text{ᵽ})\backslash\text{Ƨ}\\ \text{Ή}(\alpha) & \alpha\in\text{Ƨ}\backslash(\text{Ɉ}\cup\text{ᵽ})\\ \text{S}(\alpha)\cap\text{Ή}(\alpha) & \alpha\in(\text{Ɉ}\cup\text{ᵽ})\cap\text{Ƨ}\end{cases}$$

Thus,

$$\text{Ṅ}(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\backslash\text{Ᵽ})\backslash\text{Ƨ}=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}'\\ \mathfrak{C}(\alpha) & \alpha\in(\text{Ᵽ}\backslash\text{Ɉ})\backslash\text{Ƨ}=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}'\\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\backslash\text{Ƨ}=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}'\\ \text{Ή}(\alpha) & \alpha\in\text{Ƨ}\backslash(\text{Ɉ}\cup\text{ᵽ})=\text{Ɉ}'\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \text{Ꞙ}(\alpha)\cap\text{H}(\alpha) & \alpha\in(\text{Ɉ}\backslash\text{Ᵽ})\cap\text{Ƨ}=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \mathfrak{C}(\alpha)\cap\text{H}(\alpha) & \alpha\in(\text{Ᵽ}\backslash\text{Ɉ})\cap\text{Ƨ}=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cap\text{H}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap\text{Ƨ}=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

Now let's handle the RHS, and let ($\mathfrak{C}$,Ᵽ)$\cap_\varepsilon$(Ή,Ƨ)=(R,ᵽUƧ), where for all α∈ᵽUƧ,

$$\text{R}(\alpha)=\begin{cases}\mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\backslash\text{Ƨ}\\ \text{Ή}(\alpha) & \alpha\in\text{Ƨ}\backslash\text{Ᵽ}\\ \mathfrak{C}(\alpha)\cap\text{H}(\alpha) & \alpha\in\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

Let (Ꞙ,Ɉ)$\cap_\varepsilon$(R,ᵽUƧ) =(L,ɈU(ᵽUƧ)), where for all α∈ɈUᵽUƧ,

$$\text{L}(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\backslash(\text{Ᵽ}\cup\text{Ƨ})\\ \text{R}(\alpha) & \alpha\in(\text{Ᵽ}\cup\text{Ƨ})\backslash\text{Ɉ}\\ \text{Ꞙ}(\alpha)\cap\text{R}(\alpha) & \alpha\in\text{Ɉ}\cap(\text{Ᵽ}\cup\text{Ƨ})\end{cases}$$

Thus,

$$\text{L}(\alpha)=\begin{cases}\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\backslash(\text{Ᵽ}\cup\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}'\\ \mathfrak{C}(\alpha) & \alpha\in(\text{Ᵽ}\backslash\text{Ƨ})\backslash\text{Ɉ}=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}'\\ \text{Ή}(\alpha) & \alpha\in(\text{Ƨ}\backslash\text{Ᵽ})\backslash\text{Ɉ}=\text{Ɉ}'\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \mathfrak{C}(\alpha)\cap\text{Ή}(\alpha) & \alpha\in(\text{Ᵽ}\cap\text{Ƨ})\backslash\text{Ɉ}=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}\\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ɉ}\cap(\text{Ᵽ}\backslash\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}'\\ \text{Ꞙ}(\alpha)\cap\text{H}(\alpha) & \alpha\in\text{Ɉ}\cap(\text{Ƨ}\backslash\text{Ᵽ})=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}\\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in\text{Ɉ}\cap(\text{Ᵽ}\cap\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}\end{cases}$$

It is seen that (Ṅ,(ɈUᵽ)UƧ)=(L,ɈU(ᵽUƧ)). That is, $\cap_\varepsilon$ is associative in S_E(U).

Proposition 23 Let $(\text{Ғ},\text{Ꞁ})$, $(\mathfrak{C},\text{Ꞁ})$ and $(\text{Ꮋ},\text{Ꞁ})$ be SSs over U. Then, $[(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon}(\mathfrak{C},\text{Ꞁ})] \cap_{\varepsilon} (\text{Ꮋ},\text{Ꞁ}) = (\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon} [(\mathfrak{C},\text{Ꞁ}) \cap_{\varepsilon}(\text{Ꮋ},\text{Ꞁ})]$.
Proof: The proof follows from Note 1 and Proposition 3. That is, $\cap_{\varepsilon}$ is associative in $S_{\text{Ꞁ}}(U)$, where Ꞁ is a fixed subset of E.

Proposition 24 Let $(\text{Ғ},\text{Ꞁ})$ and $(\mathfrak{C},\text{Ᵽ})$ be SSs over U. Then, $(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon}(\mathfrak{C},\text{Ᵽ})=(\mathfrak{C},\text{Ᵽ}) \cap_{\varepsilon} (\text{Ғ},\text{Ꞁ})$ (Qin and Hong, 2010)

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it with its detailed proof. Let $(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon} (\mathfrak{C},\text{Ᵽ})=(\text{Ꮋ},\text{Ꞁ}\cup\text{Ᵽ})$, where for all $\alpha\in\text{Ꞁ}\cup\text{Ᵽ}$,

$$\text{Ꮋ}(\alpha)=\begin{cases} \text{Ғ}(\alpha) & \alpha\in\text{Ꞁ}\setminus\text{Ᵽ} \\ \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ꞁ} \\ \text{Ғ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ꞁ}\cap\text{Ᵽ} \end{cases}$$

Let $(\mathfrak{C},\text{Ᵽ})\cap_{\varepsilon}(\text{Ғ},\text{Ꞁ})=(S,\text{Ᵽ}\cup\text{Ꞁ})$, where for all $\alpha\in\text{Ᵽ}\cup\text{Ꞁ}$,

$$S(\alpha)=\begin{cases} \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ꞁ} \\ \text{Ғ}(\alpha) & \alpha\in\text{Ꞁ}\setminus\text{Ᵽ} \\ \mathfrak{C}(\alpha)\cap\text{Ғ}(\alpha) & \alpha\in\text{Ᵽ}\cap\text{Ꞁ} \end{cases}$$

Thus, $(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon}(\mathfrak{C},\text{Ᵽ})=(\mathfrak{C},\text{Ᵽ}) \cap_{\varepsilon}(\text{Ғ},\text{Ꞁ})$. Moreover, it is evident that $(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon}(\mathfrak{C},\text{Ꞁ})=(\mathfrak{C},\text{Ꞁ})\cap_{\varepsilon}(\text{Ғ},\text{Ꞁ})$. That is, $\cap_{\varepsilon}$ is commutative in both $S_E(U)$ and $S_T(U)$.

Proposition 25 Let $(\text{Ғ},\text{Ꞁ})$ be an SS over U. Then, $(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon}(\text{Ғ},\text{Ꞁ})=(\text{Ғ},\text{Ꞁ})$ (Qin and Hong, 2010)

Proof: Qin and Hong (2010) presented this property without proof in their study; however, we provide it with its detailed proof. The proof is obtained from Note 1 and Proposition 5. That is, $\cap_{\varepsilon}$ is idempotent in $S_E(U)$.

Proposition 26 Let $(\text{Ғ},\text{Ꞁ})$ be an SS over U. Then, $(\text{Ғ},\text{Ꞁ}) \cap_{\varepsilon} U_{\text{Ꞁ}}=U_{\text{Ꞁ}} \cap_{\varepsilon}(\text{Ғ},\text{Ꞁ})=(\text{Ғ},\text{Ꞁ})$.

Proof: The proof is obtained from Note 1 and Proposition 6. That is, $U_{\text{Ꞁ}}$ is the identity element of $\cap_{\varepsilon}$ in $S_{\text{Ꞁ}}(U)$.

Theorem 3 $(S_{\text{Ꞁ}}(U), \cap_{\varepsilon})$ is a bounded semi-lattice, whose identity is $U_{\text{Ꞁ}}$.

Proof: By Proposition 21, Proposition 23, Proposition 24, Proposition 25, and Proposition 26, $(S_{\text{Ꞁ}}(U),\cap_{\varepsilon})$ is a commutative, idempotent monoid whose identity is $U_{\text{Ꞁ}}$, that is, a bounded semi-lattice.

Proposition 27 Let (Ꞙ,Ɉ) be an SS over U. Then, (Ꞙ,Ɉ)$\cap_\varepsilon$ $\emptyset_\emptyset$=(Ꞙ,Ɉ) (Ali et al., 2011).

Proof: Ali et al. (2011) presented this property without proof in their study; however, we provide it with its detailed proof. Let $\emptyset_\emptyset$=(S,∅) and (Ꞙ,Ɉ) $\cap_\varepsilon$(S, ∅)=(Ꜧ,Ɉ∪∅), where for all α∈Ɉ∪∅=Ɉ,

$$\text{Ꜧ}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in \text{Ɉ}\setminus\emptyset=\text{Ɉ} \\ S(\alpha) & \alpha\in\emptyset\setminus \text{Ɉ}=\emptyset \\ \text{Ꞙ}(\alpha)\cap S(\alpha) & \alpha\in \text{Ɉ}\cap\emptyset=\emptyset \end{cases}$$

Thus, Ꜧ(α)=Ꞙ(α), for all α∈Ɉ, and (Ꜧ,Ɉ)=(Ꞙ,Ɉ).

Proposition 28 Let (Ꞙ,Ɉ) be an SS over U. Then, $\emptyset_\emptyset$ $\cap_\varepsilon$(Ꞙ,Ɉ)=(Ꞙ,Ɉ) (Ali et al., 2011).

Proof: Ali et al. (2011) presented this property without proof in their study; however, we provide it with its detailed proof. Let $\emptyset_\emptyset$ =(S,∅) and (S,∅) $\cap_\varepsilon$(Ꞙ,Ɉ)=(Ꜧ, ∅∪Ɉ), where for all α∈∅∪Ɉ= Ɉ,

$$\text{Ꜧ}(\alpha)=\begin{cases} S(\alpha) & \alpha\in\emptyset\setminus \text{Ɉ}=\emptyset \\ \text{Ꞙ}(\alpha) & \alpha\in \text{Ɉ}\setminus\emptyset=\text{Ɉ} \\ S(\alpha)\cap \text{Ꞙ}(\alpha) & \alpha\in\emptyset\cap \text{Ɉ}=\emptyset \end{cases}$$

Thus, Ꜧ(α)=Ꞙ(α), for all α∈Ɉ, and (Ꜧ,Ɉ)=(Ꞙ,Ɉ).

By Proposition 27 and Proposition 28, the identity element of $\cap_\varepsilon$ is the SS $\emptyset_\emptyset$ in $S_E(U)$.

In classical set theory, it is well-known that A∪B=∅ if and only if A=∅ and B=∅. By this fact, there does not exist (ℭ, K) ∈ $S_E(U)$ such that (Ꞙ,Ɉ)$\cap_\varepsilon$ (ℭ, K)= (ℭ,K) $\cap_\varepsilon$ (Ꞙ,Ɉ)= $\emptyset_\emptyset$, as this requires Ɉ ∪K=∅, and so Ɉ =∅ and K=∅. Since $\emptyset_\emptyset$ is the only SS with an empty parameter in $S_E(U)$, there is not any element in $S_E(U)$, except the identity element $\emptyset_\emptyset$, which has an inverse with respect to $\cap_\varepsilon$. Of course, the inverse of $\emptyset_\emptyset$, which is the identity element, is itself, as usual.

Proposition 29 Let (Ꞙ,Ɉ) be an SS over U. Then, $\emptyset_E$ $\cap_\varepsilon$(Ꞙ,Ɉ)= (Ꞙ,Ɉ) $\cap_\varepsilon$ $\emptyset_E$ =$\emptyset_E$.

Proof: Let $\emptyset_E$ =(S,E) ve (S,E) $\cap_\varepsilon$(Ꞙ,Ɉ)=(Ꜧ, E∪Ɉ=E), where for all α∈E∪Ɉ=E,

$$\text{Ꜧ}(\alpha)=\begin{cases} S(\alpha), & \alpha\in E\setminus \text{Ɉ}=\text{Ɉ}' \\ \text{Ꞙ}(\alpha), & \alpha\in \text{Ɉ}\setminus E=\emptyset \\ S(\alpha)\cap \text{Ꞙ}(\alpha), & \alpha\in E\cap \text{Ɉ}=\text{Ɉ} \end{cases}$$

Hence, for all α∈E∪Ɉ=E,

$$\begin{cases} \emptyset, & \alpha\in E\setminus \text{Ɉ}=\text{Ɉ}' \end{cases}$$

$$Ꮋ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus E=\emptyset \\ \emptyset, & \alpha\in E\cap Ꭻ=Ꭻ \end{cases}$$

Thus, $Ꮋ(\alpha)=\emptyset$, for all $\alpha\in E$, and $(Ꮋ,Ꭻ)=\emptyset_E$. That is, $\emptyset_E$ is the absorbing element of $\cap_\varepsilon$ in $S_E(U)$.

Theorem 4 $(S_E(U),\cap_\varepsilon)$ is a bounded semi-lattice, whose identity is $\emptyset_\emptyset$ and the absorbing element is $\emptyset_E$.

Proof: By Proposition 21, Proposition 22, Proposition 24, Proposition 25, Proposition 27, Proposition 28, and Proposition 29, $(S_E(U),\cap_\varepsilon)$ is a commutative, idempotent monoid whose identity is $\emptyset_\emptyset$, that is, a bounded semi-lattice with the absorbing element is $\emptyset_E$.

Proposition 30 Let (Ꞙ,Ꭻ) be an SS over U. Then, $(Ꞙ,Ꭻ)\cap_\varepsilon(Ꞙ,Ꭻ)^r=(Ꞙ,Ꭻ)^r\cap_\varepsilon(Ꞙ,Ꭻ)=\emptyset_Ꭻ$ (Sezgin and Atagün, 2011).

Proof: Sezgin and Atagün (2011) presented this property without proof; however, we give it here with proof. The proof is obtained from Note 1 and Proposition 11.

Ali et al. (2009) used the negative complement defined by Maji et al. (2003) for De Morgan's laws for extended intersection and extended union. On the other hand, Qin and Hong (2010) adopted the more commonly used relative complement defined by Ali et al. (2009) for the complement operation and provided De Morgan's laws accordingly. Their proofs were element-based in Qin and Hong (2010), while we present a simpler proof using function equality as follows:

Proposition 31 Let (Ꞙ,Ꭻ) and (ℭ,Ꮭ) be SSs over U. Then, $[(Ꞙ,Ꭻ)\cap_\varepsilon(ℭ,Ꮭ)]^r=(Ꞙ,Ꭻ)^r\cup_\varepsilon(ℭ,Ꮭ)^r$ (De Morgan Law) (Qin and Hong, 2010)

Proof: Let $(Ꞙ,Ꭻ)\cap_\varepsilon(ℭ,Ꮭ)=(Ꮋ,Ꭻ\cup Ꮭ)$, where for all $\alpha\in Ꭻ\cup Ꮭ$,

$$Ꮋ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ꮭ \\ ℭ(\alpha) & \alpha\in Ꮭ\setminus Ꭻ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ꭻ\cap Ꮭ \end{cases}$$

Let $(Ꮋ,Ꭻ\cup Ꮭ)^r=(K,Ꭻ\cup Ꮭ)$, where for all $\alpha\in Ꭻ\cup Ꮭ$,

$$K(\alpha)=\begin{cases} Ꞙ'(\alpha) & \alpha\in Ꭻ\setminus Ꮭ \\ G'(\alpha) & \alpha\in Ꮭ\setminus Ꭻ \\ Ꞙ'(\alpha)\cup G'(\alpha) & \alpha\in Ꭻ\cap Ꮭ \end{cases}$$

Hence, $(K,Ꭻ\cup Ꮭ)=(Ꞙ,Ꭻ)^r\cup_\varepsilon(ℭ,Ꮭ)^r$.

Proposition 32 Let (Ꞙ,Ꭻ) and (ℭ,Ꮭ) be SSs over U. Then, $(Ꞙ,Ꭻ)\cap_\varepsilon(ℭ,Ꮭ)=U_{Ꭻ\cup Ꮭ}$ if and only if $(Ꞙ,Ꭻ)=U_Ꭻ$ and $(ℭ,Ꮭ)=U_Ꮭ$.

Proof: Let $(Ꞙ, Ɉ) \cap_{\varepsilon} (ℭ, Ᵽ)$ =(Ͱ,Ɉ∪Ᵽ),

$$Ͱ(\alpha)=\begin{cases} Ꞙ(\alpha), & \alpha\in Ɉ\backslash Ᵽ \\ ℭ(\alpha), & \alpha\in Ᵽ\backslash Ɉ \\ Ꞙ(\alpha)\cap ℭ(\alpha), & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

Since (Ͱ, Ɉ ∪ Ᵽ)= $U_{Ɉ\cup Ᵽ}$, Ͱ(α)= U, for all α∈Ɉ∪Ᵽ. Hence, if α∈Ɉ\Ᵽ, then Ꞙ(α)=U, if α∈ Ᵽ\Ɉ, then ℭ(α)=U, and if α∈Ɉ∩Ᵽ, then Ꞙ(α)∩ℭ(α)=U, implying that Ꞙ(α)=ℭ(α)=U. Thus, Ꞙ(α)=U, for all α∈Ɉ, and ℭ(α)=U, for all α∈Ᵽ. Hence, $(Ꞙ, Ɉ) = U_{Ɉ}$ and $(ℭ, Ᵽ) = U_{Ᵽ}$.

Conversely let (Ꞙ,Ɉ)= $U_{Ɉ}$ and $(ℭ, Ᵽ) = U_{Ᵽ}$. Then, Ꞙ(α)=U, for all α∈Ɉ, and ℭ(α)=U, for all α∈Ᵽ. Then,

$$Ͱ(\alpha)=\begin{cases} U, & \alpha\in Ɉ\backslash Ᵽ \\ U, & \alpha\in Ᵽ\backslash Ɉ \\ U\cap U, & \alpha\in Ɉ\cap Ᵽ \end{cases}$$

for all α∈Ɉ∪Ᵽ. Therefore, (H, Ɉ∪Ᵽ)=(Ꞙ,Ɉ) $\cap_{\varepsilon}$ (ℭ, Ᵽ)= $U_{Ɉ\cup Ᵽ}$.

Proposition 33 Let (Ꞙ,Ɉ) and (ℭ,Ᵽ) be SSs over U. Then, $\emptyset_{Ɉ} \tilde{\subseteq}(Ꞙ,Ɉ)\cap_{\varepsilon}(ℭ,Ᵽ)$, $\emptyset_{Ᵽ} \tilde{\subseteq}(Ꞙ,Ɉ) \cap_{\varepsilon} (ℭ,Ᵽ)$. Also, $(Ꞙ,Ɉ)\cap_{\varepsilon} (ℭ,Ᵽ) \tilde{\subseteq} U_{Ɉ\cup Ᵽ}$.

Proof: The proof is obvious since the empty set is a subset of every set and the universal set includes every set.

Proposition 34 Let (Ꞙ,Ɉ) and (ℭ,Ɉ) be SSs over U. Then, $(Ꞙ, Ɉ) \cap_{\varepsilon} (ℭ, Ɉ) \tilde{\subseteq} (Ꞙ,Ɉ)$ and $(Ꞙ, Ɉ) \cap_{\varepsilon} (ℭ, Ɉ) \tilde{\subseteq} (ℭ,Ɉ)$.

Proof: The proof is obtained from Note 1 and Proposition 15.

Proposition 35 Let (Ꞙ,Ɉ) and (ℭ, Ɉ) be SSs over U. $(Ꞙ,Ɉ) \tilde{\subseteq} (ℭ, Ɉ)$ if and only if $(Ꞙ,Ɉ) \cap_{\varepsilon} (ℭ, Ɉ)$ =(Ꞙ, Ɉ)**.**

Proof: The proof is obtained from Note 1 and Proposition 16.

Proposition 36 Let (Ꞙ,Ɉ), (ℭ,Ɉ), and (K,V) be SSs over U. If $(Ꞙ,Ɉ) \tilde{\subseteq} (ℭ, Ɉ)$, then $(Ꞙ,Ɉ)\cap_{\varepsilon}(K,V) \tilde{\subseteq}(ℭ,Ɉ)\cap_{\varepsilon}(K,V)$. However, the converse is not true.

Proof: Let $(Ꞙ,Ɉ) \tilde{\subseteq} (ℭ, Ɉ)$. Hence, Ꞙ(α)⊆ℭ(α), for all α∈Ɉ. Let $(Ꞙ,Ɉ)\cap_{\varepsilon}(K,V)$=(Ͱ,Ɉ∪V), where for all α∈ɈUV,

$$\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\backslash V \end{cases}$$

$$Ħ(\alpha)= \begin{cases} K(\alpha) & \alpha\in V\setminus Ʝ \\ Ғ(\alpha)\cap K(\alpha) & \alpha\in Ʝ\cap V \end{cases}$$

Let $(\mathfrak{C},Ʝ)\cap_\varepsilon(K,V)=(S, Ʝ\cup V)$, where for all $\alpha\in Ʝ\cup V$,

$$S(\alpha)=\begin{cases} \mathfrak{C}(\alpha) & \alpha\in Ʝ\setminus V \\ K(\alpha) & \alpha\in V\setminus Ʝ \\ \mathfrak{C}(\alpha)\cap K(\alpha) & \alpha\in Ʝ\cap V \end{cases}$$

If $\alpha\in Ʝ\setminus V$, then $H(\alpha)=Ғ(\alpha)\subseteq Ғ(\alpha)=S(\alpha)$, if $\alpha\in V\setminus Ʝ$, then $H(\alpha)=K(\alpha)\subseteq K(\alpha)=S(\alpha)$, if $\alpha\in Ʝ\cap V$, then $H(\alpha)=Ғ(\alpha)\cap K(\alpha)\subseteq \mathfrak{C}(\alpha)\cap K(\alpha)= S(\alpha)$. Thus, $Ħ(\alpha)\subseteq S(\alpha)$, for all $\alpha\in Ʝ\cup V$, implying that $(Ғ,Ʝ)\cap_\varepsilon(K,V) \tilde{\subseteq}(\mathfrak{C},Ʝ)\cap_\varepsilon(K,V)$.

Let's give an example to show that the converse is not true. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $Ʝ=\{e_1,e_3\}$, $V=\{e_1,e_3,e_5\}$ be the subsets of E and, $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the universel set. Assume that $(Ғ,Ʝ)$, $(\mathfrak{C},Ʝ)$ and (K,V) are the SSs over U defined as follows: $(Ғ,Ʝ)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$,$(\mathfrak{C},Ʝ)=\{(e_1,\{h_2\}),(e_3,\{h_1,h_2\})\}$,$(K,V)=\{ e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$.
It is obvious that $(Ғ,Ʝ)\cap_\varepsilon K,V)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$ and $(\mathfrak{C},Ʝ)\cap_\varepsilon(K,V)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$. Hence, $(Ғ,Ʝ)\cap_\varepsilon(K,V)\tilde{\subseteq}(\mathfrak{C},Ʝ)\cap_\varepsilon(K,V)$; however, $(Ғ,Ʝ)$ is not soft subset of $(\mathfrak{C},Ʝ)$.

Proposition 37 Let $(Ғ,Ʝ)$, $(\mathfrak{C},Ʝ)$, (K,V), and (L,V) be SSs over U. If $(Ғ,Ʝ)\tilde{\subseteq}(\mathfrak{C},Ʝ)$ and $(K,V)\tilde{\subseteq}(L,V)$, then $(Ғ,Ʝ)\cap_\varepsilon(K,V)\tilde{\subseteq}(\mathfrak{C},Ʝ)\cap_\varepsilon(L,V)$.

Proof: Let $(Ғ,Ʝ)\tilde{\subseteq}(\mathfrak{C},Ʝ)$ and $(K,V)\tilde{\subseteq}(L,V)$. Hence, $Ғ(\alpha)\subseteq\mathfrak{C}(\alpha)$, for all $\alpha\in Ʝ$ and $K(\alpha)\subseteq L(\alpha)$, and for all $\alpha\in V$. Let $(Ғ,Ʝ)\cap_\varepsilon(K,V)=(Ħ, Ʝ\cup V)$, where for all $\alpha\in Ʝ\cup V$,

$$Ħ(\alpha)=\begin{cases} Ғ(\alpha) & \alpha\in Ʝ\setminus V \\ K(\alpha) & \alpha\in V\setminus Ʝ \\ Ғ(\alpha)\cap K(\alpha) & \alpha\in Ʝ\cap V \end{cases}$$

Let $(\mathfrak{C},Ʝ)\cap_\varepsilon(L,V)=(S, Ʝ\cup V)$, where for all $\alpha\in Ʝ\cup V$,

$$S(\alpha)=\begin{cases} \mathfrak{C}(\alpha) & \alpha\in Ʝ\setminus V \\ L(\alpha) & \alpha\in V\setminus Ʝ \\ \mathfrak{C}(\alpha)\cap L(\alpha) & \alpha\in Ʝ\cap V \end{cases}$$

If $\alpha\in Ʝ\setminus V$, then $H(\alpha)=Ғ(\alpha)\subseteq\mathfrak{C}(\alpha)=S(\alpha)$, if $\alpha\in V\setminus Ʝ$, then $H(\alpha)=K(\alpha)\subseteq L(\alpha)=S(\alpha)$, and if $\alpha\in\in Ʝ\cap V$, $H(\alpha)=Ғ(\alpha)\cap K(\alpha)\subseteq\mathfrak{C}(\alpha)\cap L(\alpha)= S(\alpha)$. Thus, $H(\alpha)\subseteq S(\alpha)$, for all $\alpha\in Ʝ\cup V$, implying that $(Ғ,Ʝ)\cap_\varepsilon(K,V)\tilde{\subseteq}(\mathfrak{C},Ʝ)\cap_\varepsilon(L,V)$.

Proposition 38 Let (Ꞙ,J) and (ℭ,J) be SSs over U. Then, $(Ꞙ,J) \widetilde{\subseteq} (ℭ,J)^{r}$ if and only if $(Ꞙ,J) \cap_{\varepsilon} (ℭ,J) = \emptyset_{J}$.

Proof: The proof is obtained from Note 1 and Proposition 20.

3.2.1. The distributions of the extended intersection operation over other SS operations:

In this subsection, the distributions of the extended intersection operation over restricted SS operations, extended operations, and soft binary partition operations have been examined.

3.2.1.1. The distributions of the extended intersection operation over restricted SS operations:

Here, the distributions of the extended intersection operation to restricted operations have been examined. First, distributions from the left, followed by distributions from the right, have been investigated. It is worth noting an important point here. In the study by Ali et al. (2011), the distributions of the extended intersection operation over the restricted union from the left were examined without proof, and demonstrated with an example that the extended intersection operation does not satisfy the property of distributions over the restricted intersection from the left. Singh and Onyeozili (2012c) showed that the extended intersection operation does not satisfy the property of distributions over the restricted difference from the left. Sezgin and Atagün (2011), although showed that the extended intersection operation distributes to the restricted union from both the right and the left, they overlooked some points in their proof. In this study, the distributive properties are presented with detailed proofs, considering the cases where the intersection of the parameter sets of the SSs involved in restricted operations is empty as well. Additionally, for those that do not satisfy the distributive property, the conditions under which they do satisfy the distributive property are also provided with detailed proofs.

a) LHS distributions of extended intersection over restricted SS operations:

Let (Ꞙ,J), (ℭ,Ꝕ), and (Ή,Ƨ) be SSs over U. Then, we have the following distributions:

i) $(Ꞙ,J) \cap_{\varepsilon}[(ℭ,Ꝕ) \cup_{R} (Ή,Ƨ)]=[(Ꞙ,J) \cap_{\varepsilon} (ℭ,Ꝕ)]\cup_{R}[(Ꞙ,J) \cap_{\varepsilon}(Ή,Ƨ)]$ (Sezgin and Atagün, 2011).

Proof: In the proof by Sezgin and Atagün (2011), a case in the parameter partitioning on the right-hand side was overlooked. Moreover, the proof emphasized that the intersection of the parameter sets of the SSs involved in restricted operations must be non-empty. However, even if the intersection of the parameter sets of the SSs involved in restricted operations is empty, this distributive property still holds. Therefore, in our proof, these cases are specifically considered and addressed by taking these situations into consideration.

First, let's consider the LHS, and let $(ℭ,Ꝕ) \cup_{R}(Ή,Ƨ)=(S,Ꝕ\capƧ)$. Hence, for all $\alpha\inꝔ\capƧ$, $S(\alpha)=ℭ(\alpha)\cupΉ(\alpha)$. Let $(Ꞙ,J)\cap_{\varepsilon}(S,Ꝕ\capƧ)=(\dot{N},J\cup(Ꝕ\capƧ))$, where for all $\alpha\in J\cup(Ꝕ\capƧ)$,

$$\dot{N}(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in J\setminus(Ꝕ\capƧ) \\ S(\alpha) & \alpha\in(Ꝕ\capƧ)\setminus J \\ Ꞙ(\alpha)\cap S(\alpha) & \alpha\in J\cap(Ꝕ\capƧ) \end{cases}$$

Hence,

$$\text{Ǹ}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in \text{Ʝ}\setminus(\text{Ᵽ}\cap\text{Ƨ}) \\ \mathfrak{C}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in(\text{Ᵽ}\cap\text{Ƨ})\setminus\text{Ʝ}=\text{Ʝ}'\cap\text{Ᵽ}\cap\text{Ƨ} \\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cup\text{Ή}(\alpha)] & \alpha\in\text{Ʝ}\cap(\text{Ᵽ}\cap\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ} \end{cases}$$

Now let's handle the RHS, and let $(\text{Ꞙ},\text{Ʝ})\cap_{\varepsilon}(\mathfrak{C},\text{Ᵽ})=(W,\text{Ʝ}\cup\text{Ᵽ})$, where for all $\alpha\in\text{Ʝ}\cup\text{Ᵽ}$,

$$W(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\setminus\text{Ᵽ} \\ \mathfrak{C}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ} \end{cases}$$

Let $(\text{Ꞙ},\text{Ʝ})\cap_{\varepsilon}(\text{Ή},\text{Ƨ})=(K,\text{Ʝ}\cup\text{Ƨ})$, where for all $\alpha\in\text{Ʝ}\cup\text{Ƨ}$,

$$K(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\setminus\text{Ƨ} \\ \text{Ή}(\alpha) & \alpha\in\text{Ƨ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ƨ} \end{cases}$$

Let $(W,\text{Ʝ}\cup\text{Ᵽ})\cup_{R}(K,\text{Ʝ}\cup\text{Ƨ})=(Y,(\text{Ʝ}\cup\text{Ᵽ})\cap(\text{Ʝ}\cup\text{Ƨ}))$, where for all $\alpha\in(\text{Ʝ}\cup\text{Ᵽ})\cap(\text{Ʝ}\cup\text{Ƨ})$, $Y(\alpha)=W(\alpha)\cup K(\alpha)$. Thereby,

$$Y(\alpha)=\begin{cases} \text{Ꞙ}(\alpha)\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ʝ}\setminus\text{Ᵽ})\cap(\text{Ʝ}\setminus\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\ \text{Ꞙ}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in(\text{Ʝ}\setminus\text{Ᵽ})\cap(\text{Ƨ}\setminus\text{Ʝ})=\emptyset \\ \text{Ꞙ}(\alpha)\cup[\text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in(\text{Ʝ}\setminus\text{Ᵽ})\cap(\text{Ʝ}\cap\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ} \\ \mathfrak{C}(\alpha)\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ʝ})\cap(\text{Ʝ}\setminus\text{Ƨ})=\emptyset \\ \mathfrak{C}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ʝ})\cap(\text{Ƨ}\setminus\text{Ʝ})=\text{Ʝ}'\cap\text{Ᵽ}\cap\text{Ƨ} \\ \mathfrak{C}(\alpha)\cup[\text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in(\text{Ᵽ}\setminus\text{Ʝ})\cap(\text{Ʝ}\cap\text{Ƨ})=\emptyset \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\cap(\text{Ʝ}\setminus\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup\text{Ή}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\cap(\text{Ƨ}\setminus\text{Ʝ})=\emptyset \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup[\text{Ꞙ}(\alpha)\cap\text{Ή}(\alpha)] & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\cap(\text{Ʝ}\cap\text{Ƨ})=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ} \end{cases}$$

Thus,

$$Y(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ} \\ \mathfrak{C}(\alpha)\cup\text{Ή}(\alpha) & \alpha\in\text{Ʝ}'\cap\text{Ᵽ}\cap\text{Ƨ} \\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cup\text{Ή}(\alpha)] & \alpha\in\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ} \end{cases}$$

Here, let’s consider Ʝ\(Ᵽ∩Ꝣ) in the function N. Since Ʝ\(Ᵽ∩Ꝣ)=Ʝ∩(Ᵽ∩Ꝣ)' and if an element is in the complement of (Ᵽ∩Ꝣ), it is either in Ᵽ\Ꝣ, in Ꝣ\Ᵽ or in the complement of Ᵽ∪Ꝣ, thus if α∈Ʝ\(Ᵽ∩Ꝣ), then α∈Ʝ∩Ᵽ∩Ꝣ' or α∈Ʝ∩Ᵽ'∩Ꝣ or α∈Ʝ∩Ᵽ'∩Ꝣ'. Therefore, Ṅ=Y.

Here, if Ᵽ∩Ꝣ=∅, then Ṅ(α)=W(α)=Ꞙ(α), and thus equality is satisfied again. Similarly, when (Ʝ∪Ᵽ)∩(Ʝ∪Ꝣ)=Ʝ∪(Ᵽ∩Ꝣ)=∅, i.e. Ʝ=∅ and Ᵽ∩Ꝣ=∅, then (Ṅ,Ʝ∪(Ᵽ∩Ꝣ))=(Y,(Ʝ∪Ᵽ)∩(Ʝ∪Ꝣ))=$\emptyset_{\emptyset}$. Therefore, there is no need to require these sets to be different from the empty set.

ii) (Ꞙ,Ʝ)$\cap_{\varepsilon}$ [(ℭ,Ᵽ) $\cap_{R}$(Ⱶ,Ꝣ)]≠[(Ꞙ,Ʝ)$\cap_{\varepsilon}$ (ℭ,Ᵽ)]$\cap_{R}$[(Ꞙ,Ʝ) $\cap_{\varepsilon}$(Ⱶ,Ꝣ)] (Ali et al., 2011), however
(Ꞙ,Ʝ) $\cap_{\varepsilon}$[(ℭ,Ᵽ) $\cap_{R}$ (Ⱶ,Ꝣ)]=[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)]$\cap_{R}$[(Ꞙ,Ʝ) $\cap_{\varepsilon}$(Ⱶ,Ꝣ)], where Ʝ∩(ᲀΔ Ꝣ)=∅.

Proof: Since the proof of the left distributive property of the extended intersection over the restricted intersection is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the union operation. However, for extended intersection and restricted union operations, this situation does not hold, as shown by Ali et al. (2011) with a counter-example. In this study, we show that the distributivity can be achieved under the condition Ʝ∩(ᲀΔ Ꝣ)=∅.

In classical sets, the intersection operation is left-distributive over both the difference and the symmetric difference operations. However, for extended intersection and restricted difference and restricted symmetric difference operations, this situation does not hold, as shown below:

iii) (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ)$\setminus_{R}$(Ⱶ,Ꝣ)]≠[(Ꞙ,Ʝ)$\cap_{\varepsilon}$ (ℭ,Ᵽ)]$\setminus_{R}$[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(Ⱶ,Ꝣ)] (Singh and Onyeozili, 2012c).

iv) (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ)Δ_{R}(Ⱶ,Ꝣ)]≠[(Ꞙ,Ʝ)$\cap_{\varepsilon}$ (ℭ,Ᵽ)]Δ_{R}[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(Ⱶ,Ꝣ)].

b) RHS distributions of extended intersection over restricted SS operations:
Let (Ꞙ,Ʝ), (ℭ,Ᵽ), and (Ⱶ,Ꝣ) be SSs over U. Then, we have the following distributions:

i) [(Ꞙ,Ʝ) $\cup_{R}$(ℭ,Ᵽ)]$\cap_{\varepsilon}$(Ⱶ,Ꝣ)= [(Ꞙ,Ʝ)$\cap_{\varepsilon}$(Ⱶ,Ꝣ)] $\cup_{R}$ [(ℭ,Ᵽ) $\cap_{\varepsilon}$ (Ⱶ,Ꝣ)] (Sezgin and Atagün, 2011).

Proof: Sezgin and Atagün (2011) presented this property without proof in their study; however, we provide it with its detailed proof. First, let’s consider the LHS and let (Ꞙ,Ʝ) $\cup_{R}$(ℭ,Ᵽ)=(R,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, R(α)=Ꞙ(α)∪ℭ(α). Let (R,Ʝ∩Ᵽ) $\cap_{\varepsilon}$(Ⱶ,Ꝣ)=(L,(Ʝ∩Ᵽ)∪Ꝣ), where for all α∈(Ʝ∩Ᵽ)∪Ꝣ,

$$L(\alpha)=\begin{cases} R(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\setminus\text{Ꝣ} \\ \text{Ⱶ}(\alpha) & \alpha\in\text{Ꝣ}\setminus(\text{Ʝ}\cap\text{Ᵽ}) \\ R(\alpha)\cap\text{Ⱶ}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\cap\text{Ꝣ} \end{cases}$$

Hence,

$$\begin{cases} \text{Ꞙ}(\alpha)\cup\text{ℭ}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\setminus\text{Ꝣ}=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ꝣ}' \\ & \\ & \end{cases}$$

$$L(\alpha)=\begin{cases} Ⱨ(\alpha) & \alpha\in Ƨ\setminus(Ɉ\cap Ᵽ) \\ [Ꞙ(\alpha)\cup ℭ(\alpha)]\cap Ⱨ(\alpha) & \alpha\in(Ɉ\cap Ᵽ)\cap Ƨ=Ɉ\cap Ᵽ\cap Ƨ \end{cases}$$

Now let's handle the RHS, and let $(Ꞙ,Ɉ)\cap_{\varepsilon}(Ⱨ,Ƨ)=(S,Ɉ\cup Ƨ)$, where for all $\alpha\in Ɉ\cup Ƨ$,

$$S(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ɉ\setminus Ƨ \\ Ⱨ(\alpha) & \alpha\in Ƨ\setminus Ɉ \\ Ꞙ(\alpha)\cap Ⱨ(\alpha) & \alpha\in Ɉ\cap Ƨ \end{cases}$$

Let $(ℭ,Ᵽ)\cap_{\varepsilon}(Ⱨ,Ƨ)=(K,Ᵽ\cup Ƨ)$, where for all $\alpha\in Ᵽ\cup Ƨ$,

$$K(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ᵽ\setminus Ƨ \\ Ⱨ(\alpha) & \alpha\in Ƨ\setminus Ᵽ \\ ℭ(\alpha)\cap Ⱨ(\alpha) & \alpha\in Ᵽ\cap Ƨ \end{cases}$$

Let $(S,Ɉ\cup Ƨ)\cup_R(K,Ᵽ\cup Ƨ)=(W,(Ɉ\cup Ƨ)\cap(Ᵽ\cup Ƨ))$, where for all $\alpha\in(Ɉ\cup Ƨ)\cap(Ᵽ\cup Ƨ)$, $W(\alpha)=S(\alpha)\cup K(\alpha)$. Thus,

$$W(\alpha)=\begin{cases} Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in(Ɉ\setminus Ƨ)\cap(Ᵽ\setminus Ƨ)=Ɉ\cap Ᵽ\cap Ƨ' \\ Ꞙ(\alpha)\cup Ⱨ(\alpha) & \alpha\in(Ɉ\setminus Ƨ)\cap(Ƨ\setminus Ᵽ)=\emptyset \\ Ꞙ(\alpha)\cup[ℭ(\alpha)\cap Ⱨ(\alpha)] & \alpha\in(Ɉ\setminus Ƨ)\cap(Ᵽ\cap Ƨ)=\emptyset \\ Ⱨ(\alpha)\cup ℭ(\alpha) & \alpha\in(Ƨ\setminus Ɉ)\cap(Ᵽ\setminus Ƨ)=\emptyset \\ Ⱨ(\alpha)\cup Ⱨ(\alpha) & \alpha\in(Ƨ\setminus Ɉ)\cap(Ƨ\setminus Ᵽ)=Ɉ'\cap Ᵽ'\cap Ƨ \\ Ⱨ(\alpha)\cup[ℭ(\alpha)\cap Ⱨ(\alpha)] & \alpha\in(Ƨ\setminus Ɉ)\cap(Ᵽ\cap Ƨ)=Ɉ'\cap Ᵽ\cap Ƨ \\ [Ꞙ(\alpha)\cap Ⱨ(\alpha)]\cup ℭ(\alpha) & \alpha\in(Ɉ\cap Ƨ)\cap(Ᵽ\setminus Ƨ)=\emptyset \\ [Ꞙ(\alpha)\cap Ⱨ(\alpha)]\cup Ⱨ(\alpha) & \alpha\in(Ɉ\cap Ƨ)\cap(Ƨ\setminus Ᵽ)=Ɉ\cap Ᵽ'\cap Ƨ \\ [Ꞙ(\alpha)\cap Ⱨ(\alpha)]\cup[ℭ(\alpha)\cap Ⱨ(\alpha)] & \alpha\in(Ɉ\cap Ƨ)\cap(Ᵽ\cap Ƨ)=Ɉ\cap Ᵽ\cap Ƨ \end{cases}$$

Thus,

$$W(\alpha)=\begin{cases} Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in Ɉ\cap Ᵽ\cap Ƨ' \\ Ⱨ(\alpha) & \alpha\in Ɉ'\cap Ᵽ'\cap Ƨ \\ Ⱨ(\alpha) & \alpha\in Ɉ'\cap Ᵽ\cap Ƨ \\ Ⱨ(\alpha) & \alpha\in Ɉ\cap Ᵽ'\cap Ƨ \\ [Ꞙ(\alpha)\cap Ⱨ(\alpha)]\cup[ℭ(\alpha)\cap Ⱨ(\alpha)] & \alpha\in Ɉ\cap Ᵽ\cap Ƨ \end{cases}$$

Here, let's consider $Ƨ\setminus(Ɉ\cap Ᵽ)$ in L. Since $Ƨ\setminus(Ɉ\cap Ᵽ)=Ƨ\cap(Ɉ\cap Ᵽ)'$, if an element is in the complement of $(Ɉ\cap Ᵽ)$, it is either in $Ɉ\setminus Ᵽ$, either in $Ᵽ\setminus Ɉ$ or in the complement of $Ɉ\cup$ Z. Thus, if $\alpha\in Ƨ\setminus(Ɉ\cap Ᵽ)$, then $\alpha\in Ƨ\cap Ɉ\cap Ᵽ'$ or $\alpha\in Ƨ\cap Ᵽ\cap Ɉ'$ or $\alpha\in Ƨ\cap Ɉ'\cap Ᵽ'$. Hence, L=W.

Here, if Ꭻ∩Ᵽ=∅, then the equality will still be satisfied, since L(α)=W(α)=Ԋ(α). Similarly, if (Ꭻ∪Ƨ)∩(Ᵽ∪Ƨ)=(Ꭻ∩Ᵽ)∪Ƨ=∅, that is, Ꭻ∩Ᵽ=∅ and Ƨ=∅, then (L,(Ꭻ∩Ᵽ)∪Ƨ)=(W,(Ꭻ∪Ƨ)∩(Ᵽ∪Ƨ))= $\emptyset_{\emptyset}$. That is, in the theorem, there is no need to require these sets to be different from the empty.

ii) [(Ꞙ,Ꭻ) $\cap_R$(ℭ,Ᵽ)]$\cap_\varepsilon$(Ԋ,Ƨ)=[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ԋ,Ƨ)]$\cap_R$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ԋ,Ƨ)], where (ᎫΔᵽ)∩Ƨ=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the restricted intersection is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the intersection operation. However, for extended and restricted operations, this situation does not hold, but we state that the distributivity can be satisfied under the condition of (ᎫΔᵽ)∩Ƨ=∅.

Similarly, in classical sets, the intersection operation is right-distributive over both the difference and the symmetric difference operations. However, for extended intersection and restricted difference and restricted symmetric difference operations, this situation does not hold, as given below:

iii) [(Ꞙ,Ꭻ) $\backslash_R$(ℭ,Ᵽ)]$\cap_\varepsilon$(Ԋ, Ƨ) ≠[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ԋ,Ƨ)]$\backslash_R$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ԋ,Ƨ)].

iv) [(Ꞙ,Ꭻ) Δ_R(ℭ,Ᵽ)]$\cap_\varepsilon$(Ԋ, Ƨ) ≠[(Ꞙ,Ꭻ)$\cap_\varepsilon$(Ԋ,Ƨ)]Δ_R [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ԋ,Ƨ)].

3.2.1.2. The distributions of the extended intersection operation over other extended SS operations:

Here, the distributive properties of the extended intersection operation over other extended operations are examined. First, left distributivity is considered, followed by right distributivity. It is important to note the following: In the study by Ali et al. (2011), the left distributive property of the extended intersection over the extended intersection was considered without proof, and it was shown with an example that the extended intersection does not satisfy the right distributive property over the extended union. In this study, these distributive properties are presented with detailed proofs. For those that do not satisfy the distributive property unconditionally, the conditions under which they do satisfy the distributive property are also proven.

a) LHS distributions of extended intersection over other extended SS operations:

Let (Ꞙ,Ꭻ), (ℭ,Ᵽ), and (Ԋ,Ƨ) be SSs over U. Then, we have the following distributions:

i) (Ꞙ,Ꭻ) $\cap_\varepsilon$[(ℭ,Ᵽ) $\cap_\varepsilon$(Ԋ,Ƨ)]=[(Ꞙ,Ꭻ) $\cap_\varepsilon$(ℭ,Ᵽ)]$\cap_\varepsilon$[(Ꞙ,Ꭻ)$\cap_\varepsilon$ (Ԋ,Ƨ)] (Ali et al., 2011).

Proof: Ali et al. (2011) presented this property without proof in their study; however, we provide it with its detailed proof. First, let's consider the LHS, and let (ℭ,Ᵽ) $\cap_\varepsilon$(Ԋ,Ƨ)=(R,Ᵽ∪Ƨ), where for all α∈Ᵽ∪Ƨ,

$$R(\alpha)=\begin{cases} ℭ(\alpha) & \alpha\in Ᵽ\backslash Ƨ \\ Ԋ(\alpha) & \alpha\in Ƨ\backslash Ᵽ \\ ℭ(\alpha)\cap Ԋ(\alpha) & \alpha\in Ᵽ\cap Ƨ \end{cases}$$

Let (Ꞙ,Ꭻ) $\cap_{\varepsilon}$(R,ⱣUƧ) =(Ǹ,(ꞸU(ⱣUƧ)), where for all α∈ꞸU(ⱣUƧ),

$$Ǹ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\backslash(Ᵽ\cup Ƨ) \\ R(\alpha) & \alpha\in(Ᵽ\cup Ƨ)\backslash Ꭻ \\ Ꞙ(\alpha)\cap R(\alpha) & \alpha\in Ꭻ\cap(Ᵽ\cup Ƨ) \end{cases}$$

Hence,

$$Ǹ(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\backslash(Ᵽ\cup Ƨ)=Ꭻ\cap Ᵽ'\cap Ƨ' \\ ℭ(\alpha) & \alpha\in(Ᵽ\backslash Ƨ)\backslash Ꭻ=Ꭻ'\cap Ᵽ\cap Ƨ' \\ Ꜧ(\alpha) & \alpha\in(Ƨ\backslash Ᵽ)\backslash Ꭻ=Ꭻ'\cap Ᵽ'\cap Ƨ \\ ℭ(\alpha)\cap Ꜧ(\alpha) & \alpha\in(Ᵽ\cap Ƨ)\backslash Ꭻ=Ꭻ'\cap Ᵽ\cap Ƨ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ꭻ\cap(Ᵽ\backslash Ƨ)=Ꭻ\cap Ᵽ\cap Ƨ' \\ Ꞙ(\alpha)\cap Ꜧ(\alpha) & \alpha\in Ꭻ\cap(Ƨ\backslash Ᵽ)=Ꭻ\cap Ᵽ'\cap Ƨ \\ Ꞙ(\alpha)\cap[ℭ(\alpha)\cap Ꜧ(\alpha)] & \alpha\in Ꭻ\cap(Ᵽ\cap Ƨ)=Ꭻ\cap Ᵽ\cap Ƨ \end{cases}$$

Now let's handle the RHS and let (Ꞙ,Ꭻ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)=(K,ꞸUꝐ), where for all α∈ꞸUꝐ,

$$K(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\backslash Ᵽ \\ ℭ(\alpha) & \alpha\in Ᵽ\backslash Ꭻ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ꭻ\cap Ᵽ \end{cases}$$

Let (Ꞙ,Ꭻ) $\cap_{\varepsilon}$(Ꜧ,Ƨ)=(S,ꞸUƧ), where for all α∈ꞸUƧ,

$$S(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\backslash Ƨ \\ Ꜧ(\alpha) & \alpha\in Ƨ\backslash Ꭻ \\ Ꞙ(\alpha)\cap Ꜧ(\alpha) & \alpha\in Ꭻ\cap Ƨ \end{cases}$$

Let (K,ꞸUꝐ) $\cap_{\varepsilon}$(S,ꞸUƧ)=(L,(ꞸUꝐ)U(ꞸUƧ)), where for all α∈(ꞸUꝐ)U(ꞸUƧ),

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(Ꭻ\cup Ꝑ)\backslash(Ꭻ\cup Ƨ) \\ S(\alpha) & \alpha\in(Ꭻ\cup Ƨ)\backslash(Ꭻ\cup Ꝑ) \\ K(\alpha)\cap S(\alpha) & \alpha\in(Ꭻ\cup Ꝑ)\cap(Ꭻ\cup Ƨ) \end{cases}$$

Hence,

$$
L(\alpha)=\begin{cases}
\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\setminus(\text{Ɉ}\cup\text{Ƨ})=\emptyset \\
\text{ℭ}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\setminus(\text{Ɉ}\cup\text{Ƨ})=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}' \\
\text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\setminus(\text{Ɉ}\cup\text{Ƨ})=\emptyset \\
\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ƨ})\setminus(\text{Ɉ}\cup\text{Ᵽ})=\emptyset \\
\text{Ⱨ}(\alpha) & \alpha\in(\text{Ƨ}\setminus\text{Ɉ})\setminus(\text{Ɉ}\cup\text{Ᵽ})=\text{Ɉ}'\cap\text{Ᵽ}'\cap\text{Ƨ} \\
\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ƨ})\setminus(\text{Ɉ}\cup\text{Ᵽ})=\emptyset \\
\text{Ꞙ}(\alpha)\cap\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap(\text{Ɉ}\setminus\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\
\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap(\text{Ƨ}\setminus\text{Ɉ})=\emptyset \\
\text{Ꞙ}(\alpha)\cap[\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha)] & \alpha\in(\text{Ɉ}\setminus\text{Ᵽ})\cap(\text{Ɉ}\cap\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ} \\
\text{ℭ}(\alpha)\cap\text{Ꞙ}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap(\text{Ɉ}\setminus\text{Ƨ})=\emptyset \\
\text{ℭ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap(\text{Ƨ}\setminus\text{Ɉ})=\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ} \\
\text{ℭ}(\alpha)\cap[\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha)] & \alpha\in(\text{Ᵽ}\setminus\text{Ɉ})\cap(\text{Ɉ}\cap\text{Ƨ})=\emptyset \\
[\text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha)]\cap\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ɉ}\setminus\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\
[\text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha)]\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ƨ}\setminus\text{Ɉ})=\emptyset \\
[\text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha)]\cap[\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha)] & \alpha\in(\text{Ɉ}\cap\text{Ᵽ})\cap(\text{Ɉ}\cap\text{Ƨ})=\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}
\end{cases}
$$

Hence,

$$
L(\alpha)=\begin{cases}
\text{ℭ}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ}' \\
\text{Ⱨ}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{Ᵽ}'\cap\text{Ƨ} \\
\text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\
\text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}'\cap\text{Ƨ} \\
\text{ℭ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{Ᵽ}\cap\text{Ƨ} \\
\text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\
\text{Ꞙ}(\alpha)\cap[\text{ℭ}(\alpha)\cap H(\alpha)] & \alpha\in\text{Ɉ}\cap\text{Ᵽ}\cap\text{Ƨ}
\end{cases}
$$

It is seen that Ǹ=L.

ii) (Ꞙ,Ɉ) $\cap_\varepsilon$[(ℭ,Ᵽ) $\cup_\varepsilon$(Ⱨ,Ƨ)]≠[(Ꞙ,Ɉ) $\cap_\varepsilon$(ℭ,Ᵽ)] $\cup_\varepsilon$[(Ꞙ,Ɉ) $\cap_\varepsilon$ (Ⱨ,Ƨ)] (Ali et al., 2011), and (Ꞙ,Ɉ) $\cap_\varepsilon$[(ℭ,Ᵽ) $\cup_\varepsilon$ (Ⱨ,Ƨ)] = [(Ꞙ,Ɉ) $\cap_\varepsilon$(ℭ,Ᵽ)] $\cup_\varepsilon$[(Ꞙ,Ɉ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ɉ∩(ᵽΔ Ƨ)=∅.

Proof: Since the proof of the left distributive property of the extended intersection over the extended union is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the union operation. However, for extended intersection and extended union operations, this situation does not hold, as shown by Ali et al. (2011) with an example. In this study, we show that distributivity can be satisfied under the condition Ɉ∩(ZΔM)=∅.

iii) If Ʝ∩Ᵽ’∩Ƨ’=Ʝ∩(ᵽΔ Ƨ)=∅, then (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ) $\backslash_{\varepsilon}$(Ħ,Ƨ)]=[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] $\backslash_{\varepsilon}$[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ħ,Ƨ)].

Proof: Since the proof of the left distributive property of the extended intersection over the extended difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the difference operation. However, for extended intersection and extended difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition Ʝ∩Ᵽ’∩Ƨ’=Ʝ∩(ᵽΔ Ƨ)=∅.

iv) If Ʝ∩Ᵽ’∩Ƨ’=Ʝ∩(ᵽΔ Ƨ)=∅, then (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ) Δ_{ε}(Ħ,Ƨ)]=[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] Δ_{ε}[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ħ,Ƨ)].

Proof: Since the proof of the left distributive property of the extended intersection over the extended symmetric difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is left-distributive over the symmetric difference operation. However, for extended intersection and extended symmetric difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition Ʝ∩Ᵽ’∩Ƨ’=Ʝ∩(ᵽΔ Ƨ)=∅.

v) If Ʝ∩Ᵽ’∩Ƨ’=Ʝ∩(ZΔM)=∅, then (Ꞙ,Ʝ)$\cap_{\varepsilon}$[(ℭ,Ᵽ) γ_{ε}(Ħ,Ƨ)]=[(Ꞙ,Ʝ)$\cap_{\varepsilon}$(ℭ,Ᵽ)] γ_{ε}[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (Ħ,Ƨ)].

b) RHS distributions of extended intersection operation over other extended SS operations:
Let (Ꞙ,Ʝ) and (ℭ,Ᵽ) be SSs over U. Then, we have the following distributions:

i)[(Ꞙ,Ʝ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)] $\cap_{\varepsilon}$(Ħ,Ƨ)= [(Ꞙ,Ʝ) $\cap_{\varepsilon}$(Ħ,Ƨ)] $\cap_{\varepsilon}$[(ℭ,Ᵽ) $\cap_{\varepsilon}$ (Ħ,Ƨ)].

Proof: First, let’s consider the LHS, and let (Ꞙ,Ʝ) $\cap_{\varepsilon}$ (ℭ,Ᵽ)=(R,Ʝ∪Ᵽ), where for all α∈Ʝ∪Ᵽ,

$$R(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\backslash\text{Ᵽ} \\ \text{ℭ}(\alpha) & \alpha\in\text{Ᵽ}\backslash\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ} \end{cases}$$

Let (R,Ʝ∪Ᵽ)$\cap_{\varepsilon}$(Ħ,Ƨ) =(Ṅ,(Ʝ∪Ᵽ)∪Ƨ), where for all α∈(Ʝ∪Ᵽ)∪Ƨ,

$$\dot{N}(\alpha)=\begin{cases} R(\alpha) & \alpha\in(\text{Ʝ}\cup\text{Ᵽ})\backslash\text{Ƨ} \\ \text{Ħ}(\alpha) & \alpha\in\text{Ƨ}\backslash(\text{Ʝ}\cup\text{Ᵽ}) \\ R(\alpha)\cap\text{Ħ}(\alpha) & \alpha\in(\text{Ʝ}\cup\text{Ᵽ})\cap\text{Ƨ} \end{cases}$$

Hence,

$$\dot{N}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in(\text{Ʝ}\backslash\text{Ᵽ})\backslash\text{Ƨ}=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\ \text{ℭ}(\alpha) & \alpha\in(\text{Ᵽ}\backslash\text{Ʝ})\backslash\text{Ƨ}=\text{Ʝ}'\cap\text{Ᵽ}\cap\text{Ƨ}' \\ \text{Ꞙ}(\alpha)\cap\text{ℭ}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\backslash\text{Ƨ}=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\ \text{Ħ}(\alpha) & \alpha\in\text{Ƨ}\backslash(\text{Ʝ}\cup\text{Ᵽ})=\text{Ʝ}'\cap\text{Ᵽ}'\cap\text{Ƨ} \end{cases}$$

Ꞙ(α)∩Ἡ(α) α∈(Ʝ\Ᵽ)∩Ƨ=Ʝ∩Ᵽ'∩Ƨ
ℭ(α)∩Ἡ(α) α∈(Ᵽ\Ʝ)∩Ƨ =Ʝ'∩Ᵽ∩Ƨ
[Ꞙ(α)∩ℭ(α)]∩Ἡ(α) α∈(Ʝ∩Ᵽ)∩Ƨ=Ʝ∩Ᵽ∩Ƨ

Now let's handle the RHS, and let (Ꞙ,Ʝ) ∩$_{\varepsilon}$(Ἡ,Ƨ)=(K,Ʝ∪Ƨ), where for all α∈Ʝ∪Ƨ,

K(α)= {
Ꞙ(α) α∈Ʝ\Ƨ
Ἡ(α) α∈Ƨ\Ʝ
Ꞙ(α)∩Ἡ(α) α∈Ʝ∩Ƨ

Let (ℭ,Ᵽ) ∩$_{\varepsilon}$(Ἡ,Ƨ)=(S,Ᵽ∪Ƨ), where for all α∈Ᵽ∪Ƨ,

S(α)= {
ℭ(α) α∈Ᵽ\Ƨ
Ἡ(α) α∈Ƨ\Ᵽ
ℭ(α)∩Ἡ(α) α∈Ᵽ∩Ƨ

Let (K,Ʝ∪Ƨ)∩$_{\varepsilon}$(S,Ᵽ∪Ƨ)=(L,(Ʝ∪Ƨ)∪(Ᵽ∪Ƨ)), where for all α∈(Ʝ∪Ƨ)∪(Ᵽ∪Ƨ),

L(α)= {
K(α) α∈(Ʝ∪Ƨ)\(Ᵽ∪Ƨ)
S(α) α∈(Ᵽ∪Ƨ)\(Ʝ∪Ƨ)
K(α)∩S(α) α∈(Ʝ∪Ƨ)∩(Ᵽ∪Ƨ)

Hence,

L(α)= {
Ꞙ(α) α∈(Ʝ\Ƨ)\(Ᵽ∪Ƨ)=Ʝ∩Ᵽ'∩Ƨ'
Ἡ(α) α∈(Ƨ\Ʝ)\(Ᵽ∪Ƨ)=∅
Ꞙ(α)∩Ἡ(α) α∈(Ʝ∩Ƨ)\(Ᵽ∪Ƨ)=∅
ℭ(α) α∈(Ᵽ\Ƨ)\(Ʝ∪Ƨ)=Ʝ'∩Ᵽ∩Ƨ'
Ἡ(α) α∈(Ƨ\Ᵽ)\(Ʝ∪Ƨ)=∅
ℭ(α)∩Ἡ(α) α∈(Ᵽ∩Ƨ)\(Ʝ∪Ƨ)=∅
Ꞙ(α)∩ℭ(α) α∈(Ʝ\Ƨ)∩(Ᵽ\Ƨ)=Ʝ∩Ᵽ∩Ƨ'
Ꞙ(α)∩Ἡ(α) α∈(Ʝ\Ƨ)∩(Ƨ\Ᵽ)=∅
Ꞙ(α)∩[ℭ(α)∩Ἡ(α)] α∈(Ʝ\Ƨ)∩(Ᵽ∩Ƨ)=∅
Ἡ(α)∩ℭ(α) α∈(Ƨ\Ʝ)∩(Ᵽ\Ƨ)=∅
Ἡ(α)∩Ἡ(α) α∈(Ƨ\Ʝ)∩(Ƨ\Ᵽ)= Ʝ'∩Ᵽ'∩Ƨ
Ἡ(α)∩[ℭ(α)∩Ἡ(α)] α∈(Ƨ\Ʝ)∩(Ᵽ∩Ƨ)=Ʝ'∩Ᵽ∩Ƨ
[Ꞙ(α)∩Ἡ(α)]∩ℭ(α) α∈(Ʝ∩Ƨ)∩(Ᵽ\Ƨ)=∅
[Ꞙ(α)∩Ἡ(α)]∩Ἡ(α) α∈(Ʝ∩Ƨ)∩(Ƨ\Ᵽ)=Ʝ∩Ᵽ'∩Ƨ
[Ꞙ(α)∩Ἡ(α)]∩[ℭ(α)∩Ἡ(α)] α∈(Ʝ∩Ƨ)∩(Ᵽ∩Ƨ)=Ʝ∩Ᵽ∩Ƨ

Thus,

$$
L(\alpha)=\begin{cases}
Ғ(\alpha) & \alpha\in J\cap Ҏ'\cap Ƨ' \\
\mathfrak{C}(\alpha) & \alpha\in J'\cap Ҏ\cap Ƨ' \\
Ғ(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in J\cap Ҏ\cap Ƨ' \\
Ħ(\alpha) & \alpha\in J'\cap Ҏ'\cap Ƨ \\
Ħ(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in J'\cap Ҏ\cap Ƨ \\
Ғ(\alpha)\cap Ħ(\alpha) & \alpha\in J\cap Ҏ'\cap Ƨ \\
[Ғ(\alpha)\cap Ħ(\alpha)]\cap[\mathfrak{C}(\alpha)\cap Ħ(\alpha)] & \alpha\in J\cap Ҏ\cap Ƨ
\end{cases}
$$

It is seen that Ǹ=L.

ii) [(Ғ,J) $\cup_\varepsilon$ ($\mathfrak{C}$,Ҏ)] $\cap_\varepsilon$(Ħ,Ƨ)= [(Ғ,J) $\cap_\varepsilon$(Ħ,Ƨ)] $\cup_\varepsilon$ [($\mathfrak{C}$,Ҏ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (J△Ҏ)∩Ƨ=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the extended union is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the union operation. However, for extended intersection and extended union operations, this situation does not hold. We state that distributivity can be satisfied under the condition (J△Ҏ)∩Ƨ=∅.

iii) [(Ғ,J) $\backslash_\varepsilon$($\mathfrak{C}$,Ҏ)] $\cap_\varepsilon$(Ħ,Ƨ)=[(Ғ,J) $\cap_\varepsilon$(Ħ,Ƨ)]$\backslash_\varepsilon$ [($\mathfrak{C}$,Ҏ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (J△Ҏ)∩Ƨ=J∩Ҏ'∩Ƨ'=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the extended difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the difference operation. However, for extended intersection and extended difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition (J△Ҏ)∩Ƨ=J∩Ҏ'∩Ƨ'=∅.

iv) [(Ғ,J) Δ_ε($\mathfrak{C}$,Ҏ)] $\cap_\varepsilon$(Ħ,Ƨ)=[(Ғ,J) $\cap_\varepsilon$(Ħ,Ƨ)]Δ_ε [($\mathfrak{C}$,Ҏ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (J△Ҏ)∩Ƨ=J∩Ҏ'∩Ƨ'=∅.

Proof: Since the proof of the right distributive property of the extended intersection over the extended symmetric difference is very similar to (i), it is not repeated here. However, it is worth mentioning the following point. In classical sets, the intersection operation is right-distributive over the symmetric difference operation. However, for extended intersection and extended symmetric difference operations, this situation does not hold. We state that distributivity can be satisfied under the condition (J△Ҏ)∩Ƨ=J∩Ҏ'∩Ƨ'=∅.

v) [(Ғ,J) γ_ε($\mathfrak{C}$,Ҏ)] $\cap_\varepsilon$(Ħ,Ƨ)=[(Ғ,J) $\cap_\varepsilon$(Ħ,Ƨ)]γ_ε [($\mathfrak{C}$,Ҏ) $\cap_\varepsilon$ (Ħ,Ƨ)], where (J△Ҏ)∩Ƨ=J∩Ҏ'∩Ƨ'=∅.

3.2.1.3. The distributions of the extended intersection operation over soft binary piecewise operations:

Here, the distributions of the extended intersection operation to soft binary operations are investigated. First, distributions from the left side, followed by distributions from the right side are examined.

a) LHS distributions of extended intersection operation over soft binary piecewise operations:

Let (Ꞙ,Ꭻ), (ℭ,Ᵽ) and (Ħ,Ƨ) be SSs over U. Then, we have the following distributions:

i) $(Ꞙ,Ꭻ) \cap_{\varepsilon} [(ℭ,Ᵽ) \tilde{\cup} (Ħ,Ƨ)] = [(Ꞙ,Ꭻ) \cap_{\varepsilon} (ℭ,Ᵽ)] \tilde{\cup} [(Ꞙ,Ꭻ) \cap_{\varepsilon} (Ħ,Ƨ)]$, where $Ꭻ \cap Ᵽ \cap Ƨ' = \emptyset$.

Proof: First, let's consider the LHS, and let $(ℭ,Ᵽ) \tilde{\cup} (Ħ,Ƨ) = (R,Ᵽ)$, where for all $\alpha \in Ᵽ$,

$$R(\alpha) = \begin{cases} ℭ(\alpha) & \alpha \in Ᵽ \setminus Ƨ \\ ℭ(\alpha) \cup Ħ(\alpha) & \alpha \in Ᵽ \cap Ƨ \end{cases}$$

Let $(Ꞙ,Ꭻ) \cap_{\varepsilon} (R,Ᵽ) = (Ṅ, Ꭻ \cup Ᵽ)$, where for all $\alpha \in Ꭻ \cup Ᵽ$,

$$Ṅ(\alpha) = \begin{cases} Ꞙ(\alpha) & \alpha \in Ꭻ \setminus Ᵽ \\ R(\alpha) & \alpha \in Ᵽ \setminus Ꭻ \\ Ꞙ(\alpha) \cap R(\alpha) & \alpha \in Ꭻ \cap Ᵽ \end{cases}$$

Hence,

$$Ṅ(\alpha) = \begin{cases} Ꞙ(\alpha) & \alpha \in Ꭻ \setminus Ᵽ \\ ℭ(\alpha) & \alpha \in (Ᵽ \setminus Ƨ) \setminus Ꭻ = Ꭻ' \cap Ᵽ \cap Ƨ' \\ ℭ(\alpha) \cup Ħ(\alpha) & \alpha \in (Ᵽ \cap Ƨ) \setminus Ꭻ = Ꭻ' \cap Ᵽ \cap Ƨ \\ Ꞙ(\alpha) \cap ℭ(\alpha) & \alpha \in Ꭻ \cap (Ᵽ \setminus Ƨ) = Ꭻ \cap Ᵽ \cap Ƨ' \\ Ꞙ(\alpha) \cap [ℭ(\alpha) \cup Ħ(\alpha)] & \alpha \in Ꭻ \cap (Ᵽ \cap Ƨ) = Ꭻ \cap Ᵽ \cap Ƨ \end{cases}$$

Now let's handle the RHS, and let $(Ꞙ,Ꭻ) \cap_{\varepsilon} (ℭ,Ᵽ) = (K, Ꭻ \cup Ᵽ)$, where for all $\alpha \in Ꭻ \cup Ᵽ$,

$$K(\alpha) = \begin{cases} Ꞙ(\alpha) & \alpha \in Ꭻ \setminus Ᵽ \\ ℭ(\alpha) & \alpha \in Ᵽ \setminus Ꭻ \\ Ꞙ(\alpha) \cap ℭ(\alpha) & \alpha \in Ꭻ \cap Ᵽ \end{cases}$$

Let $(Ꞙ,Ꭻ) \cap_{\varepsilon} (Ħ,Ƨ) = (S, Ꭻ \cup Ƨ)$, where for all $\alpha \in Ꭻ \cup Ƨ$,

$$S(\alpha) = \begin{cases} Ꞙ(\alpha) & \alpha \in Ꭻ \setminus Ƨ \\ Ħ(\alpha) & \alpha \in Ƨ \setminus Ꭻ \end{cases}$$

$$\text{Ꞙ}(\alpha)\cap\text{Ꜧ}(\alpha) \qquad \alpha\in\text{Ɉ}\cap\text{Ꝣ}$$

Let (K,Ɉ∪Ᵽ) $\tilde{\cup}$(S,Ɉ∪Ꝣ)=(L,Ɉ∪Ᵽ), where for all α∈Ɉ∪Ᵽ

$$L(\alpha)=\begin{cases} K(\alpha) & \alpha\in(\text{Ɉ}\cup\text{Ᵽ})\backslash(\text{Ɉ}\cup\text{Ꝣ}) \\ K(\alpha)\cup S(\alpha) & \alpha\in(\text{Ɉ}\cup\text{Ᵽ})\cap(\text{Ɉ}\cup\text{Ꝣ}) \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\backslash\text{₱})\backslash(\text{Ɉ}\cup\text{Ꝣ})=\emptyset \\ \mathfrak{C}(\alpha) & \alpha\in(\text{₱}\backslash\text{Ɉ})\backslash(\text{Ɉ}\cup\text{Ꝣ})=\text{Ɉ}'\cap\text{₱}\cap\text{Ꝣ}' \\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{₱})\backslash(\text{Ɉ}\cup\text{Ꝣ})=\emptyset \\ \text{Ꞙ}(\alpha)\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\backslash\text{₱})\cap(\text{Ɉ}\backslash\text{Ꝣ})=\text{Ɉ}\cap\text{₱}'\cap\text{Ꝣ}' \\ \text{Ꞙ}(\alpha)\cup\text{Ꜧ}(\alpha) & \alpha\in(\text{Ɉ}\backslash\text{₱})\cap(\text{Ꝣ}\backslash\text{Ɉ})=\emptyset \\ \text{Ꞙ}(\alpha)\cup[\text{Ꞙ}(\alpha)\cap\text{Ꜧ}(\alpha)] & \alpha\in(\text{Ɉ}\backslash\text{₱})\cap(\text{Ɉ}\cap\text{Ꝣ})=\text{Ɉ}\cap\text{₱}'\cap\text{Ꝣ} \\ \mathfrak{C}(\alpha)\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{₱}\backslash\text{Ɉ})\cap(\text{Ɉ}\backslash\text{Ꝣ})=\emptyset \\ \mathfrak{C}(\alpha)\cup\text{Ꜧ}(\alpha) & \alpha\in(\text{₱}\backslash\text{Ɉ})\cap(\text{Ꝣ}\backslash\text{Ɉ})=\text{Ɉ}'\cap\text{₱}\cap\text{Ꝣ} \\ \mathfrak{C}(\alpha)\cup[\text{Ꞙ}(\alpha)\cap\text{Ꜧ}(\alpha)] & \alpha\in(\text{₱}\backslash\text{Ɉ})\cap(\text{Ɉ}\cap\text{Ꝣ})=\emptyset \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup\text{Ꞙ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{₱})\cap(\text{Ɉ}\backslash\text{Ꝣ})=\text{Ɉ}\cap\text{₱}\cap\text{Ꝣ}' \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup\text{Ꜧ}(\alpha) & \alpha\in(\text{Ɉ}\cap\text{₱})\cap(\text{Ꝣ}\backslash\text{Ɉ})=\emptyset \\ [\text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha)]\cup[\text{Ꞙ}(\alpha)\cap\text{Ꜧ}(\alpha)] & \alpha\in(\text{Ɉ}\cap\text{₱})\cap(\text{Ɉ}\cap\text{Ꝣ})=\text{Ɉ}\cap\text{₱}\cap\text{Ꝣ} \end{cases}$$

Hence,

$$L(\alpha)=\begin{cases} \mathfrak{C}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{₱}\cap\text{Ꝣ}' \\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{₱}'\cap\text{Ꝣ}' \\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{₱}'\cap\text{Ꝣ} \\ \mathfrak{C}(\alpha)\cup\text{Ꜧ}(\alpha) & \alpha\in\text{Ɉ}'\cap\text{₱}\cap\text{Ꝣ} \\ \text{Ꞙ}(\alpha) & \alpha\in\text{Ɉ}\cap\text{₱}\cap\text{Ꝣ}' \\ \text{Ꞙ}(\alpha)\cap[\mathfrak{C}(\alpha)\cup\text{Ꜧ}(\alpha] & \alpha\in\text{Ɉ}\cap\text{₱}\cap\text{Ꝣ} \end{cases}$$

Here, if we consider Ɉ\₱ in the function N, since Ɉ\₱=Ɉ∩₱', if an element is in the complement of Z, it is either in Ꝣ\₱ or in the complement of Ꝣ∪Ᵽ. Hence, if α∈Ɉ\₱, then α∈Ɉ∩Ꝣ∩₱' or α∈Ɉ∩Ꝣ'∩₱'. Thus, Ǹ=L is satisfied with the condition Ɉ∩₱∩Ꝣ'=∅.

ii)(Ꞙ, Ɉ) $\cap_\varepsilon$ [(ℭ, Ᵽ) $\tilde{\cap}$ (H, Ꝣ)] = [(Ꞙ, Ɉ) $\cap_\varepsilon$ (ℭ, Ᵽ)] $\tilde{\cap}$ [(Ꞙ, Ɉ) $\cap_\varepsilon$ (H, Ꝣ)], where Ɉ ∩ Ᵽ' ∩ Ꝣ = ∅.

iii) (Ꞙ,Ɉ) $\cap_\varepsilon$[(ℭ,₱) $\tilde{\backslash}$(Ꜧ,Ꝣ)]=[(Ꞙ,Ɉ)$\cap_\varepsilon$(ℭ,₱)] $\tilde{\backslash}$[(Ꞙ,Ɉ) $\cap_\varepsilon$ (Ꜧ,Ꝣ)], where Ɉ∩₱'∩Ꝣ'=Ɉ∩(₱Δ Ꝣ)=∅.

iv) (Ꞙ,Ʝ) $\cap_\varepsilon$[(ℭ,Ᵽ) $\tilde{\gamma}$ (Ⱨ,Ƨ)] = [(Ꞙ,Ʝ)$\cap_\varepsilon$(ℭ,Ᵽ)] $\tilde{\gamma}$[(Ꞙ,Ʝ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ᵽΔ Ƨ)=∅.

v) (Ꞙ,Ʝ) $\cap_\varepsilon$[(ℭ,Ᵽ) $\tilde{\Delta}$ (Ⱨ,Ƨ)] = [(Ꞙ,Ʝ)$\cap_\varepsilon$(ℭ,Ᵽ)] $\tilde{\Delta}$[(Ꞙ,Ʝ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ʝ∩Ᵽ'∩Ƨ'=Ʝ∩(ᵽΔ Ƨ)=∅.

b) RHS distributions of extended intersection operation over soft binary piecewise operations:
Let (Ꞙ,Ʝ), (ℭ,Ᵽ) and (Ⱨ,Ƨ) be SSs over U. Then, we have the following distributions:

i) [(Ꞙ,Ʝ) $\tilde{\cup}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ⱨ,Ƨ)=[(Ꞙ,Ʝ)$\cap_\varepsilon$(Ⱨ,Ƨ)] $\tilde{\cup}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ʝ∩Ᵽ'∩Ƨ=∅.

Proof: First, let's consider the LHS, and let (Ꞙ,Ʝ) $\tilde{\cup}$ (ℭ,Ᵽ)=(R,Ʝ), where for all α∈Ʝ,

$$R(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in \text{Ʝ}\setminus\text{Ᵽ} \\ \text{Ꞙ}(\alpha)\cup\text{ℭ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ᵽ} \end{cases}$$

Let (R,Ʝ) $\cap_\varepsilon$ (Ⱨ,Ƨ) =(Ǹ,Ʝ∪Ƨ), where for all α∈Ʝ∪Ƨ,

$$\text{Ǹ}(\alpha)=\begin{cases} R(\alpha) & \alpha\in\text{Ʝ}\setminus\text{Ƨ} \\ \text{Ⱨ}(\alpha) & \alpha\in\text{Ƨ}\setminus\text{Ʝ} \\ R(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ƨ} \end{cases}$$

Hence,

$$\text{Ǹ}(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in(\text{Ʝ}\setminus\text{Ᵽ})\setminus\text{Ƨ}=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ}' \\ \text{Ꞙ}(\alpha)\cup\text{ℭ}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\setminus\text{Ƨ}=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ}' \\ \text{Ⱨ}(\alpha) & \alpha\in\text{Ƨ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in(\text{Ʝ}\setminus\text{Ᵽ})\cap\text{Ƨ}=\text{Ʝ}\cap\text{Ᵽ}'\cap\text{Ƨ} \\ [\text{Ꞙ}(\alpha)\cup\text{ℭ}(\alpha)]\cap \text{H}(\alpha) & \alpha\in(\text{Ʝ}\cap\text{Ᵽ})\cap\text{Ƨ}=\text{Ʝ}\cap\text{Ᵽ}\cap\text{Ƨ} \end{cases}$$

Now let's handle the RHS, and let (Ꞙ,Ʝ) $\cap_\varepsilon$ (Ⱨ,Ƨ)=(K,Ʝ∪Ƨ), where for all α∈Ʝ∪Ƨ,

$$K(\alpha)=\begin{cases} \text{Ꞙ}(\alpha) & \alpha\in\text{Ʝ}\setminus\text{Ƨ} \\ \text{Ⱨ}(\alpha) & \alpha\in\text{Ƨ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cap\text{Ⱨ}(\alpha) & \alpha\in\text{Ʝ}\cap\text{Ƨ} \end{cases}$$

Let (ℭ,Ᵽ) $\cap_\varepsilon$ (Ⱨ,Ƨ)=(S, Ᵽ∪Ƨ), where for all α∈Ᵽ∪Ƨ,

$$\begin{cases} \text{ℭ}(\alpha) & \alpha\in\text{Ᵽ}\setminus\text{Ƨ} \end{cases}$$

S(α)=   Ή(α)   α∈Ƨ\Ᵽ
          ℭ(α)∩Ή(α)   α∈Ᵽ∩Ƨ

Let (K,ꞀƧ) $\tilde{\cup}$ (S,Ᵽ∪Ƨ)=(L,(Ɉ∪Ƨ), where for all α∈(Ɉ∪Ƨ),

L(α)= { K(α)   α∈(Ɉ∪Ƨ)\(Ᵽ∪Ƨ)
        K(α)∪S(α)   α∈(Ɉ∪Ƨ)∩(Ᵽ∪Ƨ)

Hence,

L(α)= { Ꞙ(α)   α∈(Ɉ\Ƨ)\(Ᵽ∪Ƨ)=Ɉ∩Ᵽ’∩Ƨ’
        Ή(α)   α∈(Ƨ\Ɉ)\(Ᵽ∪Ƨ)=∅
        Ꞙ(α)∩Ή(α)   α∈(Ɉ∩Ƨ)\(Ᵽ∪Ƨ)=∅
        Ꞙ(α)∪ℭ(α)   α∈(Ɉ\Ƨ)∩(Ᵽ\Ƨ)=Ɉ∩Ᵽ∩Ƨ’
        Ꞙ(α)∪Ή(α)   α∈(Ɉ\Ƨ)∩(Ƨ\Ᵽ)=∅
        Ꞙ(α)∪[ℭ(α)∩Ή(α)]   α∈(Ɉ\Ƨ)∩(Ᵽ∩Ƨ)=∅
        Ή(α)∪ℭ(α)   α∈(Ƨ\Ɉ)∩(Ᵽ\Ƨ)=∅
        Ή(α)∪Ή(α)   α∈(Ƨ\Ɉ)∩(Ƨ\Ᵽ)= Ɉ’∩Ᵽ’∩Ƨ
        Ή(α)∪[ℭ(α)∩Ή(α)]   α∈(Ƨ\Ɉ)∩(Ᵽ∩Ƨ)=Ɉ’∩Ᵽ∩Ƨ
        [Ꞙ(α)∩Ή(α)]∪ℭ(α)   α∈(Ɉ∩Ƨ)∩(Ᵽ\Ƨ)=∅
        [Ꞙ(α)∩Ή(α)]∪Ή(α)   α∈(Ɉ∩Ƨ)∩(Ƨ\Ᵽ)=Ɉ∩Ᵽ’∩Ƨ
        [Ꞙ(α)∩Ή(α)]∪[ℭ(α)∩Ή(α)]   α∈(Ɉ∩Ƨ)∩(Ᵽ∩Ƨ)=Ɉ∩Ᵽ∩Ƨ

Hence,

L(α)= { Ꞙ(α)   α∈Ɉ∩Ᵽ’∩Ƨ’
        Ꞙ(α)∪ℭ(α)   α∈Ɉ∩Ᵽ∩Ƨ’
        Ή(α)   α∈Ɉ’∩Ᵽ’∩Ƨ
        Ή(α)   α∈Ɉ’∩Ᵽ∩Ƨ
        Ή(α)   α∈Ɉ∩Ᵽ’∩Ƨ
        [Ꞙ(α)∪ℭ(α)]∩Ή(α)   α∈Ɉ∩Ᵽ∩Ƨ

Here, if we consider Ƨ\Ɉ in the function N, since Ƨ\Ɉ=Ƨ∩Ɉ’, if an element is in the complement of Ɉ, it is either in Ᵽ\Ɉ or in the complement of Ᵽ∪Ɉ. Hence, if α∈Ƨ\Ɉ, then α∈ Ƨ∩Ᵽ∩Ɉ’ or α∈Ƨ∩Ᵽ’∩Ɉ’. Thus, Ǹ=L is satisfied with the condition Ɉ∩Ᵽ’∩Ƨ=∅.

ii)[(Ꞙ, Ɉ) $\tilde{\cap}$ (ℭ, Ᵽ)] $\cap_{\varepsilon}$ (H, Ƨ) = [(F, Ɉ) $\cap_{\varepsilon}$ (H, Ƨ)] $\tilde{\cap}$ [(, Ᵽ) $\cap_{\varepsilon}$ (H, Ƨ)], where Ɉ’ ∩ Ᵽ ∩ Ƨ = ∅.

iii) [(Ꞙ,Ʝ) $\tilde{\setminus}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ⱨ,Ƨ)=[(Ꞙ,Ʝ)$\cap_\varepsilon$(Ⱨ,Ƨ)] $\tilde{\setminus}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ʝ'∩Ᵽ'∩Ƨ=(ƝΔᵽ)∩Ƨ=∅.

iv) [(Ꞙ,Ʝ) $\tilde{\gamma}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ⱨ,Ƨ)=[(Ꞙ,Ʝ)$\cap_\varepsilon$(Ⱨ,Ƨ)] $\tilde{\gamma}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ʝ'∩Ᵽ'∩Ƨ=(ƝΔᵽ)∩Ƨ=∅.

v) [(Ꞙ,Ʝ) $\tilde{\Delta}$ (ℭ,Ᵽ)] $\cap_\varepsilon$ (Ⱨ,Ƨ)=[(Ꞙ,Ʝ)$\cap_\varepsilon$(Ⱨ,Ƨ)] $\tilde{\Delta}$ [(ℭ,Ᵽ) $\cap_\varepsilon$ (Ⱨ,Ƨ)], where Ʝ'∩Ᵽ'∩Ƨ=(ƝΔᵽ)∩Ƨ=∅.

4. ABSORPTION LAWS FOR SOFT SETS AND ALGEBRAIC STRUCTURES OF SOFT SETS

In this section, in order to find out whether the collection of SSs and restricted and extended intersection operation form lattice structures in $S_E(U)$ and $S_Ʝ(U)$, firstly the so-called absorption laws are examined with detailed proofs. Although the laws of absorption in $S_E(U)$ have been presented in previous works (Ali et al., 2009; Ali et al., 2011; Qin and Hong, 2010; Singh and Onyeozili, 2012c) presented the results only with a table without proofs, and since the proofs in other studies are element-based and relatively long proofs, they are presented here with simpler proofs. In addition, in this study, the absorption laws in $S_A(U)$ for the newly-defined operations by Aybek (2024) and Yavuz (2024) are given in detail as well. Additionally, the distributive rules obtained from Section 3.1.1 and Section 3.2.1 in $S_E(U)$ and $S_Ʝ(U)$ are presented collectively in a table. Finally, we systematically, in detail, and collectively present the unary and binary algebraic structures formed by the restricted intersection and extended intersection together with other types of SS operations in $S_E(U)$ and $S_Ʝ(U)$. We believe that this comprehensive study will fill a gap in the literature, as such an inclusive study is currently absent.

4.1. Absorption laws for SSs

4.1.1. Absorption laws in $S_E(U)$:

Let (Ꞙ,Ʝ) and (ℭ,Ᵽ) be SSs over U. Then,

i) (Ꞙ,Ʝ)$\cap_R$ [(Ꞙ,Ʝ) $\cup_\varepsilon$ (ℭ,Ᵽ)]= (Ꞙ,Ʝ) and (Ꞙ,Ʝ) $\cup_\varepsilon$ [(Ꞙ,Ʝ) $\cap_R$ (ℭ,Ᵽ)] = (Ꞙ,Ʝ) (Qin and Hong, 2010; Singh and Onyeozili, 2012c).

Proof: Here, these absorption laws are proved with a simpler proof than the proofs given in Qin and Hong (2010) and Singh and Onyeozili (2012c) First, let's handle the LHS, and let (Ꞙ,Ʝ) $\cup_\varepsilon$(ℭ, Ᵽ)=(Q,Ʝ∪Ᵽ), where for all α∈ Ʝ∪Ᵽ,

$$Q(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ʝ\setminus Ᵽ \\ ℭ(\alpha) & \alpha\in Ᵽ\setminus Ʝ \\ Ꞙ(\alpha)\cup ℭ(\alpha) & \alpha\in Ʝ\cap Ᵽ \end{cases}$$

Let (Ꞙ,Ʝ) $\cap_R$(Q,Ʝ∪Ᵽ)=(M,Ʝ∩(Ʝ ∪ Ᵽ))=(M,Ʝ), where for all α∈Ʝ, M(α)=Ꞙ(α)∩Q(α). Hence,

$$M(\alpha)=\begin{cases} Ꞙ(\alpha)\cap Ꞙ(\alpha) & \alpha\in Ʝ\cap(Ʝ\setminus Ᵽ)=Ʝ\setminus Ᵽ \\ Ꞙ(\alpha)\cap ℭ(\alpha) & \alpha\in Ʝ\cap(Ᵽ\setminus Ʝ)=\emptyset \\ & \end{cases}$$

$Ꞙ(\alpha)\cap[Ꞙ(\alpha)\cup\mathfrak{C}(\alpha)] \quad \alpha\in Ꭻ\cap(Ꭻ\capꝐ)=Ꭻ\capꝐ$

Thus,

$$M(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ꝑ \\ Ꞙ(\alpha) & \alpha\in Ꭻ\cap Ꝑ \end{cases}$$

Hence, $(Ꞙ,Ꭻ)\cap_R[(Ꞙ,Ꭻ)\cup_\varepsilon(\mathfrak{C},Ꝑ)] = (Ꞙ,Ꭻ)$.

Now, show that $(Ꞙ,Ꭻ)\cup_\varepsilon[(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꝑ)]) = (Ꞙ,Ꭻ)$. Let $(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꝑ)=(L,Ꭻ\capꝐ)$, where for all $\alpha\in Ꭻ\capꝐ$, $L(\alpha)=Ꞙ(\alpha)\cap\mathfrak{C}(\alpha)$. Let $(Ꞙ,Ꭻ)\cup_\varepsilon(L,Ꭻ\capꝐ)=(W,Ꭻ\cup(Ꭻ\capꝐ))=(W,Ꭻ)$, where for all $\alpha\in Ꭻ$,

$$W(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus(Ꭻ\capꝐ)=Ꭻ\setminus Ꝑ \\ L(\alpha) & \alpha\in(Ꭻ\capꝐ)\setminus Ꭻ=\emptyset \\ Ꞙ(\alpha)\cup L(\alpha) & \alpha\in Ꭻ\cap(Ꭻ\capꝐ)=Ꭻ\capꝐ \end{cases}$$

Thus, for all $\alpha\in Ꭻ$,

$$W(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ꝑ \\ [Ꞙ(\alpha)\cup[Ꞙ(\alpha)\cap\mathfrak{C}(\alpha)] & \alpha\in Ꭻ\capꝐ \end{cases}$$

Hence, for all $\alpha\in Ꭻ$

$$W(\alpha)=\begin{cases} Ꞙ(\alpha) & \alpha\in Ꭻ\setminus Ꝑ \\ Ꞙ(\alpha) & \alpha\in Ꭻ\capꝐ \end{cases}$$

That is, $(Ꞙ,Ꭻ)\cup_\varepsilon[(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꝑ)] = (Ꞙ,Ꭻ)$. Thus, the absorption law is valid for the operations $\cup_\varepsilon$ and $\cap_R$ in $S_E(U)$ as well. Here, even if $Ꭻ\capꝐ=\emptyset$, the equality still holds in every case because $W(\alpha)=Ꞙ(\alpha)$ for all $\alpha\in Ꭻ$.

ii) $(Ꞙ,Ꭻ)\cup_R[(Ꞙ,Ꭻ)\cap_\varepsilon(\mathfrak{C},Ꝑ)] = (Ꞙ,Ꭻ)$ and $(Ꞙ,Ꭻ)\cap_\varepsilon[(Ꞙ,Ꭻ)\cup_R(\mathfrak{C},Ꝑ)]= (Ꞙ,Ꭻ)$ (Qin and Hong, 2010; Singh and Onyeozili, 2012c).

Remark 1 Absorption laws do not hold for the following cases. Here note that these cases and their proofs were given in Singh and Onyeozili (2012c); however, we present here once again, since there are some mathematical typos in the proofs of Singh and Onyeozili (2012c).

i) (Ꞙ,Ʝ)$\cap_R$ [(Ꞙ,Ʝ) $\cup_R$ (ℭ,Ᵽ)] $\widetilde{\subseteq}$ (Ꞙ,Ʝ) and (Ꞙ,Ʝ) $\cup_R$ [(Ꞙ,Ʝ) $\cap_R$ (ℭ,Ᵽ)] $\widetilde{\subseteq}$ (Ꞙ,Ʝ) (Singh and Onyeozili, 2012c).

Proof: First, let's consider the LHS, and let (Ꞙ,Ʝ) $\cup_R$(ℭ,Ᵽ)=(Q,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, Q(α)=Ꞙ(α)∪ℭ(α). Let (Ꞙ,Ʝ) $\cap_R$(Q,Ʝ∩Ᵽ)=(M,Ʝ∩ (Ʝ ∩ Ᵽ))=(M,Ʝ∩Ᵽ), where for all α∈Ʝ∩Ᵽ, M(α)=Ꞙ(α)∩Q(α). Hence, for all α∈Ʝ∩Ᵽ, M(α)=Ꞙ(α)∩[Ꞙ(α)∪ℭ(α)]=Ꞙ(α). Thus, (M,Ʝ∩Ᵽ) $\widetilde{\subseteq}$ (Ꞙ,Ʝ).That is, (Ꞙ,Ʝ)$\cap_R$ [(Ꞙ,Ʝ) $\cup_R$ (ℭ,Ᵽ) $\widetilde{\subseteq}$ (Ꞙ,Ʝ). Here, note that (Ꞙ,Ʝ)$\cap_R$ [(Ꞙ,Ʝ) $\cup_R$ (ℭ,Ᵽ) can not be soft equal to (Ꞙ,Ʝ), as they have different parameter sets. Similarly, one can show that (Ꞙ,Ʝ) $\cup_R$ [(Ꞙ,Ʝ) $\cap_R$ (ℭ,Ᵽ)]) $\widetilde{\subseteq}$ (Ꞙ,Ʝ).

Hence, the absorption law does not hold for the SS operations $\cup_R$ and $\cap_R$ in $S_E(U)$.

ii) (Ꞙ,Ʝ)$\widetilde{\subseteq}$ (Ꞙ,Ʝ) $\cap_\varepsilon$ [(Ꞙ,Ʝ) $\cup_\varepsilon$ (ℭ,Ᵽ)] and (Ꞙ,Ʝ) $\widetilde{\subseteq}$ (Ꞙ,Ʝ) $\cup_\varepsilon$ [(Ꞙ,Ʝ) $\cap_\varepsilon$ (ℭ,Ᵽ)] (Singh and Onyeozili, 2012c).

Proof: Let us show that (Ꞙ,Ʝ)$\widetilde{\subseteq}$ (Ꞙ,Ʝ) $\cap_\varepsilon$ [(Ꞙ,Ʝ) $\cup_\varepsilon$ (ℭ,Ᵽ)]. Let (Ꞙ,Ʝ) $\cup_\varepsilon$(ℭ,Ᵽ)=(Q,Ʝ∪Ᵽ), where for all α∈ Ʝ∪Ᵽ,

$$Q(\alpha)=\begin{cases} \text{Ꞙ}(\alpha), & \alpha\in \text{Ʝ}\setminus\text{Ᵽ} \\ \mathfrak{C}(\alpha), & \alpha\in \text{Ᵽ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cup\mathfrak{C}(\alpha), & \alpha\in \text{Ʝ}\cap\text{Ᵽ} \end{cases}$$

Let (Ꞙ,Ʝ) $\cap_\varepsilon$(Q,Ʝ∪Ᵽ)=(M,Ʝ∪(Ʝ ∪ Ᵽ))=(M,Ʝ∪Ᵽ), where for all α∈Ʝ∪Ᵽ,

$$M(\alpha)=\begin{cases} \text{Ꞙ}(\alpha), & \alpha\in \text{Ʝ}\setminus(\text{Ʝ}\cup\text{Ᵽ})=\emptyset \\ Q(\alpha), & \alpha\in (\text{Ʝ}\cup\text{Ᵽ})\setminus\text{Ʝ}=\text{Ᵽ} \\ \text{Ꞙ}(\alpha)\cap Q(\alpha), & \alpha\in \text{Ʝ}\cap(\text{Ʝ}\cup\text{Ᵽ})=\text{Ʝ} \end{cases}$$

Thus,

$$M(\alpha)=\begin{cases} \text{Ꞙ}(\alpha), & \alpha\in (\text{Ʝ}\setminus\text{Ᵽ})\setminus\text{Ʝ}=\emptyset \\ \mathfrak{C}(\alpha), & \alpha\in (\text{Ᵽ}\setminus\text{Ʝ})\setminus\text{Ʝ}=\text{Ᵽ}\setminus\text{Ʝ} \\ \text{Ꞙ}(\alpha)\cup\mathfrak{C}(\alpha), & \alpha\in (\text{Ʝ}\cap\text{Ᵽ})\setminus\text{Ʝ}=\emptyset \\ \text{Ꞙ}(\alpha)\cap\text{Ꞙ}(\alpha), & \alpha\in \text{Ʝ}\cap(\text{Ʝ}\setminus\text{Ᵽ})=\text{Ʝ}\setminus\text{Ᵽ} \\ \text{Ꞙ}(\alpha)\cap\mathfrak{C}(\alpha), & \alpha\in \text{Ʝ}\cap(\text{Ᵽ}\setminus\text{Ʝ})=\emptyset \\ \text{Ꞙ}(\alpha)\cap[\text{Ꞙ}(\alpha)\cup\mathfrak{C}(\alpha)], & \alpha\in \text{Ʝ}\cap(\text{Ʝ}\cap\text{Ᵽ})=\text{Ʝ}\cap\text{Ᵽ} \end{cases}$$

Thereby, for all α∈Ʝ∪Ᵽ,

$$M(\alpha)=\begin{cases}\mathfrak{C}(\alpha), & \alpha\in Ⱂ\setminus Ꭻ\\ Ꞙ(\alpha), & \alpha\in Ꭻ\setminus Ⱂ\\ Ꞙ(\alpha), & \alpha\in Ꭻ\cap Ⱂ\end{cases}$$

Thus, $(Ꞙ,Ꭻ)\cap_{\varepsilon}[(Ꞙ,Ꭻ)\cup_{\varepsilon}(\mathfrak{C},Ⱂ)]\neq(Ꞙ,Ꭻ)$. Since $Ꞙ(\alpha)\subseteq M(\alpha)$, for all $\alpha\in Ꭻ$, it is evident that $(Ꞙ,Ꭻ)\widetilde{\subseteq}(Ꞙ,Ꭻ)\cap_{\varepsilon}[(Ꞙ,Ꭻ)\cup_{\varepsilon}(\mathfrak{C},Ⱂ)]$. Similarly, one can show that $(Ꞙ,Ꭻ)\widetilde{\subseteq}(Ꞙ,Ꭻ)\cup_{\varepsilon}[(Ꞙ,Ꭻ)\cap_{\varepsilon}(\mathfrak{C},Ⱂ)]$. Thereby, the absorption law does not hold for the SS operations $\cup_{\varepsilon}$ and $\cap_{\varepsilon}$ in $S_E(U)$.

When the absorption laws in Subsection 4.1.1 are considered, the following absorption laws exist in $S_E(U)$. In the table below, 1 indicates that the absorption law is satisfied, while 0 indicates that it is not.

	$\cap_R$	$\cup_R$	$\cap_{\varepsilon}$	$\cup_{\varepsilon}$
$\cap_R$	0	0	0	1
$\cup_R$	0	0	1	0
$\cap_{\varepsilon}$	0	1	0	0
$\cup_{\varepsilon}$	1	0	0	0

Table 1 Absorption Laws in $S_E(U)$ (Ali et al., 2011)

In the study by Ali et al. (2011), this table was provided without proving any of the absorption laws. In our study, before presenting the table, we have detailed the properties with thorough proofs.

4.1.2. Absorption laws in $S_{Ꭻ}(U)$:

Let $(Ꞙ,Ꭻ),(\mathfrak{C},Ꭻ)$ be soft sets over U. Then,

i)The following absorption laws are valid for $\cap_R$ in $S_{Ꭻ}(U)$:

- $(Ꞙ,Ꭻ)\cap_R[(Ꞙ,Ꭻ)\cup_R(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$ and $(Ꞙ,Ꭻ)\cup_R[(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$.
- $(Ꞙ,Ꭻ)\cap_R[(Ꞙ,Ꭻ)\cup_{\varepsilon}(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$ and $(Ꞙ,Ꭻ)\cup_{\varepsilon}[(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$.
- $(Ꞙ,Ꭻ)\cap_R[(Ꞙ,Ꭻ)\overset{*}{\cup}_{\varepsilon}(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$ and $(Ꞙ,Ꭻ)\overset{*}{\cup}_{\varepsilon}[(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$.
- $(Ꞙ,Ꭻ)\cap_R[(Ꞙ,Ꭻ)\tilde{\cup}(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$ and $(Ꞙ,Ꭻ)\tilde{\cup}[(Ꞙ,Ꭻ)\cap_R(\mathfrak{C},Ꭻ)]=(Ꞙ,Ꭻ)$.

Proof: Since the operations of restricted union, extended union, complementary extended union, soft binary piecewise union are coincident, and these operations are commutative in $S_{Ꭻ}(U)$, the proof follows the Subsection of 4.1.1.

ii) The following absorption laws are valid for $\cap_{\varepsilon}$ in $S_{Л}(U)$:

- $(\text{Ꞙ},Л)\cap_{\varepsilon} [(\text{Ꞙ},Л) \cup_{R} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \cup_{R} [(\text{Ꞙ},Л) \cap_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cap_{\varepsilon} [(\text{Ꞙ},Л) \cup_{\varepsilon} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \cup_{\varepsilon} [(\text{Ꞙ},Л) \cap_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cap_{\varepsilon} [Л \overset{*}{\cup}_{\varepsilon} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \overset{*}{\cup}_{\varepsilon} [(\text{Ꞙ},Л) \cap_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cap_{\varepsilon} [(\text{Ꞙ},Л) \tilde{\cup} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \tilde{\cup} [(\text{Ꞙ},Л) \cap_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.

iii) The following absorption laws are valid for $\cup_{R}$ in $S_{Л}(U)$:

- $(\text{Ꞙ},Л)\cup_{R} [(\text{Ꞙ},Л) \cap_{R} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \cap_{R} [(\text{Ꞙ},Л) \cup_{R} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cup_{R} [(\text{Ꞙ},Л) \cap_{\varepsilon} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \cap_{\varepsilon} [(\text{Ꞙ},Л) \cup_{R} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cup_{R} [(\text{Ꞙ},Л) \overset{*}{\cap}_{\varepsilon} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \overset{*}{\cap}_{\varepsilon} [(\text{Ꞙ},Л) \cup_{R} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cup_{R} [(\text{Ꞙ},Л) \tilde{\cap} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \tilde{\cap} [(\text{Ꞙ},Л) \cup_{R} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.

iv) The following absorption laws are valid for $\cup_{\varepsilon}$ in $S_{Л}(U)$:

- $(\text{Ꞙ},Л)\cup_{\varepsilon} [(\text{Ꞙ},Л) \cap_{R} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \cap_{R} [(\text{Ꞙ},Л) \cup_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cup_{\varepsilon} [(\text{Ꞙ},Л) \cap_{\varepsilon} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \cap_{\varepsilon} [(\text{Ꞙ},Л) \cup_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cup_{\varepsilon} [(\text{Ꞙ},Л) \overset{*}{\cap}_{\varepsilon} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \overset{*}{\cap}_{\varepsilon} [(\text{Ꞙ},Л) \cup_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.
- $(\text{Ꞙ},Л)\cup_{\varepsilon} [(\text{Ꞙ},Л) \tilde{\cap} (\mathfrak{C},Л)]= (\text{Ꞙ},Л)$ and $(\text{Ꞙ},Л) \tilde{\cap} [(\text{Ꞙ},Л) \cup_{\varepsilon} (\mathfrak{C},Л)] = (\text{Ꞙ},Л)$.

When the absorption laws in Subsection 4.1.2. are considered, the following absorption laws exist in $S_{Л}(U)$. In the table below, 1 indicates that the absorption law is satisfied, while 0 indicates that it is not.

	$\cap_R$	$\cup_R$	$\cap_\varepsilon$	$\cup_\varepsilon$	$\tilde{\cap}$	$\tilde{\cup}$	$\overset{*}{\cap}_{\varepsilon}$	$\overset{*}{\cup}_{\varepsilon}$
$\cap_R$	0	1	0	1	0	1	0	1
$\cup_R$	1	0	1	0	1	0	1	0
$\cap_\varepsilon$	1	0	0	1	0	1	0	1
$\cup_\varepsilon$	1	0	1	0	1	0	1	0
$\tilde{\cap}$	0	1	0	1	0	1	0	1

$\tilde{\cup}$	1	0	1	0	1	0	1	0
$\overset{*}{\cap}_{\varepsilon}$	0	1	0	1	0	1	0	1
$\overset{*}{\cup}_{\varepsilon}$	1	0	1	0	1	0	1	0

Table 2 Absorption Laws in $S_{J}(U)$

In addition, this table includes the latest SS operations introduced in the literature in 2023 and 2024, such as complementary extended SS operations and soft binary piecewise operations,

4.2. Algebraic Structures of SSs Formed by Restricted and Extended Intersection SS Operations

In this subsection, it is examined in detail which algebraic structures are formed by the restricted and extended intersection SS operations together with other SS operations in $S_E(U)$ and $S_J(U)$, respectively. First of all, algebraic structures with one binary operation (restricted intersection and extended intersection), and then algebraic structures with two binary operations (respectively, one of them is restricted intersection SS operation and the other is other SS operations, then one of them is extended intersection SS operation and the other is other SS operations) are explored. In line with this aim, by considering all distributions in Section 3.1 and 3.2, the tables for the distributive laws in $S_E(U)$ and $S_J(U)$ are provided.

For the algebraic structures with one binary operation, all the properties such as the identity element, if any, the inverse element, the absorbing element, idempotent, and the commutative property of the algebraic structures are presented in detail. For the algebraic structures with two binary operations, the properties of the algebraic structures, such as the identity element (if any), commutative and idempotent properties for the first and second operations, and the zero element (if any), are also presented in detail without omission. Additionally, for the structures that form a lattice, it is specified whether the lattice is bounded or not. If it is bounded, the lower and upper bounds are given, as well as whether it is distributive, and if it satisfies the De Morgan properties or not. In this regard, we emphasize the importance of our study, as it is comprehensive, covering the works of Ali et al. (2011), Qin and Hong (2010), and Sen (2014), and serves as a handbook for those newly interested in SSs.

Now, first by considering all distributions in Section 3.1 and 3.2, we present the table for distributive laws in $S_E(U)$ and $S_J(U)$, respectively. In these tables, '1' indicates that the distributive law holds; '0' indicates that it does not. It is important to note the following: places marked with '1' indicate full distributivity, meaning both right and left distributivity are satisfied; places marked with '1*' indicate only right distributivity is satisfied; places marked with '0' indicate that neither right nor left distributivity is satisfied.

	$\cap_R$	$\cup_R$	$\backslash_R$	Δ_R	γ_R	$\cap_\varepsilon$	$\cup_\varepsilon$	$\backslash_\varepsilon$	Δ_ε	γ_ε	$\tilde{\cap}$	$\tilde{\cup}$	$\tilde{\backslash}$	$\tilde{\Delta}$	$\tilde{\gamma}$
$\cap_R$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\cap_\varepsilon$	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0

Table 3 Distributive laws in $S_E(U)$ for restricted and extended intersection operations

(*: Just right distributions)

Here note that in the study by Ali et al. (2011), they provided this table without proving any of the distributive laws, demonstrating only those that do not hold with examples. In our study, before presenting the table, we provided detailed proofs in Section 3.1 and Section 3.2. Additionally, we have included the soft binary piecewise operations, which are newly introduced in the literature in 2023 and 2024, in the first row of this table.

	$\cap_R$	$\cup_R$	$\backslash_R$	Δ_R	γ_R	$\cap_\varepsilon$	$\cup_\varepsilon$	$\backslash_\varepsilon$	Δ_ε	γ_ε	$\tilde{\cap}$	$\tilde{\cup}$	$\tilde{\backslash}$	$\tilde{\Delta}$	$\tilde{\gamma}$
$\cap_R$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\cap_\varepsilon$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 3 Distributive laws in $S_{\mathbb{J}}(U)$ for restricted and extended intersection operations

(*: Just right distributions)

4.2.1. Algebraic structures with one binary operation in set $S_E(U)$ and $S_{\mathbb{A}}(U)$ formed by for restricted and extended intersection operations

In this subsection, algebraic structures with one binary operation, specifically the binary operation is restricted intersection operation and extended intersection SS operation, respectively are examined in $S_E(U)$ and $S_{\mathbb{A}}(U)$, respectively.

<u>4.2.1.1. Algebraic structures with one binary operation in $S_E(U)$ formed by restricted and extended intersection operations</u>

1) $(S_E(U),\cap_R)$ is a commutative idempotent monoid with the identity U_E, namely, a bounded semi-lattice with the absorbing element $\emptyset_\emptyset$.

2) $(S_E(U), \cap_\varepsilon)$ is a commutative idempotent monoid with the identity $\emptyset_\emptyset$, namely, a bounded semi-lattice with the absorbing element $\emptyset_E$.

<u>4.2.1.2. Algebraic structures with one binary operation in $S_{\mathbb{A}}(U)$ formed by restricted and extended intersection operations</u>

1) $(S_{\mathbb{A}}(U), \cap_R)$ and $(S_{\mathbb{A}}(U), \cap_\varepsilon)$ are commutative idempotent monoids with the identity element $U_{\mathbb{A}}$, namely, a bounded semi-lattice with the absorbing element $\emptyset_{\mathbb{A}}$.

4.2.2. Algebraic structures with two binary operations in $S_E(U)$ and $S_{\mathbb{A}}(U)$ formed by restricted and extended intersection operations

In this subsection, algebraic structures with two binary operations, the second binary operation of which is restricted intersection operation and extended intersection operation, respectively are examined in $S_E(U)$

and $S_{Ⱥ}(U)$, respectively. Additionally, four mathematically incorrect algebraic structures in the study by Ali et al. [9] are corrected.

4.2.2.1. Algebraic structures with two binary operations in $S_E(U)$ formed by restricted and extended intersection operations

i) Algebraic structures in $S_E(U)$ with two binary operations, the second binary operation of which is the restricted intersection operation:

Let (Ғ,Ⱥ), (Ȼ,Ᵽ) and (Ͱ,Ƨ) be SSs over U. Then,

1) $(S_E(U),\cap_R,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E and without zero.

2) $(S_E(U),\cup_R,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E and without zero.

Here, we also want to correct an error made in a previous study by Ali et al. (2011). It was stated that $(S_E(U),\cup_R,\cap_R)$ is a hemiring with the identity U_E. However, since (Ғ,Ⱥ)$\cup_R \emptyset_E=\emptyset_E \cup_R$(Ғ,Ⱥ)=(Ғ,Ⱥ) and (Ғ,Ⱥ)$\cap_R \emptyset_E = \emptyset_E \cap_R$(Ғ,Ⱥ)$\neq \emptyset_E$ (since (Ғ,Ⱥ)$\cap_R \emptyset_E = \emptyset_E \cap_R$(Ғ,Ⱥ)$= \emptyset_{Ⱥ}$), $(S_E(U),\cup_R,\cap_R)$ cannot be a hemiring.

3) $(S_E(U),\Delta_R,\cap_R)$ is an additively and multiplicatively commutative, multiplicatively idempotent semiring with the identity element U_E and without zero.

4) $(S_E(U),\cap_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E.

Moreover, since (Ғ,Ⱥ)$\cap_\varepsilon \emptyset_\emptyset=\emptyset_\emptyset \cap_\varepsilon$(Ғ,Ⱥ)=(Ғ,Ⱥ) and (Ғ,Ⱥ)$\cap_R \emptyset_\emptyset=\emptyset_\emptyset \cap_R$(Ғ,Ⱥ)$=\emptyset_\emptyset$, $\emptyset_\emptyset$ is the zero of $(S_E(U),\cap_\varepsilon,\cap_R)$, and thus $(S_E(U),\cap_\varepsilon,\cap_R)$ is a hemiring.

5) $(S_E(U),\cup_\varepsilon,\cap_R)$is an additively and multiplicatively commutative and idempotent semiring with the identity element U_E.

Moreover, since (Ғ,Ⱥ) $\cup_\varepsilon \emptyset_\emptyset=\emptyset_\emptyset \cup_\varepsilon$(Ғ,Ⱥ)=(Ғ,Ⱥ) and (Ғ,Ⱥ)$\cap_R \emptyset_\emptyset=\emptyset_\emptyset \cap_R$(Ғ,Ⱥ)$=\emptyset_\emptyset$, $\emptyset_\emptyset$ is the zero of $(S_E(U),\cup_\varepsilon,\cap_R)$, and thus $(S_E(U),\cup_\varepsilon,\cap_R)$ is a hemiring. (Ali et al., 2011)

6) $(S_E(U),\Delta_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E.

Moreover, since (Ғ,Ⱥ)$\Delta_\varepsilon\emptyset_\emptyset=\emptyset_\emptyset\Delta_\varepsilon$(Ғ,Ⱥ)=(Ғ,Ⱥ) and (Ғ,Ⱥ)$\cap_R \emptyset_\emptyset=\emptyset_\emptyset \cap_R$(Ғ,Ⱥ)$=\emptyset_\emptyset$, $\emptyset_\emptyset$ is the zero of $(S_E(U),\Delta_\varepsilon,\cap_R)$, and thus $(S_E(U),\Delta_\varepsilon,\cap_R)$ is a hemiring. (Sezgin and Çağman, 2025)

7) $(S_E(U),\setminus_{\varepsilon},\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

Although $(Ꞙ,Ⱥ)\setminus_{\varepsilon}\emptyset_{\emptyset}=\emptyset_{\emptyset}\setminus_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{\emptyset}=\emptyset_{\emptyset} \cap_R(Ꞙ,Ⱥ)=\emptyset_{\emptyset}$, namely, $\emptyset_{\emptyset}$ is the zero of $(S_E(U),\setminus_{\varepsilon},\cap_R)$, $(S_E(U),\setminus_{\varepsilon},\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

8) $(S_E(U),\gamma_{\varepsilon},\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

Although $(Ꞙ,Ⱥ)\gamma_{\varepsilon}\emptyset_{\emptyset}=\emptyset_{\emptyset}\gamma_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{\emptyset}=\emptyset_{\emptyset} \cap_R(Ꞙ,Ⱥ)=\emptyset_{\emptyset}$, namely, $\emptyset_{\emptyset}$ is the zero of $(S_E(U),\gamma_{\varepsilon},\cap_R)$, $(S_E(U),\setminus_{\varepsilon},\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

9) $(S_E(U),+_{\varepsilon},\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

Although $(Ꞙ,Ⱥ)+_{\varepsilon}\emptyset_{\emptyset}=\emptyset_{\emptyset}+_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{\emptyset}=\emptyset_{\emptyset} \cap_R(Ꞙ,Ⱥ)=\emptyset_{\emptyset}$, namely, $\emptyset_{\emptyset}$ is the zero of $(S_E(U),+_{\varepsilon},\cap_R)$, $(S_E(U),+_{\varepsilon},\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

10) $(S_E(U),\lambda_{\varepsilon},\cap_R)$ is a multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where $Ⱥ \cap Ᵽ\cap Ƨ=\emptyset$.

Although $(Ꞙ,Ⱥ)\lambda_{\varepsilon}\emptyset_{\emptyset}=\emptyset_{\emptyset}\lambda_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{\emptyset}=\emptyset_{\emptyset} \cap_R(Ꞙ,Ⱥ)=\emptyset_{\emptyset}$, namely, $\emptyset_{\emptyset}$ is the zero of $(S_E(U),\lambda_{\varepsilon},\cap_R)$, $(S_E(U),\lambda_{\varepsilon},\cap_R)$ cannot be a hemiring, since it is not additively commutative, but it is semiring with zero.

11) $(S_E(U),\theta_{\varepsilon},\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent semiring with the identity element U_E, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

Moreover, since $(Ꞙ,Ⱥ)\theta_{\varepsilon}\emptyset_{\emptyset}=\emptyset_{\emptyset}\theta_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{\emptyset}=\emptyset_{\emptyset} \cap_R(Ꞙ,Ⱥ)=\emptyset_{\emptyset}$, namely, $\emptyset_{\emptyset}$ is the zero of $(S_E(U),\theta_{\varepsilon},\cap_R)$, $(S_E(U),\theta_{\varepsilon},\cap_R)$ is a hemiring, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

12) $(S_E(U),*_{\varepsilon},\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent a semiring with the identity element U_E, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

Moreover, since $(Ꞙ,Ⱥ)*_{\varepsilon}\emptyset_{\emptyset}=\emptyset_{\emptyset}*_{\varepsilon}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R \emptyset_{\emptyset}=\emptyset_{\emptyset} \cap_R(Ꞙ,Ⱥ)=\emptyset_{\emptyset}$, namely, $\emptyset_{\emptyset}$ is the zero of $(S_E(U),*_{\varepsilon},\cap_R)$, $(S_E(U),*_{\varepsilon},\cap_R)$ is a hemiring, where $Ⱥ\cap Ᵽ\cap Ƨ=\emptyset$.

13) (S_E(U),$\tilde{\cap}$, $\cap_R$) is an additively and multiplicatively idempotent, multiplicatively commutative semiring with the identity element U_E, where Ⱥ∩Ꝑ'∩Ƨ=∅.

14) (S_E(U),$\tilde{\cup}$,$\cap_R$) is an additively and multiplicatively idempotent, multiplicatively commutative semiring with the identity element U_E, where Ⱥ∩Ꝑ'∩Ƨ=∅.

15) (S_E(U),$\tilde{\Delta}$,$\cap_R$) is a multiplicatively commutative and idempotent semiring with the identity element U_E, where Ⱥ∩Ꝑ'∩Ƨ=∅ (Sezgin and Yavuz, 2023b).

16) (S_E(U),$\cup_\varepsilon$,$\cap_R$) is a bounded distributive lattice with the lower bound $\emptyset_\emptyset$ and the upper bound U_E.

In fact, **(**S_E(U),$\cup_\varepsilon$) and **(**S_E(U),$\cap_R$) are commutative idempotent monoids with the identity element $\emptyset_\emptyset$ and U_E, respectively (Ali et al., 2011) and restricted intersection distributes over extended intersection from both left and right sides in S_E(U). Thus, (S_E(U),$\cup_\varepsilon$,$\cap_R$) is a bounded distributive lattice with the lower bound $\emptyset_\emptyset$ and the upper bound U_E. Since (Ꞙ,Ⱥ)$\cup_\varepsilon$(Ꞙ,Ⱥ)$^r \neq U_E$ and (Ꞙ,Ⱥ)$\cap_R$(Ꞙ,Ⱥ)$^r \neq \emptyset_\emptyset$, the algebraic structure (S_E(U),$\cup_\varepsilon$,$\cap_R$) is not complemented, thus it is not a Boolean algebra. (Ali et al., 2011; Qin and Hong, 2010)

<u>ii) Algebraic structures in S_E(U) with two binary operations, the second binary operation of which is the extended intersection operation</u>

Let (Ꞙ,Ⱥ), (ℭ,Ꝑ) and (Ħ,Ƨ) be SSs over U. Then,

1) (S_E(U),$\cup_R$,$\cap_\varepsilon$) is an additively and multiplicatively commutative and idempotent semiring with the identity element $\emptyset_\emptyset$.

Moreover, since (Ꞙ,Ⱥ)$\cup_R$ $\emptyset_E$=$\emptyset_E$ $\cup_R$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_\varepsilon$ $\emptyset_E$=$\emptyset_E$ $\cap_\varepsilon$(Ꞙ,Ⱥ)=$\emptyset_E$, the algebraic structure **(**S_E(U),$\cup_R$,$\cap_\varepsilon$) is a hemiring (Ali et al., 2011).

2) (S_E(U),$\cap_R$,$\cap_\varepsilon$) is an additively and multiplicatively commutative and idempotent semiring with the identity element $\emptyset_\emptyset$ where (ȺΔꝐ)∩Ƨ=∅ and Ⱥ ∩(ꝐΔ Ƨ)=∅.

Here, we also want to correct an error made in a previous study by Ali et al. (2011). It was stated that (S_E(U),$\cap_R$,$\cap_\varepsilon$) is a hemiring with the identity $\emptyset_\emptyset$. However, since (Ꞙ,Ⱥ)$\cap_R$ U_E=U_E $\cap_R$(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ) and (Ꞙ,Ⱥ)$\cap_\varepsilon$ U_E = U_E $\cap_\varepsilon$(Ꞙ,Ⱥ)$\neq U_E$, (S_E(U),$\cap_R$,$\cap_\varepsilon$) cannot be a hemiring. Moreover, since extended intersection distributes over restricted intersection from LHS and RHS, respectively where (ȺΔꝐ)∩Ƨ=∅ and Ⱥ ∩(ꝐΔ Ƨ)=∅. (S_E(U),$\cap_R$,$\cap_\varepsilon$) cannot be a hemiring.

3) (S_E(U),$\cap_\varepsilon$,$\cap_\varepsilon$) is an additively and multiplicatively commutative and idempotent semiring with the identity element $\emptyset_\emptyset$ and without zero.

4) $(S_E(U),\tilde{\cup},\cap_\varepsilon)$ is a multiplicatively commutative, additively and multiplicatively idempotent semiring with the identity element $\emptyset_\emptyset$ and without zero, where $Ⱥ\cap(ⱣΔƧ)=\emptyset$.

5) $(S_E(U),\tilde{\cap},\cap_\varepsilon)$ is a multiplicatively commutative, additively and multiplicatively idempotent semiring with the identity element $\emptyset_\emptyset$ and without zero, where $(ȺΔⱣ)\cap Ƨ=\emptyset$.

6) $(S_E(U),\cup_R,\cap_\varepsilon)$ is a bounded distributive lattice with the lower bound $\emptyset_E$ and the upper bound $\emptyset_\emptyset$. Since $(Ꞙ,Ⱥ)\cup_R(Ꞙ,Ⱥ)^r\neq\emptyset_\emptyset$ and $(Ꞙ,Ⱥ)\cap_\varepsilon(Ꞙ,Ⱥ)^r\neq U_E$, the algebraic structure $(S_E(U),\cup_R,\cap_\varepsilon)$ is non-complemented, bounded and distributive lattice; thus, it is not a Boolean algebra. (Ali et al., 2011; Qin and Hong, 2010)

<u>4.2.2.2. Algebraic structures with two binary operations in $S_Ⱥ(U)$:</u>

<u>i) Algebraic structures in $S_Ⱥ(U)$ with two binary operations, the second binary operation of which is the restricted intersection operation:</u>

1) $(S_Ⱥ(U),\cap_R,\cap_R)$, $(S_Ⱥ(U),\cap_\varepsilon,\cap_R)$, $(S_Ⱥ(U),\tilde{\cap},\cap_R)$ are additively and multiplicatively commutative and idempotent semirings with the identity element $U_Ⱥ$ and without zero.

2) $(S_Ⱥ(U),\cup_R,\cap_R)$ are additively and multiplicatively commutative and idempotent semirings with the identity element $U_Ⱥ$.

Moreover, since $(Ꞙ,Ⱥ)\cup_R\emptyset_Ⱥ=\emptyset_Ⱥ\cup_R(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R\emptyset_Ⱥ=\emptyset_Ⱥ\cap_R(Ꞙ,Ⱥ)=\emptyset_Ⱥ$, namely $\emptyset_Ⱥ$ is the zero of $(S_Ⱥ(U),\cup_R,\cap_R)$, $(S_Ⱥ(U),\cup_R,\cap_R)$ is a hemiring (Ali et al., 2011).

3) $(S_Ⱥ(U),\Delta_R,\cap_R)$ is an additively and multiplicatively commutative and multiplicatively idempotent semiring with the identity element $U_Ⱥ$.

Moreover, since $(Ꞙ,Ⱥ)\Delta_R\emptyset_Ⱥ=\emptyset_Ⱥ\Delta_R(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R\emptyset_Ⱥ=\emptyset_Ⱥ\cap_R(Ꞙ,Ⱥ)=\emptyset_Ⱥ$, namely $\emptyset_Ⱥ$ is the zero of $(S_Ⱥ(U),\Delta_R,\cap_R)$, $(S_Ⱥ(U),\Delta_R,\cap_R)$ is a hemiring.

Additionally, $(S_Ⱥ(U),\Delta_R,\cap_R)$ is a ring with the identity element, and since $(Ꞙ,Ⱥ)^2=(Ꞙ,Ⱥ)\cap_R(Ꞙ,Ⱥ)$, $(S_Ⱥ(U),\Delta_R,\cap_R)$ is a Boolean Ring. The fact that $(Ꞙ,Ⱥ)\Delta_R(Ꞙ,Ⱥ)=\emptyset_Ⱥ$ and $(Ꞙ,Ⱥ)\cap_R(\mathfrak{C},Ⱥ)=(\mathfrak{C},Ⱥ)\cap_R(Ꞙ,Ⱥ)$ is a natural consequence of the algebraic structure $(S_Ⱥ(U),\Delta_R,\cap_R)$ being a Bool ring (Eren and Çalışıcı, 2019).

4) $(S_Ⱥ(U),\cup_\varepsilon,\cap_R)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $U_Ⱥ$,

Moreover, since $(Ꞙ,Ⱥ)\cup_\varepsilon\emptyset_Ⱥ=\emptyset_Ⱥ\cup_\varepsilon(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_R\emptyset_Ⱥ=\emptyset_Ⱥ\cap_R(Ꞙ,Ⱥ)=\emptyset_Ⱥ$, namely $\emptyset_Ⱥ$ is the zero of $(S_Ⱥ(U),\cup_\varepsilon,\cap_R)$, $(S_Ⱥ(U),\cup_\varepsilon,\cap_R)$ is a hemiring.

5) $(S_{\mathcal{A}}(U),\Delta_{\varepsilon},\cap_{R})$ is an additively and multiplicatively commutative, multiplicatively idempotent semiring with the identity element $U_{\mathcal{A}}$,

Moreover, since $(\mathcal{F},\mathcal{A})\Delta_{\varepsilon}\emptyset_{\mathcal{A}}=\emptyset_{\mathcal{A}}\Delta_{\varepsilon}(\mathcal{F},\mathcal{A})=(\mathcal{F},\mathcal{A})$ and $(\mathcal{F},\mathcal{A})\cap_{R}\emptyset_{\mathcal{A}}=\emptyset_{\mathcal{A}}\cap_{R}(\mathcal{F},\mathcal{A})=\emptyset_{\mathcal{A}}$, namely $\emptyset_{\mathcal{A}}$ is the zero of $(S_{\mathcal{A}}(U),\Delta_{\varepsilon},\cap_{R})$, $(S_{\mathcal{A}}(U),\Delta_{\varepsilon},\cap_{R})$ is a hemiring (Sezgin and Çağman, 2025).

6) $(S_{\mathcal{A}}(U),\tilde{\cup},\cap_{R})$ is an additively and multiplicatively commutative and idempotent is a semiring with the identity element $U_{\mathcal{A}}$.

Moreover, since $(\mathcal{F},\mathcal{A})\tilde{\cup}\emptyset_{\mathcal{A}}=\emptyset_{\mathcal{A}}\tilde{\cup}(\mathcal{F},\mathcal{A})=(\mathcal{F},\mathcal{A})$ and $(\mathcal{F},\mathcal{A})\cap_{R}\emptyset_{\mathcal{A}}=\emptyset_{\mathcal{A}}\cap_{R}(\mathcal{F},\mathcal{A})=\emptyset_{\mathcal{A}}$, namely $\emptyset_{\mathcal{A}}$ is the zero of $(S_{\mathcal{A}}(U),\tilde{\cup},\cap_{R})$, $(S_{\mathcal{A}}(U),\tilde{\cup},\cap_{R})$ is a hemiring.

7) $(S_{\mathcal{A}}(U),\tilde{\Delta},\cap_{R})$ is an additively and multiplicatively commutative, multiplicatively idempotent semiring with the identity element $U_{\mathcal{A}}$.

Moreover, since $(\mathcal{F},\mathcal{A})\tilde{\Delta}\emptyset_{\mathcal{A}}=\emptyset_{\mathcal{A}}\tilde{\Delta}(\mathcal{F},\mathcal{A})=(\mathcal{F},\mathcal{A})$ and $(\mathcal{F},\mathcal{A})\cap_{R}\emptyset_{\mathcal{A}}=\emptyset_{\mathcal{A}}\cap_{R}(\mathcal{F},\mathcal{A})=\emptyset_{\mathcal{A}}$, namely $\emptyset_{\mathcal{A}}$ is the zero of $(S_{\mathcal{A}}(U),\tilde{\Delta},\cap_{R})$, $(S_{\mathcal{A}}(U),\tilde{\Delta},\cap_{R})$ is a hemiring (Sezgin and Yavuz, 2023b).

8) $(S_{\mathcal{A}}(U),\cup_{\varepsilon},\cap_{R})$ is a complemented, bounded, distributive lattice with the lower bound $\emptyset_{\mathcal{A}}$ and the upper bound $U_{\mathcal{A}}$.

In fact, it was presented that $(S_{\mathcal{A}}(U),\cup_{\varepsilon})$ and $(S_{\mathcal{A}}(U),\cap_{R})$ are commutative idempotent monoids with the identity element $\emptyset_{\mathcal{A}}$ and $U_{\mathcal{A}}$, respectively, $\cup_{\varepsilon}$ ve $\cap_{R}$ hold distributive laws in $S_{\mathcal{A}}(U)$, and restricted intersection distributes over extended union from both left and right sides in $S_{\mathcal{A}}(U)$.

Furthermore, since $(\mathcal{F},\mathcal{A})\cup_{\varepsilon}(\mathcal{F},\mathcal{A})^{r}=U_{\mathcal{A}}$ and $(\mathcal{F},\mathcal{A})\cap_{R}(\mathcal{F},\mathcal{A})^{r}=\emptyset_{\mathcal{A}}$, $(S_{\mathcal{A}}(U),\cup_{\varepsilon},\cap_{R},{}^{r})$ is a complemented, bounded and distributive lattice; thus, it is a Boolean algebra. Moreover, since it satisfies the De Morgan law, that is, $[(\mathcal{F},\mathcal{A})\cap_{R}(\mathfrak{C},\mathcal{A})]^{r}=(\mathcal{F},\mathcal{A})^{r}\cup_{\varepsilon}(\mathfrak{C},\mathcal{A})^{r}$ and $[(\mathcal{F},\mathcal{A})\cup_{\varepsilon}G,\mathcal{A})]^{r}=(\mathcal{F},\mathcal{A})^{r}\cap_{R}(\mathfrak{C},\mathcal{A})^{r}$. Thus, $(S_{\mathcal{A}}(U),\cup_{\varepsilon},\cap_{R},{}^{c})$ is a De Morgan Algebra.

Additionally, $(\mathcal{F},\mathcal{A})\cap_{R}(\mathcal{F},\mathcal{A})^{r}=\emptyset_{\mathcal{A}}\,\tilde{\subseteq}(\mathfrak{C},\mathcal{A})\cup_{\varepsilon}(\mathfrak{C},\mathcal{A})^{r}=U_{\mathcal{A}}$ for all $(\mathcal{F},\mathcal{A}),(\mathfrak{C},\mathcal{A})\in S_{A}(U)$, thus $(S_{\mathcal{A}}(U),\cup_{\varepsilon},\cap_{R},{}^{r})$ is a Kleene Algebra.

Additionally, it is known that $(\mathcal{F},\mathcal{A})\cap_{R}(\mathcal{F},\mathcal{A})^{r}=\emptyset_{\mathcal{A}}$ and if $(\mathcal{F},\mathcal{A})\cap_{R}(\mathfrak{C},\mathcal{A})=\emptyset_{\mathcal{A}}$, then $(\mathfrak{C},\mathcal{A})\,\tilde{\subseteq}\,(\mathcal{F},\mathcal{A})^{r}$. This shows that $(\mathcal{F},\mathcal{A})^{r}$ is the pseudo-complement of $(\mathcal{F},\mathcal{A})$. Furthermore, since $(\mathcal{F},\mathcal{A})^{r}\cup_{\varepsilon}((\mathcal{F},\mathcal{A})^{r})^{r}=U_{\mathcal{A}}$, $(S_{\mathcal{A}}(U),\cup_{\varepsilon},\cap_{R},{}^{r})$ satisfies Stone's unit property and thus, the algebraic structure $(S_{\mathcal{A}}(U),\cup_{\varepsilon},\cap_{R},{}^{r})$ is a Stone Algebra (Ali et al., 2011).

9) $(S_{A}(U),\cup_R,\cap_R\,,^r)$ and $(S_{A}(U),\tilde{\cup},\cap_R\,,^r)$ is a complemented, bounded, distributive lattice with the lower bound $\emptyset_{A}$ and the upper bound U_{A}, is therefore a Boolean algebra, De Morgan algebra, besides, Kleene algebra and Stone algebra.

Additionally $(S_{A}(U),^r,\cap_R)$ is an MV-algebra with the constant U_{A} (Ali et al., 2011).

To show that $(S_{A}(U),^r,\cap_R,U_{A})$ is an MV-algebra, we need to show that it satisfies the MV-algebra conditions.

- (MV1) $(S_{A}(U),\cap_R)$ is commutative monoid with the identity element U_{A}.
- (MV2) $((F,A)^r)^r=(F,A)$.
- (MV3) $(U_{A})^r\cap_R(F,A)=\emptyset_{A}\cap_R(F,A)=\emptyset_{A}=(U_{A})^r$.
- (MV4) $[(F,A)^r\cap_R(\mathfrak{C},A)]^r\cap_R(\mathfrak{C},A)=((\mathfrak{C},A)^r\cap_R(F,A)^r)^r\cap_R(F,A)$. Indeed,

$$\begin{aligned}
[(F,A)^r\cap_R(\mathfrak{C},A)]^r\cap_R(\mathfrak{C},A) &=[((F,A)^r)^r\cup_R(\mathfrak{C},A)^r]\cap_R(\mathfrak{C},A)\\
&=[(F,A)\cup_R(\mathfrak{C},A)^r]\cap_R(\mathfrak{C},A)\\
&=[(F,A)\cap_R(\mathfrak{C},A)]\cup_R[(\mathfrak{C},A)^r\cap_R G,A)]\\
&=[(F,A)\cap_R(\mathfrak{C},A)]\cup_R[(F,A)\cap_R(F,A)^r]\\
&=(F,A)\cap_R[(\mathfrak{C},A)\cup_R(F,A)^r]\\
&=(F,A)\cap_R[(\mathfrak{C},A)^r\cap_R(F,A)]^r\\
&=[(\mathfrak{C},A)^r\cap_R(F,A)]^r\cap_R(F,A)
\end{aligned}$$

Thus, $(S_{A}(U),^r,\cap_R)$ is an MV-algebra with the constant U_{A}.

<u>ii) Algebraic structures in $S_{A}(U)$ with two binary operations, the second binary operation of which is the extended intersection operation:</u>

1) $(S_{A}(U),\cap_\varepsilon,\cap_\varepsilon)$, $(S_{A}(U),\tilde{\cap},\cap_\varepsilon)$, $(S_{A}(U),\cap_R,\cap_\varepsilon)$ are additively and multiplicatively commutative and idempotent semirings with the identity element U_{A} and without zero.

2) $(S_{A}(U),\cup_R,\cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_A.

Moreover, since $(F,A)\cup_R\emptyset_{A}=\emptyset_{A}\cup_R(F,A)=(F,A)$ and $(F,A)\cap_\varepsilon\emptyset_{A}=\emptyset_{A}\cap_\varepsilon(F,A)=\emptyset_{A}$, namely $\emptyset_{A}$ is the zero of $(S_{A}(U),\cup_R,\cap_\varepsilon)$, $(S_{A}(U),\cup_R,\cap_\varepsilon)$ is a hemiring.

3) $(S_{A}(U),\cup_\varepsilon,\cap_\varepsilon)$ is an additively and multiplicatively commutative and idempotent semiring with the identity element U_{A}.

Moreover, since $(F,A)\cup_\varepsilon\emptyset_{A}=\emptyset_{A}\cup_\varepsilon(F,A)=(F,A)$ and $(F,A)\cap_\varepsilon\emptyset_{A}=\emptyset_{A}\cap_\varepsilon(F,A)=\emptyset_{A}$, namely $\emptyset_{A}$ is the zero of $(S_{A}(U),\cup_\varepsilon,\cap_\varepsilon)$, $(S_{A}(U),\cup_\varepsilon,\cap_\varepsilon)$ is a hemiring.

4) $(S_{Ⱥ}(U),\tilde{\cup},\cap_{\varepsilon})$ is an additively and multiplicatively commutative and idempotent semiring with the identity element $U_{Ⱥ}$.

Moreover, since $(Ꞙ,Ⱥ)\tilde{\cup}\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\tilde{\cup}(Ꞙ,Ⱥ)=(Ꞙ,Ⱥ)$ and $(Ꞙ,Ⱥ)\cap_{\varepsilon}\emptyset_{Ⱥ}=\emptyset_{Ⱥ}\cap_{\varepsilon}(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}$, namely $\emptyset_{Ⱥ}$ is the zero of $(S_{Ⱥ}(U),\tilde{\cup},\cap_{\varepsilon})$, $(S_{Ⱥ}(U),\tilde{\cup},\cap_{\varepsilon})$ is a hemiring.

5) $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_{\varepsilon})$ is a complemented, bounded, distributive lattice with the lower bound $\emptyset_{Ⱥ}$ and the upper bound $U_{Ⱥ}$.

In fact, it was presented that $(S_{Ⱥ}(U),\cup_{\varepsilon})$ and $(S_{Ⱥ}(U),\cap_{\varepsilon})$ are commutative idempotent monoids with the identity element $\emptyset_{Ⱥ}$ and $U_{Ⱥ}$, respectively, $\cup_{\varepsilon}$ ve $\cap_{\varepsilon}$ hold distributive laws in $S_{Ⱥ}(U)$, and extended intersection distributes over extended union from both left and right sides in $S_{Ⱥ}(U)$.

Furthermore, since $(Ꞙ,Ⱥ)\cup_{\varepsilon}(Ꞙ,Ⱥ)^{r}= U_{Ⱥ}$ and $(Ꞙ,Ⱥ)\cap_{\varepsilon}(Ꞙ,Ⱥ)^{r}= \emptyset_{Ⱥ}$, $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_{\varepsilon},{}^{r})$ is a complemented, bounded and distributive lattice; thus a Boolean algebra.

Moreover, since De Morgan law, that is $[(Ꞙ,Ⱥ)\cap_{\varepsilon}(\mathfrak{C},Ⱥ)]^{r}=(Ꞙ,Ⱥ)^{r}\cup_{\varepsilon}(\mathfrak{C},Ⱥ)^{r}$ and $[(Ꞙ,Ⱥ)\cup_{\varepsilon}G,Ⱥ)]^{r}=(Ꞙ,Ⱥ)^{r}\cap_{\varepsilon}(\mathfrak{C},Ⱥ)^{r}$ is satisfied, $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_{\varepsilon},{}^{c})$ is a De Morgan Algebra.

Additionally, $(Ꞙ,Ⱥ)\cap_{\varepsilon}(Ꞙ,Ⱥ)^{r}=\emptyset_{Ⱥ}\,\tilde{\subseteq}(\mathfrak{C},Ⱥ)\cup_{\varepsilon}(\mathfrak{C},Ⱥ)^{r}=U_{Ⱥ}$, for all $(Ꞙ,Ⱥ),(\mathfrak{C},Ⱥ)\in S_{Ⱥ}(U)$, thus $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_{\varepsilon},{}^{r})$ is a Kleene Algebra.

Additionally, it is known that $(Ꞙ,Ⱥ)\cap_{\varepsilon}(Ꞙ,Ⱥ)^{r}=\emptyset_{Ⱥ}$ and if $(Ꞙ,Ⱥ)\cap_{\varepsilon}(\mathfrak{C},Ⱥ)=\emptyset_{Ⱥ}$, then $(\mathfrak{C},Ⱥ)\,\tilde{\subseteq}(Ꞙ,Ⱥ)^{r}$. This shows that $(Ꞙ,Ⱥ)^{r}$ is the pseudo-complement of $(Ꞙ,Ⱥ)$. Furthermore, since $(Ꞙ,Ⱥ)^{r}\cup_{\varepsilon}((Ꞙ,Ⱥ)^{r})^{r}=Ⱥ$, $(S_{A}(U),\cup_{\varepsilon},\cap_{\varepsilon},{}^{r})$ satisfies Stone's unit property and thus, $(S_{Ⱥ}(U),\cup_{\varepsilon},\cap_{\varepsilon},{}^{r})$ is a Stone Algebra.

6) $(S_{Ⱥ}(U),\cup_{R},\cap_{\varepsilon},{}^{r})$ and $(S_{Ⱥ}(U),\tilde{\cup},\cap_{\varepsilon},{}^{r})$ are complemented, bounded, distributive lattice with the lower bound $\emptyset_{Ⱥ}$ and the upper bound $U_{Ⱥ}$, is therefore, a Boolean algebra, De Morgan algebra, Kleene algebra, and Stone algebra. Additionally $(S_{Ⱥ}(U),{}^{r},\cap_{\varepsilon})$ is an MV-algebra the constant element $U_{Ⱥ}$.

To show that $(S_{Ⱥ}(U),{}^{r},\cap_{\varepsilon},U_{Ⱥ})$ is an MV-algebra, we neeed show that it satisfies the MV-algebra conditions.

- (MV1) $(S_{Ⱥ}(U),\cap_{\varepsilon},U_{Ⱥ})$ commutative monoid with $U_{Ⱥ}$.
- (MV2) $((Ꞙ,Ⱥ)^{r})^{r}=(Ꞙ,Ⱥ)$.
- (MV3) $(U_{A})^{r}\cap_{\varepsilon}(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}\cap_{\varepsilon}(Ꞙ,Ⱥ)=\emptyset_{Ⱥ}=(U_{Ⱥ})^{r}$.
- (MV4) $[(Ꞙ,Ⱥ)^{r}\cap_{\varepsilon}(\mathfrak{C},Ⱥ)]^{r}\cap_{\varepsilon}(\mathfrak{C},Ⱥ)=((\mathfrak{C},Ⱥ)^{r}\cap_{\varepsilon}(Ꞙ,Ⱥ)^{r})^{r}\cap_{\varepsilon}(Ꞙ,Ⱥ)$. Indeed,

$$
\begin{aligned}
[(Ꞙ,Ⱥ)^{r}\cap_{\varepsilon}(\mathfrak{C},Ⱥ)]^{r}\cap_{\varepsilon}(\mathfrak{C},Ⱥ) &=[((Ꞙ,Ⱥ)^{r})^{r}\cup_{\varepsilon}(\mathfrak{C},Ⱥ)^{r}]\cap_{\varepsilon}(\mathfrak{C},Ⱥ)\\
&=[(Ꞙ,Ⱥ)\cup_{\varepsilon}(\mathfrak{C},Ⱥ)^{r}]\cap_{\varepsilon}(\mathfrak{C},Ⱥ)\\
&=[(Ꞙ,Ⱥ)\cap_{\varepsilon}(\mathfrak{C},Ⱥ)]\cup_{\varepsilon}[(\mathfrak{C},Ⱥ)^{r}\cap_{\varepsilon}(\mathfrak{C},Ⱥ)]
\end{aligned}
$$

$=[(\text{Ꞙ},\text{Ꞁ})\cap_{\varepsilon}(\mathfrak{C},\text{Ꞁ})]\cup_{\varepsilon}[(\text{Ꞙ},\text{Ꞁ})\cap_{\varepsilon}(\text{Ꞙ},\text{Ꞁ})^{r}]$

$= (\text{Ꞙ},\text{Ꞁ})\cap_{\varepsilon}[(\mathfrak{C},\text{Ꞁ})\cup_{\varepsilon}(\text{Ꞙ},\text{Ꞁ})^{r}]$

$= (\text{Ꞙ},\text{Ꞁ})\cap_{\varepsilon}[(\mathfrak{C},\text{Ꞁ})^{r}\cap_{\varepsilon}(\text{Ꞙ},\text{Ꞁ})]^{r}$

$= [(\mathfrak{C},\text{Ꞁ})^{r}\cap_{\varepsilon}(\text{Ꞙ},\text{Ꞁ})]^{r}\cap_{\varepsilon}(\text{F},\text{Ꞁ})$

Thus, $(S_{\text{Ꞁ}}(U),{}^{r},\cap_{\varepsilon})$ is an MV-algebra with the constant $U_{\text{Ꞁ}}$.

5. CONCLUSION

This work presents a thorough examination of all of the characteristics of restricted intersection and extended intersection operations, which are key concepts in SS theory. First of all, the intersection operations are viewed historically, demonstrating the incompleteness of the restricted intersection definition by Ali et al., 2009 and Ali et al., 2011. As the definition has rough edges, the claims in all papers examining the characteristics of the concept and applying it suffer from some problematic circumstances, as it is neglected that the parameter sets of the SSs contained in restricted intersection may also be disjoint. Following the inadequate definition of restricted intersection, some theorems and assertions in previous research on restricted and extended intersection operations were presented without proofs, or the proofs were wrong or missing sections. First and foremost, the presentation of the concept of restricted union is renewed in this study in a new manner that eliminates any incorrectness. This study typically gives proofs based on function equality and corrects any faulty parts in these studies. When evaluating the properties and distributive rules of restricted and extended intersection operations, the case in which the intersection of the parameter sets of the SSs is empty is always considered in the statements and proofs. Moreover, the relationships between restricted and extended intersection operations and the soft subset proposed by Pei and Miao (2005) are also examined in relation to their classical set counterparts. We also add many more properties to the properties that were previously supplied in this topic. In the set of SSs with a fixed parameter set and in the set of sets over the universe, the distribution rules and absorption laws are thoroughly investigated, and the algebraic structures formed by these operations individually and in combination with other SS operations are thoroughly examined with their detailed proofs by also correcting the incorrect parts in the literature in this regard. Boolean algebra, De Morgan algebra, MV-algebra, Kleene algebra, Stone algebra, semiring, hemiring, bounded distributive lattice, monoid, and bounded semi-lattice are some examples of these algebraic structures associated with restricted and extended intersection operations. Furthermore, if a distribution rule does not hold, we specify the condition(s) under which the assertions do. According to these perspectives, this paper represents the most comprehensive analysis of SSs in the literature that is currently available in terms of restricted and extended intersection operations, taking into account all of the earlier research on the topic such as Ali et al., 2009; Ali et al., 2011; Maji et al., 2003; Pei and Miao, 2005; Qin and Hong, 2010; Sen, 2014; Sezgin and Atagün, 2011; Singh and Onyeozili, 2012c as well as Neog and Sut, 2011; Fu, 2011; Ge and Yang S, 2011; Zhu and Wen, 2013; Onyeozili and Gwary, 2014; Husain and Shivani, 2018)), as there isn't any literature available at the moment with such a thorough analysis. As SS operations serve as the theoretical foundation for several approaches to soft computing, which open the door to a variety of applications, such as the development of new SS-based cryptography techniques and decision-making processes and the studies on soft algebraic structures have been the basis for understanding the applications of SS algebra in both classical and non-classical logic, this paper fills a significant gap for the past and future literature by advancing both the theoretical and practical aspects of SS theory. Future research can be employed from the perspective of this study to address other basic SS operations, such as restricted and extended union, difference, and symmetric difference operations.

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