



On generalized Matsumoto metrics with a special π -form

Amr Soleiman¹, Ebtsam H. Taha^{*2}

¹*Department of Mathematics, Faculty of Science, Jouf University, Skaka, KSA*

²*Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*

Abstract

We explore a generalization of Matsumoto metric intrinsically. Given a Finsler manifold (M, F) which admits a concurrent π -vector field $\bar{\varphi}$, we consider the change $\hat{F}(x, y) = \frac{F^2(x, y)}{F(x, y) - \Phi(x, y)}$, where Φ is the associated concurrent π -form with $F(x, y) > \Phi(x, y)$ for all $(x, y) \in \mathcal{T}M$. We find the condition under which the generalized ϕ -Matsumoto metric $\hat{F}$ is a Finsler metric. Moreover, the relations between the associated Finslerian geometric objects of $\hat{F}$ and F are obtained, namely, the relations between angular metric tensors, metric tensors, Cartan torsions, geodesic sprays, Barthel connections (along with its curvature) and Berwald connections. Further, we prove that the Finsler metrics F and $\hat{F}$ can never be projectively related. Also, a condition for the π -vector field $\bar{\varphi}$ to be concurrent with respect to $\hat{F}$ is acquired. Moreover, an example of a rational Finsler metric admitting a concurrent π -vector field together with the associated change $\hat{F}$ is provided. Finally, we find the conditions that preserve the almost rationality property of a Finsler metric F under the ϕ -Matsumoto change.

Mathematics Subject Classification (2020). 53C60, 53B40, 58B20.

Keywords. Finsler metric, generalized Matsumoto metric, geodesic spray, concurrent vector field, Berwald connection, almost rational Finsler metric

1. Introduction

The theory of special Finsler spaces constitutes a remarkable domain of mathematical inquiry, characterized by its profound structural depth and its wide-ranging utility in modelling some physical and biological systems. The richness of special Finsler spaces stems primarily from the behaviour of π -tensor fields, specifically the torsion and curvature tensors (e.g., associated with the Cartan connection). When these tensors satisfy particular geometric identities or take on certain forms, they define the various classes of special Finsler spaces. This flexibility allows for a much broader classification of manifold structures than is possible in Riemannian geometry, where the rigid dependence on a quadratic metric restricts the variety of obtainable spaces. Historically, much of the progress

*Corresponding Author.

Email addresses: asoliman@ju.edu.sa, amrsoleiman@yahoo.com (A. Soleiman) ,
ebtsam@sci.cu.edu.eg, ebtsam.h.taha@hotmail.com (E. H. Taha)

Received: 05.01.2025; Accepted: 24.09.2025

in this area has relied on local coordinate expressions, a framework pioneered and extensively documented by M. Matsumoto [7, 9, 12] and his contemporaries. However, while local methods are powerful for computation, they often obscure the underlying global geometric properties. In response to this, a more modern, intrinsic approach has emerged to treat these spaces as unified geometric objects. Although such coordinate-free investigation remain comparatively rare in the literature, the works of A. Tamim, L. Youssef, and their collaborators (see e.g., [13–15, 26, 28, 29]) have provided critical advancements, establishing a rigorous basis for the intrinsic study of Finslerian structures.

The Randers metric is defined by $F = \alpha + B$, with α is defined by a Riemannian metric and B is a 1-form on the manifold M . It named after G. Randers, 1941, who introduced a simple Finslerian metric by this change. It is used to construct a generalized field theory that would comprise gravity and electromagnetism. Randers metrics are important as they represent solutions to Zermelo’s navigation problem in the case of a weak wind (see, e.g. [1] for a wide perspective of Randers or the more general notion of a wind Finsler metric). In 1974, M. Matsumoto studied Randers spaces in a more general setting, by assuming that α is a Finsler metric [7]. Later, generalized Randers metrics were investigated in [20]. Another interesting metric is the Matsumoto (Matsumoto’s slope of a mountain) metric defined by $F = \frac{\alpha^2}{\alpha - B}$, where B was originally taken to be induced by earth’s gravity. It was introduced by Matsumoto as a realization of the idea of a slope measure of a mountain with respect to a time measure [8, 11]. A further local study of Matsumoto metric and its generalizations can be found in [4, 12, 22–24]. On the other hand, the existence of a concurrent vector field on Finsler spaces has been studied firstly by Tachibana [19]. The existence of a concurrent vector field is a very rigid property as, for example, a 3-dimensional Finsler manifold, a Finsler surface, a Landsberg space, a C -reducible Finsler space admitting a concurrent vector field is Riemannian [9]. Also, a generalization of a concurrent vector field, which is called a semi-concurrent vector field, has been investigated in [31].

In this paper, we provide an intrinsic investigation of what we called a *generalized ϕ -Matsumoto metric*. Our intrinsic formulation and index-free proofs in the first three sections give rise to simple compact results that hold globally on the manifold. More precisely, in §2, we provide a coordinate-free study of a generalized Matsumoto metric with a special π -form. By a generalized Matsumoto metric we mean the change of a Finsler metric F (not necessarily Riemannian) by a 1-form B , defined by, $\hat{F} = \frac{F^2}{F - B}$, with $F(x, y) > \Phi(x, y)$ for all $(x, y) \in \mathcal{T}M$. We consider a Finsler manifold (M, F) that admits a concurrent π -vector field $\bar{\varphi}$ and find the corresponding π -form Φ . Thus the associated 1-form $\Phi(x, y)$ is used to define the ϕ -Matsumoto change (3.5). We analyse intrinsically some of the geometric objects associated with $\hat{F}$, namely, the supporting form $\hat{\ell}$, the angular metric tensor $\hat{h}$, the Finsler metric tensor $\hat{g}$ and the Cartan torsion $\hat{\mathbf{T}}$. Also, we characterize the non-degeneracy property of the metric tensor $\hat{g}$, that is, $\hat{g}$ is non-degenerate if and only if $F(1 + 2g(\bar{\varphi}, \bar{\varphi})) - 3\Phi \neq 0$. Then, in §3, the related geodesic sprays corresponding to this change as well as the relation between the two Barthel connections Γ and $\hat{\Gamma}$ are obtained. As a consequence, we prove that two sprays are projectively related if and only if $\bar{\varphi} = 0$, which contradicts the assumption of π -concurrent vector field. Then, the generalized Matsumoto change of Berwald connection implies that the π -vector field $\bar{\varphi}$ is never concurrent with respect to the Finsler metric $\hat{F}$. Finally, in §4, an example of a rational Finsler metric F that admits a π -concurrent vector field is given and its ϕ -Matsumoto change is obtained. Finally, we tried to answer the question: if F is an almost rational Finsler metric, under what conditions $\hat{F}$ is an almost rational Finsler metric?

2. Preliminaries of Finsler geometry

Let M be an n -dimensional para-compact smooth manifold and $\pi : TM \rightarrow M$ its tangent bundle. The vertical subbundle $V(TM)$ is defined to be $\ker(d\pi)$. Let $\mathcal{T}M := TM/\{0\}$ be the slit tangent bundle, $\dot{\pi} : \mathcal{T}M \rightarrow M$ and the pullback bundle of the tangent bundle be denoted by $\dot{\pi}^{-1}(TM)$ over $\mathcal{T}M$. It is called also the Finsler bundle [18]. Further, $\mathfrak{F}(\mathcal{T}M)$ denotes the algebra of smooth functions on $\mathcal{T}M$ and $\mathfrak{X}(\pi)$ the $\mathfrak{F}(\mathcal{T}M)$ -module of differentiable sections of $\dot{\pi}^{-1}(\mathcal{T}M)$. The elements of $\mathfrak{X}(\pi)$ will be called π -vector fields and denoted by barred letters $\overline{X}$.

We have the short exact sequence [3, 27]

$$0 \rightarrow \dot{\pi}^{-1}(\mathcal{T}M) \xrightarrow{\gamma} T\mathcal{T}M \xrightarrow{\rho} \dot{\pi}^{-1}(\mathcal{T}M) \rightarrow 0,$$

where γ is the natural injection ($\gamma : \dot{\pi}^{-1}(\mathcal{T}M) \rightarrow V(\mathcal{T}M)$ is an isomorphism) and $\rho := (\dot{\pi}, d\pi)$. The vertical endomorphism J is the map $J : T\mathcal{T}M \rightarrow T\mathcal{T}M$ defined by $J = \gamma \circ \rho$. For all $f \in \mathfrak{F}(\mathcal{T}M)$, $W \in \mathfrak{X}(\mathcal{T}M)$, J satisfies:

$$[fW, J] = f[W, J] + df \wedge i_W J - d_J f \otimes W, \quad (2.1)$$

where df is the exterior derivative of f , the derivative $d_J := [i_J, d] = i_J \circ d - d \circ i_J$ and i_W is the interior product with respect to W defined by $i_W f = 0$, $i_W J = J(W)$. Moreover, we have:

$$i_{\mathcal{C}} J = 0 \quad \text{and} \quad [\mathcal{C}, J] = -J, \quad (2.2)$$

where the vector field defined by $\mathcal{C} := \gamma \bar{\eta}$ is called Liouville vector field and $\bar{\eta}(u) = (u, u)$ for all u in $\mathcal{T}M$. Moreover, a spray on M is a smooth vector field G on $\mathcal{T}M$ such that $JG = \mathcal{C}$ and $[\mathcal{C}, G] = G$.

For a linear connection D on $\dot{\pi}^{-1}(\mathcal{T}M)$, we have $K : T\mathcal{T}M \rightarrow \dot{\pi}^{-1}(\mathcal{T}M)$ which is defined by $K(W) = D_W \bar{\eta}$. Thereby, the horizontal space at $u \in TM$ is $H_u(\mathcal{T}M) := \{W \in T_u(\mathcal{T}M) : K(W) = 0\}$. The connection D is said to be regular if for all $u \in TM$, we have $T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M)$. For a regular connection D , the vector bundle maps $\rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are isomorphisms. In this case, the map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ is called the horizontal map of D . A famous regular connection is the Berwald connection which is defined by [3], [25, Proposition 4.4]

$$\gamma D^\circ_{hZ} \overline{W} := v[hZ, JW], \quad D^\circ_{\gamma \overline{Z}} \rho W := \rho[\gamma \overline{Z}, \beta \overline{W}], \quad (2.3)$$

where $h := \beta \circ \rho$ is the horizontal projector of D and $v := I - \beta \circ \rho$ is the vertical projector of D .

A nonlinear connection on M is a vector 1-form Γ on TM which is smooth on $\mathcal{T}M$ and continuous on TM such that $J\Gamma = J$ and $\Gamma J = -J$ [3]. Consequently, the horizontal and vertical projectors associated with Γ are given, respectively, by

$$h := \frac{1}{2}(I + \Gamma), \quad v := \frac{1}{2}(I - \Gamma), \quad (2.4)$$

Moreover, the curvature of Γ is defined by $\mathfrak{R} := -\frac{1}{2}[h, h]$, which can be computed using Frölicher-Nijenhuis bracket $[\mathbb{K}, \mathbb{L}]$ of two vector 1-forms $\mathbb{K}$ and $\mathbb{L}$ as follows [2]:

$$\begin{aligned} [\mathbb{K}, \mathbb{L}](W, Z) &= [\mathbb{K}W, \mathbb{L}Z] + [\mathbb{L}W, \mathbb{K}Z] + \mathbb{K}\mathbb{L}[W, Z] + \mathbb{L}\mathbb{K}[W, Z] \\ &\quad - \mathbb{K}[\mathbb{L}W, Z] - \mathbb{K}[W, \mathbb{L}Z] - \mathbb{L}[\mathbb{K}W, Z] - \mathbb{L}[W, \mathbb{K}Z]. \end{aligned} \quad (2.5)$$

In particular, the vector 2-form $N_{\mathbb{L}}$ defined by

$$N_{\mathbb{L}}(W, Z) := \frac{1}{2}[\mathbb{L}, \mathbb{L}](W, Z) = [\mathbb{L}W, \mathbb{L}Z] + \mathbb{L}^2[W, Z] - \mathbb{L}[\mathbb{L}W, Z] - \mathbb{L}[W, \mathbb{L}Z] \quad (2.6)$$

is said to be the Nijenhuis torsion of a vector 1-form $\mathbb{L}$. For example, we have $N_J = 0$ and $J^2 = 0$ which give

$$[JW, JZ] = J[W, JZ] + J[JW, Z]. \quad (2.7)$$

Definition 2.1. A Finsler structure (or Finsler metric) on M is a mapping

$$F : TM \longrightarrow [0, \infty)$$

such that:

- (a) F is C^∞ on $\mathcal{T}M$ and C^0 on TM ,
- (b) F is positively homogeneous of degree 1 in the directional argument y , that is $\mathcal{L}_W F = F$, where $\mathcal{L}_W$ is the Lie derivative in the direction of W ,
- (c) the Hilbert 2-form $dd_J E$ has a maximal rank, where $E = \frac{1}{2}F^2$ is the Finsler energy function.

The Finsler metric tensor g induced by F on $\pi^{-1}(TM)$ is defined as follows [27]

$$g(\rho W, \rho Z) := dd_J E(JW, Z), \quad \forall W, Z \in \mathfrak{X}(TM). \quad (2.8)$$

In this case, the pair (M, F) is called a Finsler manifold and F is a regular Finsler metric or simply a Finsler metric.

According to [6], the Finsler metric tensor g defined in Definition 2.1 is positive definite. The main reference for the notion of a conic pseudo-Finsler structure is [5].

Definition 2.2. A conic sub-bundle of TM is a non-empty open subset $\mathcal{D} \subset \mathcal{T}M$ that is invariant under scaling of its tangent vectors by positive real numbers and satisfies $\pi(\mathcal{D}) = M$. Assume that, $\forall x \in M$, $\mathcal{D}_x := D \in T_x M$ is a connected set. A conic pseudo-Finsler structure F on M is a smooth mapping such that

$$F : \mathcal{D} \longrightarrow [0, \infty),$$

F is positively homogeneous of degree 1 in the directional argument y and the Hilbert 2-form $dd_J E$ has a maximal rank. The Finsler metric tensor g induced by F on $\pi_{\mathcal{D}}^{-1}(TM)$ [†] is defined by

$$g(\rho W, \rho Z) := dd_J E(JW, Z), \quad \forall W, Z \in \mathfrak{X}(\mathcal{D}).$$

The triple $(M, \mathcal{D}, F)$ (or simply, (M, F)) is called a conic pseudo-Finsler manifold.

Remark 2.3. On a conic domain $\mathcal{D}$, we have again the short exact sequence (see e.g., [10])

$$0 \longrightarrow \pi_{\mathcal{D}}^{-1}(\mathcal{T}M) \xrightarrow{\gamma} T\mathcal{D} \xrightarrow{\rho} \pi_{\mathcal{D}}^{-1}(\mathcal{T}M) \longrightarrow 0,$$

with the obvious modifications of the definitions of γ and ρ .

Further, the normalized supporting element (or supporting form) is $\ell := F^{-1}i_{\bar{\eta}} g$ and the angular metric tensor $\bar{h} := g - \ell \otimes \ell$. For a Finsler manifold, its geodesic spray G satisfies $i_G dd_J E = -dE$. Moreover, the Barthel connection Γ can be written in terms of the geodesic spray as $\Gamma = [J, G]$ (see, [3]). Another interesting regular connection is the Cartan connection ∇ which is the uniquely determined by [25]

- (i) $2g(\nabla_{\gamma \bar{X}} \bar{Y}, \bar{Z}) = \gamma \bar{X} \cdot g(\bar{Y}, \bar{Z}) + g(\bar{Y}, \rho[\beta \bar{Z}, \gamma \bar{X}]) + g(\bar{Z}, \rho[\gamma \bar{X}, \beta \bar{Y}]),$
- (ii) $2g(\nabla_{\beta \bar{X}} \rho Y, \rho Z) = \beta \bar{X} \cdot g(\bar{Y}, \bar{Z}) + \beta \bar{Y} \cdot g(\bar{Z}, \bar{X}) - \beta \bar{Z} \cdot g(\bar{X}, \bar{Y})$
 $- g(\bar{X}, \rho[\beta \bar{Y}, \beta \bar{Z}]) + g(\bar{Y}, \rho[\beta \bar{Z}, \beta \bar{X}]) + g(\bar{Z}, \rho[\beta \bar{X}, \beta \bar{Y}]).$

3. ϕ -Matsumoto change

In this section, we give an intrinsic investigation of what we call generalized ϕ -Matsumoto change $F \longrightarrow \hat{F}$. We find out the relation between the supporting forms (ℓ and $\hat{\ell}$), the angular metric tensors ($\bar{h}$ and $\hat{\bar{h}}$), Finsler metric tensors (g and $\hat{g}$) and the Cartan torsions ($\mathbf{T}$ and $\hat{\mathbf{T}}$), corresponding to this change. Moreover, the condition which makes $\hat{F}$ non-degenerate is derived.

[†] $\pi_{\mathcal{D}}$ is the restriction of π on $\mathcal{D}$

Definition 3.1 ([27]). Let (M, F) be a Finsler manifold. A non-vanishing π -vector field $\bar{\varphi}$ is called a concurrent π -vector field if it satisfies

$$\nabla_{\beta\bar{W}} \bar{\varphi} = -\bar{W} = D_{\beta\bar{W}}^\circ \bar{\varphi}, \quad \nabla_{\gamma\bar{W}} \bar{\varphi} = 0 = D_{\gamma\bar{W}}^\circ \bar{\varphi}. \quad (3.1)$$

where ∇ (respectively, D°) is the Cartan (respectively, Berwald) connection associated with F .

Consequently, the π -form $\phi := i_{\bar{\varphi}} g$ associated with $\bar{\varphi}$ has the properties

$$(\nabla_{\beta\bar{W}} \phi)(\bar{Z}) = -g(\bar{W}, \bar{Z}) = (D_{\beta\bar{W}}^\circ \phi)(\bar{Z}), \quad (\nabla_{\gamma\bar{W}} \phi)(\bar{Z}) = 0 = (D_{\gamma\bar{W}}^\circ \phi)(\bar{Z}). \quad (3.2)$$

Let us fix our notation throughout the whole paper:

- $\bar{\varphi}$ denotes a concurrent π -vector field with respect to F ,
- ϕ is the π -form associated with $\bar{\varphi}$,
- $\Phi := g(\bar{\varphi}, \bar{\eta}) = \phi(\bar{\eta})$.

Remark 3.2. For a Finsler manifold (M, F) , a π -vector field $\bar{Z} \in \mathfrak{X}(\pi)$ is independent of the directional argument y if, and only if, $D_{\gamma\bar{W}}^\circ \bar{Z} = 0$ for all $\bar{W} \in \mathfrak{X}(\pi)$. Similarly, a scalar (vector) π -form $\bar{\omega}$ is independent of the directional argument y if, and only if, $D_{\gamma\bar{W}}^\circ \bar{\omega} = 0$ for all $\bar{W} \in \mathfrak{X}(\pi)$. Thus, a concurrent π -vector field $\bar{\varphi}$ and its associated π -form ϕ are independent of y [27, Theorem 3.7]. Moreover, we have

$$i_{\gamma\bar{\varphi}} J = 0, \quad d_J p^2 = 0, \quad p^2 := \phi(\bar{\varphi}) = g(\bar{\varphi}, \bar{\varphi}). \quad (3.3)$$

Lemma 3.3 ([16, 27]). Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$ with associated π -form ϕ . Then, for all $X \in \mathfrak{X}(TM)$ and $\bar{W} \in \mathfrak{X}(\pi)$ we have:

- (a) $d_J \Phi(\gamma\bar{W}) = 0$, $d_J \Phi(\beta\bar{W}) = d\Phi(\gamma\bar{W}) = D_{\gamma\bar{W}}^\circ \Phi = \phi(\bar{W})$,
 $d\Phi(X) = \phi(KX) - F\ell(\rho X)$,
- (b) $d_J F(\gamma\bar{W}) = 0$, $d_J F(\beta\bar{W}) = dF(\gamma\bar{W}) = D_{\gamma\bar{W}}^\circ F = \ell(\bar{W})$,
 $dF(X) = dF(\gamma KX) = \ell(KX)$,
- (c) $d_h \Phi(\beta\bar{W}) = d\Phi(\beta\bar{W}) = D_{\beta\bar{W}}^\circ \Phi = -F\ell(\bar{W})$, $d\Phi(G) = -F^2$,
 $(D_G^\circ \phi)(\bar{W}) = -g(\bar{W}, \bar{\eta}) = -F\ell(\bar{W})$,
- (d) $d_h F(\beta\bar{W}) = dF(\beta\bar{W}) = D_{\beta\bar{W}}^\circ F = 0$,
- (e) $(D_{\gamma\bar{W}}^\circ \ell)(\bar{Z}) = (\nabla_{\gamma\bar{W}} \ell)(\bar{Z}) = F^{-1}h(\bar{W}, \bar{Z})$, $(D_G^\circ \ell)(\bar{W}) = (\nabla_G \ell)(\bar{W}) = 0$,
- (f) $dd_J E(\gamma\bar{W}, \beta\bar{Z}) = g(\bar{W}, \bar{Z})$, $\rho[G, X] = D_G^\circ \rho X - KX$,
- (g) for an arbitrary smooth function f of two variables F and Φ , we have

$$D_{\gamma\bar{W}}^\circ f(F, \Phi) = d_J f(\beta\bar{W}) = \frac{\partial f}{\partial F} \ell(\bar{W}) + \frac{\partial f}{\partial \Phi} \phi(\bar{W}). \quad (3.4)$$

Definition 3.4. Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$ with the associated π -form ϕ . Consider the following change

$$\hat{F}(x, y) = \frac{F^2(x, y)}{F(x, y) - \Phi(x, y)}, \quad (3.5)$$

with $\Phi = g(\bar{\varphi}, \bar{\eta}) = \phi(\bar{\eta})$ defined on the conic domain $\mathcal{D} := \{(x, y) \in TM \mid F(x, y) > \Phi(x, y)\}$. If $\hat{F}$ defines a conic pseudo-Finsler structure on M , then $\hat{F}$ is referred to as generalized ϕ -Matsumoto metric.

If, in particular, F is a Riemannian metric and ϕ is an 1-form on M , then $\hat{F}$ is called a Matsumoto metric. Also, when F is a Finsler metric and ϕ is an 1-form on M , $\hat{F}$ is called a generalized Matsumoto metric. The geometric objects corresponding to $\hat{F}$ will be

denoted by hat letters such as $\hat{\beta}, \hat{\ell}, \hat{g}, \dots$, etc. Thus, clearly we get

$$G \cdot \hat{F} = d\hat{F}(G) = \frac{F(F-2\Phi)}{(F-\Phi)^2} dF(G) + \frac{F^2}{(F-\Phi)^2} d\Phi(G) = -\frac{F^4}{(F-\Phi)^2}, \quad (3.6)$$

$$W \cdot \hat{F} = d\hat{F}(W) = \frac{F(F-2\Phi)}{(F-\Phi)^2} dF(W) + \frac{F^2}{(F-\Phi)^2} d\Phi(W). \quad (3.7)$$

Proposition 3.5. *Let (M, F) be a Finsler manifold admitting concurrent π -vector field $\bar{\varphi}$. Under the ϕ -Matsumoto change (3.5), we have:*

(1) *The total derivative of the Finsler energy functions $d\hat{E}$ and dE are related by*

$$d\hat{E} = \frac{F^2(F-2\Phi)}{(F-\Phi)^3} dE + \frac{F^4}{(F-\Phi)^3} d\Phi. \quad (3.8)$$

(2) *The supporting forms $\hat{\ell}$ and ℓ are related by*

$$\hat{\ell} = \frac{F(F-2\Phi)}{(F-\Phi)^2} \ell + \frac{F^2}{(F-\Phi)^2} \phi. \quad (3.9)$$

(3) *The vertical counterpart for Berwald connections $\hat{D}_{\gamma\bar{W}}^\circ \bar{Z}$ and $D_{\gamma\bar{W}}^\circ \bar{Z}$ are related by*

$$\hat{D}_{\gamma\bar{W}}^\circ \bar{Z} = D_{\gamma\bar{W}}^\circ \bar{Z}. \quad (3.10)$$

(4) *The angular metric tensors $\hat{h}$ and h are related by*

$$\begin{aligned} \hat{h}(\bar{W}, \bar{Z}) &= \frac{F^2(F-2\Phi)}{(F-\Phi)^3} h(\bar{W}, \bar{Z}) + \frac{2F^4}{(F-\Phi)^4} \phi(\bar{W}) \phi(\bar{Z}) + \frac{2\Phi^2 F^2}{(F-\Phi)^4} \ell(\bar{W}) \ell(\bar{Z}) \\ &\quad - \frac{2\Phi F^3}{(F-\Phi)^4} \left\{ \phi(\bar{W}) \ell(\bar{Z}) + \phi(\bar{Z}) \ell(\bar{W}) \right\}. \end{aligned} \quad (3.11)$$

Proof. Under the ϕ -Matsumoto change (3.5):

(1) We have

$$\begin{aligned} d\hat{E}(W) &= \hat{F} d\hat{F}(W) = \frac{F^2}{F-\Phi} \left\{ \frac{F(F-2\Phi)}{(F-\Phi)^2} dF(W) + \frac{F^2}{(F-\Phi)^2} d\Phi(W) \right\} \\ &= \frac{F^3(F-2\Phi)}{(F-\Phi)^3} dF(W) + \frac{F^4}{(F-\Phi)^3} d\Phi(W). \end{aligned} \quad (3.12)$$

(2) Taking into account Lemma 3.3 (a), (b), it follows that

$$\begin{aligned} \hat{\ell}(\bar{W}) &= d_J \hat{F}(\hat{\beta}\bar{W}) = d_J \hat{F}(\beta\bar{W}) \\ &= \frac{\partial \hat{F}}{\partial F} d_J F(\beta\bar{W}) + \frac{\partial \hat{F}}{\partial \Phi} d_J \Phi(\beta\bar{W}) \frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\bar{W}) + \frac{F^2}{(F-\Phi)^2} \phi(\bar{W}). \end{aligned}$$

(3) Since the difference between the horizontal maps $\hat{\beta}$ and β is a vertical vector field, that is, $\hat{\beta} = \beta + \gamma\bar{\mu}$, for some $\bar{\mu} \in \mathfrak{X}(\pi)$. Using the facts that $\rho \circ \gamma = 0$, the vertical distribution is completely integrable and $D_{\gamma\bar{W}}^\circ \bar{Z} = \rho[\gamma\bar{W}, \beta\bar{Z}]$ [17], hence we get

$$\hat{D}_{\gamma\bar{W}}^\circ \bar{Z} = \rho[\gamma\bar{W}, \hat{\beta}\bar{Z}] = \rho[\gamma\bar{W}, \beta\bar{Z}] + \rho[\gamma\bar{W}, \gamma\bar{\mu}] = \rho[\gamma\bar{W}, \beta\bar{Z}] = D_{\gamma\bar{W}}^\circ \bar{Z}.$$

(4) Using items (2), (3) above, Lemma 3.3 (a), (b), (e), together with Definition 3.1, one can show that

$$\begin{aligned}
 \hat{h}(\overline{W}, \overline{Z}) &= \hat{F} \left(\hat{D}_{\gamma \overline{W}}^{\circ} \hat{\ell} \right) (\overline{Z}) = \hat{F} \left(D_{\gamma \overline{W}}^{\circ} \hat{\ell} \right) (\overline{Z}) \\
 &= \hat{F} D_{\gamma \overline{W}}^{\circ} \left(\frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\overline{Z}) + \frac{F^2}{(F-\Phi)^2} \phi(\overline{Z}) \right) \\
 &= \hat{F} \left\{ \left(D_{\gamma \overline{W}}^{\circ} \frac{F(F-2\Phi)}{(F-\Phi)^2} \right) \ell(\overline{Z}) + \left(D_{\gamma \overline{W}}^{\circ} \frac{F^2}{(F-\Phi)^2} \right) \phi(\overline{Z}) \right\} \\
 &\quad + \hat{F} \left\{ \frac{F(F-2\Phi)}{(F-\Phi)^2} \left(D_{\gamma \overline{W}}^{\circ} \ell \right) (\overline{Z}) + \frac{F^2}{(F-\Phi)^2} \left(D_{\gamma \overline{W}}^{\circ} \phi \right) (\overline{Z}) \right\} \\
 &= \frac{F^2}{F-\Phi} \left\{ -\frac{2\Phi^2}{(\Phi-F)^3} \ell(\overline{W}) \ell(\overline{Z}) + \frac{2\Phi F}{(\Phi-F)^3} \phi(\overline{W}) \ell(\overline{Z}) \right. \\
 &\quad \left. + \frac{2\Phi F}{(\Phi-F)^3} \ell(\overline{W}) \phi(\overline{Z}) + \frac{2F^2}{(F-\Phi)^3} \phi(\overline{W}) \phi(\overline{Z}) \right\} \\
 &\quad + \frac{F^2}{F-\Phi} \frac{F(F-2\Phi)}{(F-\Phi)^2} F^{-1} \hat{h}(\overline{W}, \overline{Z}).
 \end{aligned}$$

Hence, the result follows. $\square$

Proposition 3.6. *The Finsler metric tensor $\hat{g}$ associated with the generalized ϕ -Matsumoto metric $\hat{F}$ is given by*

$$\begin{aligned}
 \hat{g}(\overline{W}, \overline{Z}) &= \frac{F^2(F-2\Phi)}{(F-\Phi)^3} g(\overline{W}, \overline{Z}) + \frac{3F^4}{(F-\Phi)^4} \phi(\overline{W}) \phi(\overline{Z}) + \frac{F^2 \Phi(4\Phi-F)}{(F-\Phi)^4} \ell(\overline{W}) \ell(\overline{Z}) \\
 &\quad + \frac{F^3(F-4\Phi)}{(F-\Phi)^4} \left\{ \phi(\overline{W}) \ell(\overline{Z}) + \phi(\overline{Z}) \ell(\overline{W}) \right\}.
 \end{aligned} \tag{3.13}$$

Proof. From the definition of the angular metric tensor $\hat{h} := \hat{g} - \hat{\ell} \otimes \hat{\ell}$ and using Proposition 3.5 (2), (4), we obtain

$$\begin{aligned}
 \hat{g}(\overline{W}, \overline{Z}) &= \frac{F^2(F-2\Phi)}{(F-\Phi)^3} \hat{h}(\overline{W}, \overline{Z}) + \frac{2F^4}{(F-\Phi)^4} \phi(\overline{W}) \phi(\overline{Z}) \\
 &\quad + \frac{2\Phi^2 F^2}{(F-\Phi)^4} \ell(\overline{W}) \ell(\overline{Z}) - \frac{2\Phi F^3}{(F-\Phi)^4} \left\{ \phi(\overline{W}) \ell(\overline{Z}) + \phi(\overline{Z}) \ell(\overline{W}) \right\} \\
 &\quad + \left\{ \frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\overline{W}) + \frac{F^2}{(F-\Phi)^2} \phi(\overline{W}) \right\} \\
 &\quad \times \left\{ \frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\overline{Z}) + \frac{F^2}{(F-\Phi)^2} \phi(\overline{Z}) \right\} \\
 &= \frac{F^2(F-2\Phi)}{(F-\Phi)^3} g(\overline{W}, \overline{Z}) + \frac{3F^4}{(F-\Phi)^4} \phi(\overline{W}) \phi(\overline{Z}) + \frac{F^2 \Phi(4\Phi-F)}{(F-\Phi)^4} \ell(\overline{W}) \ell(\overline{Z}) \\
 &\quad + \frac{F^3(F-4\Phi)}{(F-\Phi)^4} \left\{ \phi(\overline{W}) \ell(\overline{Z}) + \phi(\overline{Z}) \ell(\overline{W}) \right\}.
 \end{aligned}$$

$\square$

Consequently, for any π -vector field $\overline{\mu}$ and $W \in \mathfrak{X}(TM)$, we get

$$\begin{aligned}
 \hat{g}(\overline{\mu}, \rho W) &= \frac{F^2(F-2\Phi)}{(F-\Phi)^3} g(\overline{\mu}, \rho W) + \frac{F^3(F-4\Phi)}{(F-\Phi)^4} \left\{ \phi(\overline{\mu}) \ell(\rho W) + \phi(\rho W) \ell(\overline{\mu}) \right\} \\
 &\quad + \frac{3F^4}{(F-\Phi)^4} \phi(\overline{\mu}) \phi(\rho W) + \frac{F^2 \Phi(4\Phi-F)}{(F-\Phi)^4} \ell(\overline{\mu}) \ell(\rho W).
 \end{aligned} \tag{3.14}$$

Theorem 3.7. Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$ with associated π -form ϕ . The function $\hat{F}$ defined by (3.5) is a conic pseudo-Finsler structure if and only if

$$F(1 + 2p^2) - 3\Phi \neq 0. \quad (3.15)$$

That is, the Finsler metric tensor $\hat{g}$ of $\hat{F}$ is non-degenerate if and only if the function $F(1 + 2p^2) - 3\Phi$ does not vanish.

Proof. The metric $\hat{g}$ associated with $\hat{F}$ is non-degenerate if and only if

$$\hat{g}(\bar{W}, \bar{Z}) = 0 \quad \forall \bar{W} \in \mathfrak{X}(\pi) \implies \bar{Z} = 0.$$

Assume that $\hat{g}(\bar{W}, \bar{Z}) = 0, \quad \forall \bar{W} \in \mathfrak{X}(\pi)$. Then, relation (3.13) gives rise to

$$\begin{aligned} 0 &= \frac{F^2(F - 2\Phi)}{(F - \Phi)^3} g(\bar{W}, \bar{Z}) + \frac{3F^4}{(F - \Phi)^4} \phi(\bar{W}) \phi(\bar{Z}) + \frac{F^2 \Phi (4\Phi - F)}{(F - \Phi)^4} \ell(\bar{W}) \ell(\bar{Z}) \\ &\quad + \frac{F^3(F - 4\Phi)}{(F - \Phi)^4} \left\{ \phi(\bar{W}) \ell(\bar{Z}) + \phi(\bar{Z}) \ell(\bar{W}) \right\}. \end{aligned} \quad (3.16)$$

Setting $\bar{W} = \bar{\varphi}$ in (3.16), noting that $\ell(\bar{\varphi}) = \frac{\Phi}{F}$ and $\phi(\bar{\varphi}) = p^2$, one can show that

$$\chi_1 \ell(\bar{Z}) + \Upsilon_1 \phi(\bar{Z}) = 0, \quad (3.17)$$

where

$$\chi_1 := \frac{F(4\Phi - F)(\Phi^2 - F^2 p^2)}{(\Phi - F)^4}, \quad \Upsilon_1 := \frac{F^2(-2\Phi^2 - 2\Phi F + F^2(3p^2 + 1))}{(\Phi - F)^4}.$$

Similarly, setting by $\bar{W} = \bar{\eta}$ in (3.16), taking into account $\ell(\bar{\eta}) = F$ and $\phi(\bar{\eta}) = \Phi$, we obtain

$$\chi_2 \ell(\bar{Z}) + \Upsilon_2 \phi(\bar{Z}) = 0, \quad (3.18)$$

with

$$\chi_2 := \frac{F^3(F - 2\Phi)}{(F - \Phi)^3}, \quad \Upsilon_2 := \frac{F^4}{(F - \Phi)^3}.$$

The system of the algebraic equations (3.17) and (3.18), in ℓ and ϕ , has a non-trivial solution if and only if

$$\frac{F^6(F(2p^2 + 1) - 3\Phi)}{(\Phi - F)^6} = 0.$$

Hence, as $\hat{F}(x, y) = \frac{F^2(x, y)}{F(x, y) - \Phi(x, y)} \neq 0$ over $\mathcal{TM}$, then we conclude that $F(2p^2 + 1) - 3\Phi = 0$.

Consequently,

$$\bar{Z} \neq 0 \iff F(2p^2 + 1) - 3\Phi = 0.$$

Therefore, $\bar{Z} = 0$ if and only if the Finsler structure F and the π -form Φ satisfy the condition

$$F(2p^2 + 1) - 3\Phi \neq 0.$$

This means that the metric tensor $\hat{g}$ of $\hat{F}$ is non-degenerate if and only if the condition (3.15) is satisfied. Hence, the proof is complete. $\square$

Form now on, we consider that the generalized ϕ -Matsumoto metric $\hat{F}$, defined by (3.5), satisfies the condition (3.15).

Proposition 3.8. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$. Under the ϕ -Matsumoto change (3.5), the Cartan torsion tensor $\hat{\mathbf{T}}$ of $\hat{F}$ can be written in terms of the Cartan torsion tensor $\mathbf{T}$ of F as follows*

$$\begin{aligned} 2\hat{\mathbf{T}}(\bar{W}, \bar{Y}, \bar{Z}) &= \frac{2F^2(F-2\Phi)}{(F-\Phi)^3} \mathbf{T}(\bar{W}, \bar{Y}, \bar{Z}) \\ &+ \frac{F\Phi(4\Phi-F)}{(F-\Phi)^4} \left\{ \hbar(\bar{W}, \bar{Z}) \ell(\bar{Y}) + \hbar(\bar{Y}, \bar{Z}) \ell(\bar{W}) \right\} \\ &+ \frac{F^2(F-4\Phi)}{(F-\Phi)^4} \left\{ \hbar(\bar{W}, \bar{Z}) \phi(\bar{Y}) + \hbar(\bar{Y}, \bar{Z}) \phi(\bar{W}) \right\} \\ &+ \left(D_{\gamma\bar{Z}}^\circ \frac{F^2(F-2\Phi)}{(F-\Phi)^3} \right) g(\bar{W}, \bar{Y}) + D_{\gamma\bar{Z}}^\circ \left(\frac{3F^4}{(F-\Phi)^4} \phi(\bar{W}) \phi(\bar{Y}) \right) \\ &+ \left(D_{\gamma\bar{Z}}^\circ \frac{F^2\Phi(4\Phi-F)}{(F-\Phi)^4} \right) \ell(\bar{W}) \ell(\bar{Y}) \\ &+ \left(D_{\gamma\bar{Z}}^\circ \frac{F^3(F-4\Phi)}{(F-\Phi)^4} \right) \left\{ \phi(\bar{W}) \ell(\bar{Y}) + \phi(\bar{Y}) \ell(\bar{W}) \right\}. \end{aligned}$$

Proof. Using the expression of the metric $\hat{g}$ obtained in Proposition 3.6, taking into account the fact that $(D_{\gamma\bar{Z}}^\circ g)(\bar{W}, \bar{Y}) = 2\mathbf{T}(\bar{W}, \bar{Y}, \bar{Z})$, the result follows. $\square$

4. The change of the spray and Berwald connection

Here, we find out the expression of the geodesic spray $\hat{G}$ of the generalized ϕ -Matsumoto metric $\hat{F}$ in terms of the geodesic spray G of F . Consequently, we prove that the two geodesic sprays G and $\hat{G}$ can not be projectively related. Moreover, the relationship between the two Barthel connections Γ and $\hat{\Gamma}$ is obtained, as well as the relations between Barthel curvature tensors $\mathfrak{R}$ and $\hat{\mathfrak{R}}$ and Berwald connections D° and $\hat{D}^\circ$ are derived. We conclude that the π -vector field $\bar{\varphi}$ is never concurrent with respect to $\hat{F}$, then we end this section by providing a condition that makes $\bar{\varphi}$ concurrent with respect to $\hat{F}$.

Theorem 4.1. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$ with associated π -form ϕ . If G is the geodesic spray of F , then the geodesic spray $\hat{G}$ of the generalized ϕ -Matsumoto metric $\hat{F}$ is given by*

$$\hat{G} = G - f_1 \mathcal{C} + f_2 \gamma \bar{\varphi},$$

$$\text{where } f_1 := \frac{F(4\Phi-F)}{F(1+2p^2)-3\Phi} \quad \text{and} \quad f_2 := \frac{2F^3}{F(1+2p^2)-3\Phi}.$$

Proof. Since the geodesic spray $\hat{G}$ of $\hat{F}$ satisfies [3]

$$-d\hat{E} = \frac{1}{2} i_{\hat{G}} dd_J \hat{F}^2,$$

where $d\hat{E}$ is expressed in (3.8) in terms of dE . Due to the fact that the difference between two sprays is a vertical vector field (i.e. $\hat{G} = G + \gamma\bar{\mu}$, for some π -vector field $\bar{\mu}$), we get

$$-d\hat{E}(W) = \frac{1}{2} i_{G+\gamma\bar{\mu}} dd_J \hat{F}^2(W) = \frac{1}{2} i_G dd_J \hat{F}^2(W) + \frac{1}{2} i_{\gamma\bar{\mu}} dd_J \hat{F}^2(W), \quad (4.1)$$

Using $\beta\bar{\eta} = G$ and $W = hW + vW = \beta\rho W + \gamma KW$, together with Lemma 3.3, we obtain

$$\begin{aligned} \frac{1}{2} i_G dd_J \hat{F}^2(W) &= \frac{1}{2} \{ dd_J \hat{F}^2(\beta\bar{\eta}, W) \} = \frac{1}{2} \left\{ G \cdot d_J \hat{F}^2(W) - W \cdot d_J \hat{F}^2(G) - d_J \hat{F}^2[G, W] \right\} \\ &= \frac{1}{2} \left\{ G \cdot (2\hat{F}\hat{\ell}(\rho W)) - W \cdot (2\hat{F}\hat{\ell}(\bar{\eta})) - 2\hat{F}\hat{\ell}(\rho[G, W]) \right\} \\ &= (G \cdot \hat{F}) \hat{\ell}(\rho W) + \hat{F} G \cdot \hat{\ell}(\rho W) - (W \cdot \hat{F}^2) - \hat{F} \hat{\ell}(\rho[G, W]). \end{aligned} \quad (4.2)$$

Taking into account Lemmas 3.3, Proposition 3.5 (2), Formula (3.6) and (3.7), relation (4.2) reduces to

$$\begin{aligned}
\frac{1}{2} i_G dd_J \widehat{F}^2(W) &= -\frac{F^4}{(F-\Phi)^2} \left(\frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\rho W) + \frac{F^2}{(F-\Phi)^2} \phi(\rho W) \right) \\
&\quad + \frac{F^2}{(F-\Phi)} G \cdot \left(\frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\rho W) + \frac{F^2}{(F-\Phi)^2} \phi(\rho W) \right) \\
&\quad - \frac{2F^2}{(F-\Phi)} \left(\frac{F(F-2\Phi)}{(F-\Phi)^2} dF(W) + \frac{F^2}{(F-\Phi)^2} d\Phi(W) \right) \\
&\quad - \frac{F^2}{(F-\Phi)} \left(\frac{F(F-2\Phi)}{(F-\Phi)^2} \ell(\rho[G, W]) + \frac{F^2}{(F-\Phi)^2} \phi(\rho[G, W]) \right) \\
&= -\frac{F^5(F-4\Phi)}{(\Phi-F)^4} \ell(\rho W) - \frac{3F^6}{(\Phi-F)^4} \phi(\rho W) \\
&\quad - \frac{F^3(F-2\Phi)}{(F-\Phi)^3} dF(W) - \frac{F^4}{(F-\Phi)^3} d\Phi(W). \tag{4.3}
\end{aligned}$$

Plugging the relations (4.3) and (3.14) (which expresses $\frac{1}{2} i_{\gamma\bar{\mu}} dd_J \widehat{F}^2(W)$ as $\widehat{g}(\bar{\mu}, \rho W)$) into Equation (4.1), after some calculation, it follows that

$$\begin{aligned}
&-\frac{F^3(F-2\Phi)}{(F-\Phi)^3} dF(W) - \frac{F^4}{(F-\Phi)^3} d\Phi(W) \\
&= -\frac{F^5(F-4\Phi)}{(\Phi-F)^4} \ell(\rho W) - \frac{3F^6}{(\Phi-F)^4} \phi(\rho W) - \frac{F^3(F-2\Phi)}{(F-\Phi)^3} dF(W) - \frac{F^4}{(F-\Phi)^3} d\Phi(W) \\
&\quad + \frac{F^2(F-2\Phi)}{(F-\Phi)^3} g(\bar{\mu}, \rho W) + \frac{3F^4}{(F-\Phi)^4} \phi(\bar{\mu}) \phi(\rho W) + \frac{F^2 \Phi(4\Phi-F)}{(F-\Phi)^4} \ell(\bar{\mu}) \ell(\rho W) \\
&\quad + \frac{F^3(F-4\Phi)}{(F-\Phi)^4} \{ \phi(\bar{\mu}) \ell(\rho W) + \phi(\rho W) \ell(\bar{\mu}) \}.
\end{aligned}$$

Using the non-degeneracy property of Finsler metric g , the above relation reduces to

$$\begin{aligned}
\frac{F^2(F-2\Phi)}{(F-\Phi)^3} \bar{\mu} &= \left\{ \frac{F^4(F-4\Phi)}{(F-\Phi)^4} - \frac{F^2(F-4\Phi)}{(F-\Phi)^4} \phi(\bar{\mu}) - \frac{F \Phi(4\Phi-F)}{(F-\Phi)^4} \ell(\bar{\mu}) \right\} \bar{\eta} \\
&\quad + \left\{ \frac{3F^6}{(F-\Phi)^4} - \frac{3F^4}{(F-\Phi)^4} \phi(\bar{\mu}) - \frac{F^3(F-4\Phi)}{(F-\Phi)^4} \ell(\bar{\mu}) \right\} \bar{\varphi}, \tag{4.4}
\end{aligned}$$

where $\ell(\bar{\mu})$ and $\phi(\bar{\mu})$ are determined by the following two equations

$$F^2(F-2\Phi)\ell(\bar{\mu}) + F^3\phi(\bar{\mu}) = F^5, \tag{4.5}$$

$$\begin{aligned}
&F^4(-4\Phi^2 + \Phi F + 3F^2 p^2) = \\
&F(4\Phi - F)(\Phi^2 - F^2 p^2)\ell(\bar{\mu}) + F^2(2\Phi^2 - 2\Phi F + F^2(1 + 3p^2))\phi(\bar{\mu}). \tag{4.6}
\end{aligned}$$

Thus, the condition (3.15) leads to

$$\ell(\bar{\mu}) = \frac{F^2(2\Phi - F)}{3\Phi - F(2p^2 + 1)}, \quad \phi(\bar{\mu}) = \frac{F(-4\Phi^2 + \Phi F + 2F^2 p^2)}{-3\Phi + 2Fp^2 + F}. \tag{4.7}$$

Consequently, in view of Equation (4.4) and the assumption $\widehat{G} = G + \gamma\bar{\mu}$, it follows that the geodesic sprays $\widehat{G}$ is given by

$$\widehat{G} = G - \frac{F(4\Phi - F)}{F(1 + 2p^2) - 3\Phi} \mathcal{C} + \frac{2F^3}{F(1 + 2p^2) - 3\Phi} \gamma\bar{\varphi}. \tag{4.8}$$

Hence, the proof is complete. $\square$

Theorem 4.2. *Let (M, F) be a Finsler manifold admitting concurrent π -vector field $\bar{\varphi}$. Under the ϕ -Matsumoto change (3.5), the geodesic sprays G and $\hat{G}$ can never be projectively related.*

Proof. Let G and $\hat{G}$ be the geodesic sprays of F and the generalized ϕ -Matsumoto metric $\hat{F}$, respectively. The Finsler metrics F and $\hat{F}$ are projectively related means that

$$\hat{G} = G - 2A(x, y) \mathbb{C},$$

with a projective factor $A(x, y)$ being positively homogeneous function of degree 1 in the directional argument y . As F is a non-zero function, in view of the relation (4.8), we get G and $\hat{G}$ are projectively related if and only if $\gamma\bar{\varphi} = 0$. Thus, $A(x, y) = \frac{1}{2}f_1$, which leads to $\bar{\varphi} = 0$. This contradicts our assumption that the π -vector field $\bar{\varphi}$ is everywhere nonzero. $\square$

Proposition 4.3. *The Barthel connection $\hat{\Gamma}$ associated with the generalized ϕ -Matsumoto metric $\hat{F}$ can be expressed in terms of the Barthel connection Γ associated with F as follows*

$$\hat{\Gamma} = \Gamma - f_1 J - d_J f_1 \otimes \gamma\bar{\eta} + d_J f_2 \otimes \gamma\bar{\varphi}.$$

Consequently, the horizontal map $\hat{\beta}$ associated with the $\hat{F}$ has the form

$$\hat{\beta}\bar{W} = \beta\bar{W} - \frac{1}{2} \left\{ f_1 \gamma\bar{W} + d_J f_1 (\beta\bar{W}) \gamma\bar{\eta} - d_J f_2 (\beta\bar{W}) \gamma\bar{\varphi} \right\}.$$

Proof. In view of Theorem 4.1 and (2.1) along with (2.2), we obtain

$$\begin{aligned} \hat{\Gamma} &= [J, \hat{G}] = [J, G - f_1 \gamma\bar{\eta} + f_2 \gamma\bar{\varphi}] = [J, G] + [f_1 \gamma\bar{\eta} - f_2 \gamma\bar{\varphi}, J] \\ &= [J, G] + f_1 [\gamma\bar{\eta}, J] + df_1 \wedge i_{\gamma\bar{\eta}} J - d_J f_1 \otimes \gamma\bar{\eta} - f_2 [\gamma\bar{\varphi}, J] - df_2 \wedge i_{\gamma\bar{\varphi}} J + d_J f_2 \otimes \gamma\bar{\varphi}. \end{aligned}$$

From (2.2) and (3.3) along with

$$\begin{aligned} [\gamma\bar{\varphi}, J]W &= [\gamma\bar{\varphi}, JW] - J[\gamma\bar{\varphi}, W] \\ &= \gamma\{\nabla_{\gamma\bar{\varphi}} \rho W - \nabla_{JW} \bar{\varphi}\} - \gamma\{\nabla_{\gamma\bar{\varphi}} \rho W - T(\bar{\varphi}, \rho W)\} = 0, \end{aligned}$$

we get

$$\hat{\Gamma} = \Gamma - f_1 J - d_J f_1 \otimes \gamma\bar{\eta} + d_J f_2 \otimes \gamma\bar{\varphi}.$$

Consequently, using the fact that $\Gamma = 2\beta \circ \rho - I$, the horizontal map $\hat{\beta}$ associated with the generalized ϕ -Matsumoto metric has the form

$$\hat{\beta}\bar{W} = \beta\bar{W} - \frac{1}{2} \left\{ f_1 \gamma\bar{W} + d_J f_1 (\beta\bar{W}) \gamma\bar{\eta} - d_J f_2 (\beta\bar{W}) \gamma\bar{\varphi} \right\}.$$

$\square$

Corollary 4.4. *The horizontal projector $\hat{h}$ and vertical projector $\hat{v}$ associated with the generalized ϕ -Matsumoto metric $\hat{F}$ can be written, in terms of the horizontal projector h and vertical projector v associated with F , in the form*

$$\hat{h} = h + \mathbb{S}, \quad \hat{v} = v - \mathbb{S}, \quad (4.9)$$

where $\mathbb{S}$ is a semi-basic vector 1-form [3] given by

$$\mathbb{S} := -\frac{1}{2} \{ f_1 J + d_J f_1 \otimes \gamma\bar{\eta} - d_J f_2 \otimes \gamma\bar{\varphi} \}. \quad (4.10)$$

Moreover, the Barthel curvature tensor $\hat{\mathfrak{R}}$ associated with the generalized ϕ -Matsumoto metric (3.5) is determined by

$$\hat{\mathfrak{R}} = \mathfrak{R} - [h, \mathbb{S}] - N_{\mathbb{S}}.$$

Proof. The relation (4.9) follows from Theorem 4.3 and formula (2.4). Now, since the Barthel curvature tensor of $\widehat{F}$ is defined by $\widehat{\mathfrak{R}} = -\frac{1}{2}[\widehat{h}, \widehat{h}]$, the proof follows from (4.9) and formula (2.6) together with the properties of the Frölicher-Nijenhuis bracket. More precisely,

$$\widehat{\mathfrak{R}} = -\frac{1}{2}[h + \mathbb{S}, h + \mathbb{S}] = -\frac{1}{2}([h, h] + [h, \mathbb{S}] + [\mathbb{S}, h] + [\mathbb{S}, \mathbb{S}]) = \mathfrak{R} - [h, \mathbb{S}] - N_{\mathbb{S}}.$$

□

Proposition 4.5. *For the generalized ϕ -Matsumoto metric (3.5), we have:*

- (1) *The vertical counterpart of Berwald connection can be expressed as*

$$\widehat{D}^{\circ}_{\gamma\overline{W}}\overline{Z} = D^{\circ}_{\gamma\overline{W}}\overline{Z}.$$

- (2) *The horizontal counterpart of Berwald connection can be expressed as*

$$\begin{aligned} \widehat{D}^{\circ}_{\widehat{\beta}\overline{W}}\overline{Z} &= D^{\circ}_{\beta\overline{W}}\overline{Z} - \frac{1}{2}\{f_1 D^{\circ}_{\gamma\overline{W}}\overline{Z} + d_J f_1(\beta\overline{W}) D^{\circ}_{\gamma\overline{\eta}}\overline{Z} \\ &\quad - d_J f_1(\beta\overline{W})\overline{Z} - d_J f_1(\beta\overline{Z})\overline{W} - d_J f_2(\beta\overline{W}) D^{\circ}_{\gamma\overline{\varphi}}\overline{Z}\} \\ &\quad + \frac{1}{2}\{dd_J f_1(\gamma\overline{Z}, \beta\overline{W})\overline{\eta} - dd_J f_2(\gamma\overline{Z}, \beta\overline{W})\overline{\varphi}\}. \end{aligned}$$

Proof. Under the ϕ -Matsumoto change (3.5), we have:

- (1) The vertical counterpart for Berwald connection $D^{\circ}_{\gamma\overline{W}}\overline{Z}$ is invariant by (3.10).
 (2) Using the facts that $v := \gamma \circ K$, $h := \beta \circ \rho$, the Berwald v-curvature $\widehat{S}^{\circ} = 0$, together with formulae (2.3), (2.5) and (2.7), we obtain

$$\begin{aligned} \gamma\widehat{D}^{\circ}_{hW}\rho Z &= \widehat{v}[\widehat{h}W, JZ] \stackrel{(4.9)}{=} (v - \mathbb{S})[(h + \mathbb{S})W, JZ] \\ &= v[hW, JZ] + v[\mathbb{S}W, JZ] - \mathbb{S}[hW, JZ] - \mathbb{S}[\mathbb{S}W, JZ] \\ &\stackrel{(4.10)}{=} \gamma D^{\circ}_{hW}\overline{Z} \\ &\quad - \frac{\gamma}{2}\{f_1 K[JW, JZ] + d_J f_1(W) K[\gamma\overline{\eta}, JZ] - d_J f_2(W) K[\gamma\overline{\varphi}, JZ]\} \\ &\quad + \frac{\gamma}{2}\{(JZ \cdot f_1)\rho W + (JZ \cdot d_J f_1(W))\overline{\eta} - (JZ \cdot d_J f_2(W))\overline{\varphi}\} \\ &\quad + \frac{\gamma}{2}\{f_1 \rho([hW, JZ]) + d_J f_1([hW, JZ])\overline{\eta} - d_J f_2([hW, JZ])\overline{\varphi}\} \\ &= \gamma D^{\circ}_{hW}\rho Z - \frac{\gamma}{2}\{f_1 D^{\circ}_{\beta\overline{W}}\rho Z + d_J f_1(W) D^{\circ}_{\gamma\overline{\eta}}\rho Z \\ &\quad - d_J f_1(W)\rho Z - d_J f_1(Z)\rho W - d_J f_2(W) D^{\circ}_{\gamma\overline{\varphi}}\rho Z\} \\ &\quad + \frac{\gamma}{2}\{dd_J f_1(JZ, W)\overline{\eta} - dd_J f_2(JZ, W)\overline{\varphi}\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \widehat{D}^{\circ}_{\widehat{\beta}\overline{W}}\overline{Z} &= D^{\circ}_{\beta\overline{W}}\overline{Z} - \frac{1}{2}\{f_1 D^{\circ}_{\gamma\overline{W}}\overline{Z} + d_J f_1(\beta\overline{W}) D^{\circ}_{\gamma\overline{\eta}}\overline{Z} \\ &\quad - d_J f_1(\beta\overline{W})\overline{Z} - d_J f_1(\beta\overline{Z})\overline{W} - d_J f_2(\beta\overline{W}) D^{\circ}_{\gamma\overline{\varphi}}\overline{Z}\} \\ &\quad + \frac{1}{2}\{dd_J f_1(\gamma\overline{Z}, \beta\overline{W})\overline{\eta} - dd_J f_2(\gamma\overline{Z}, \beta\overline{W})\overline{\varphi}\}. \end{aligned}$$

□

Corollary 4.6. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\overline{\varphi}$. Under the ϕ -Matsumoto change (3.5), the π -vector field $\overline{\varphi}$ can not be concurrent with respect to the Finsler metric $\widehat{F}$.*

Proof. It follows directly from Definition 3.1 and Proposition 4.5 as

$$\widehat{D}^\circ_{\widehat{\beta}\overline{W}}\overline{Z} \neq D^\circ_{\beta\overline{W}}\overline{Z}.$$

□

Nevertheless, the following result gives a condition under which the property of the π -vector field $\overline{\varphi}$ being concurrent is preserved under ϕ -Matsumoto change.

Theorem 4.7. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\overline{\varphi}$. Let $\widehat{F}$ be the generalized ϕ -Matsumoto metric defined by (3.5). A necessary and sufficient condition for $\overline{\varphi}$ to be concurrent with respect to $\widehat{F}$ is that*

$$[d_J f_1(\beta\overline{X}) - dd_J f_2(\gamma\overline{\varphi}, \beta\overline{X})]\overline{\varphi} - d_J f_1(\beta\overline{\varphi})\overline{X} + dd_J f_1(\gamma\overline{\varphi}, \beta\overline{X})\overline{\eta} = 0 \quad (4.11)$$

Proof. In view of Proposition 4.5, we have

$$\widehat{D}^\circ_{\gamma\overline{X}}\overline{\varphi} = D^\circ_{\gamma\overline{X}}\overline{\varphi} \quad (4.12)$$

and

$$\begin{aligned} \widehat{D}^\circ_{\widehat{\beta}\overline{X}}\overline{\varphi} &= D^\circ_{\beta\overline{X}}\overline{\varphi} + \frac{1}{2} \left\{ f_1 D^\circ_{\gamma\overline{X}}\overline{\varphi} + d_J f_1(\beta\overline{X}) D^\circ_{\gamma\overline{\eta}}\overline{\varphi} - d_J f_2(\beta\overline{X}) D^\circ_{\gamma\overline{\varphi}}\overline{\varphi} \right\} \\ &\quad + \frac{1}{2} \left\{ d_J f_1(\beta\overline{X})\overline{\varphi} - d_J f_1(\beta\overline{\varphi})\overline{X} + dd_J f_1(\gamma\overline{\varphi}, \beta\overline{X})\overline{\eta} - dd_J f_2(\gamma\overline{\varphi}, \beta\overline{X})\overline{\varphi} \right\}. \end{aligned} \quad (4.13)$$

Now, assume that the condition (4.11) is satisfied, then (4.13) reduces to

$$\widehat{D}^\circ_{\widehat{\beta}\overline{X}}\overline{\varphi} = D^\circ_{\beta\overline{X}}\overline{\varphi} + \frac{1}{2} \left\{ f_1 D^\circ_{\gamma\overline{X}}\overline{\varphi} + d_J f_1(\beta\overline{X}) D^\circ_{\gamma\overline{\eta}}\overline{\varphi} - d_J f_2(\beta\overline{X}) D^\circ_{\gamma\overline{\varphi}}\overline{\varphi} \right\}. \quad (4.14)$$

Since $\overline{\varphi}$ is a concurrent π -vector field with respect to F , i.e., $D^\circ_{\beta\overline{X}}\overline{\varphi} = -\overline{X}$ and $D^\circ_{\gamma\overline{X}}\overline{\varphi} = 0$, then, by (4.14), $\widehat{D}^\circ_{\widehat{\beta}\overline{X}}\overline{\varphi} = -\overline{X}$. This, together with (4.12), imply that $\overline{\varphi}$ is a concurrent π -vector field with respect to $\widehat{F}$.

Now, if $\overline{\varphi}$ is concurrent π -vector field with respect to $\widehat{F}$, i.e.,

$$\widehat{D}^\circ_{\gamma\overline{X}}\overline{\varphi} = 0 \quad \text{and} \quad \widehat{D}^\circ_{\widehat{\beta}\overline{X}}\overline{\varphi} = -\overline{X},$$

then, by (4.12) and (4.14),

$$f_1 D^\circ_{\gamma\overline{X}}\overline{\varphi} + d_J f_1(\beta\overline{X}) D^\circ_{\gamma\overline{\eta}}\overline{\varphi} - d_J f_2(\beta\overline{X}) D^\circ_{\gamma\overline{\varphi}}\overline{\varphi} = 0.$$

□

5. Preservation of (almost) rationality

Now, we give an example of a conic non-Riemannian Finsler metric that admits a concurrent π -vector field and find $\widehat{F}$.

Example 5.1. *Let $M = \mathbb{R}^3$ and F be a conic Finsler metric defined by*

$$F(x, y) := F(x_1, x_2, x_3; y_1, y_2, y_3) = \sqrt{x_3^2 \left(\frac{x_1^2 y_2^2 + 2y_1 y_2}{y_1} \right)^2 + y_3^2} \quad (5.1)$$

on the domain $D_F = \{(x_1, x_2, x_3; y_1, y_2, y_3) \in T\mathbb{R}^3 \mid x_1 \neq 0, x_3 \neq 0, y_1 \neq 0, y_2 \neq 0\}$.

By straightforward calculations or using the Maple Finsler package [30], the non-vanishing components of the Finsler metric tensor g_{ij} are given by

$$\begin{aligned} g_{11} &= \frac{x_3^2 x_1^2 y_2^3 (3x_1^2 y_2 + 4y_1)}{y_1^4}, & g_{12} &= -\frac{2x_3^2 x_1^2 y_2^2 (2x_1^2 y_2 + 3y_1)}{y_1^3}, \\ g_{22} &= \frac{2x_3^2 (3x_1^4 y_2^2 + 6x_1^2 y_1 y_2 + 2y_1^2)}{y_1^2}, & g_{33} &= 1. \end{aligned}$$

The non-vanishing components g^{ij} of the inverse metric tensor are the following:

$$\begin{aligned} g^{11} &= \frac{(3x_1^4 y_2^2 + 6x_1^2 y_1 y_2 + 2y_1^2) y_1^4}{x_3^2 x_1^2 y_2^3 (x_1^6 y_2^3 + 6x_1^4 y_1 y_2^2 + 12x_1^2 y_1^2 y_2 + 8y_1^3)}, \\ g^{12} &= \frac{(2x_1^2 y_2 + 3y_1) y_1^3}{x_3^2 y_2 (x_1^6 y_2^3 + 6x_1^4 y_1 y_2^2 + 12x_1^2 y_1^2 y_2 + 8y_1^3)}, \\ g^{22} &= \frac{1}{2} \frac{(3x_1^2 y_2 + 4y_1) y_1^2}{x_3^2 (x_1^6 y_2^3 + 6x_1^4 y_1 y_2^2 + 12x_1^2 y_1^2 y_2 + 8y_1^3)}, \quad g^{33} = 1. \end{aligned}$$

Consequently, the non-vanishing components of the Cartan torsion C_{ijk} are

$$\begin{aligned} C_{111} &= -\frac{6x_1^2 x_3^2 y_2^3 (x_1^2 y_2 + y_1)}{y_1^5}, & C_{112} &= \frac{6x_1^2 x_3^2 y_2^2 (x_1^2 y_2 + y_1)}{y_1^4}, \\ C_{122} &= -\frac{6x_1^2 x_3^2 y_2 (x_1^2 y_2 + y_1)}{y_1^3}, & C_{222} &= \frac{6x_1^2 x_3^2 (x_1^2 y_2 + y_1)}{y_1^2}. \end{aligned}$$

Moreover, the geodesic spray coefficients are given by

$$G^1 = \frac{(x_1 y_3 - x_3 y_1) y_1}{x_1 x_3}, \quad G^2 = \frac{y_2 y_3}{x_3}, \quad G^3 = -\frac{x_3 y_2^2 (x_1^4 y_2^2 + 4x_1^2 y_1 y_2 + 4y_1^2)}{2y_1^2}.$$

Some of the coefficients of Cartan connection are

$$\Gamma_{13}^1 = \frac{1}{x_3}, \quad \Gamma_{23}^2 = \frac{1}{x_3}, \quad \Gamma_{33}^3 = 0.$$

One can choose a concurrent π -vector field $\bar{\varphi} = \varphi^i(x) \bar{\partial}_i = x_3 \bar{\partial}_3$ with respect to F , where $\bar{\partial}_i$ are the basis of fibres of $\pi^{-1}(TM)$. Clearly, $\varphi^i C_{ijk} = 0$ and

$$\varphi^1|_1 = \delta_1 \varphi^1 + \varphi^1 \Gamma_{11}^1 + \varphi^2 \Gamma_{12}^1 + \varphi^3 \Gamma_{13}^1 = 1.$$

Similarly, $\varphi^2|_2 = 1$, $\varphi^3|_3 = 1$ and all other components of $\varphi^i|_j$ vanish identically.

Moreover, the components of the corresponding π -form ϕ are $\phi_1 = \phi_2 = 0$, $\phi_3 = x_3$. Consequently, $\Phi(x, y) = x_3 y_3$.

We then have

$$\hat{F}(x, y) = \frac{F^2(x, y)}{F(x, y) - \Phi(x, y)} = \frac{x_3^2 \left(\frac{x_1^2 y_2^2 + 2y_1 y_2}{y_1} \right)^2 + y_3^2}{\sqrt{x_3^2 \left(\frac{x_1^2 y_2^2 + 2y_1 y_2}{y_1} \right)^2 + y_3^2} - x_3 y_3}.$$

Since $\hat{F}$ satisfies (3.15), as it can be verified using the Maple Finsler package [30], then $\hat{F}$ defines a generalized ϕ -Matsumoto metric over D , where

$$D := D_F \cap \left\{ \sqrt{x_3^2 \left(\frac{x_1^2 y_2^2 + 2y_1 y_2}{y_1} \right)^2 + y_3^2} > x_3 y_3 \right\}.$$

Let us recall the following definition.

Definition 5.2. [21] A conic-Finsler metric F on $\mathcal{D}$ is called an **almost rational** Finsler metric if all its Finsler metric tensor components $g_{ij}(x, y)$ can be expressed in the form

$$g_{ij}(x, y) = \theta(x, y) a_{ij}(x, y),$$

where $\theta : \mathcal{D} \rightarrow (0, \infty)$ is a smooth function and the matrix $(a_{ij}(x, y))_{1 \leq i, j \leq n}$ is symmetric non-degenerate and each of the functions $a_{ij}(x, y)$ is rational in the directional argument y .

If in addition, θ is a rational function in y , the Finsler metric F is said to be a **rational** Finsler metric.

Remark 5.3. The Finsler metric F defined by (5.1) is a rational Finsler metric since g_{ij} can be written as $g_{ij}(x, y) = \theta(x, y) a_{ij}(x, y)$ with $\theta(x, y) = (\frac{x_3}{y_1})^2$ being a rational function in the variable y and

$$\begin{aligned} a_{11} &= \frac{x_1^2 y_2^3 (3x_1^2 y_2 + 4y_1)}{y_1^2}, & a_{12} &= -\frac{2x_1^2 y_2^2 (2x_1^2 y_2 + 3y_1)}{y_1^2}, \\ a_{22} &= 2(3x_1^4 y_2^2 + 6x_1^2 y_1 y_2 + 2y_1^2), & a_{33} &= \left(\frac{y_1}{x_3}\right)^2 \end{aligned}$$

all are rational functions in the variable y .

This motivate us to study the following problem:

If g_{ij} are written as $g_{ij} = \theta a_{ij}$ with a_{ij} are rational functions in y , when can $\hat{g}_{ij}$ be written as $\hat{g}_{ij} = \hat{\theta} \hat{a}_{ij}$ with $\hat{a}_{ij}$ rational functions in y ? Or, equivalently, under what conditions the ϕ -Matsumoto change preserves the almost rationality property of Finsler metrics?

Lemma 5.4. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$. If F is a rational function in y , then F is a rational Finsler metric.*

Proof. Suppose that the Finsler structure F is a rational function in y . It is clear that F^2 will be a rational function in y . Since, the partial differentiation $\frac{\partial}{\partial y^i}$ of a geometric object that is rational in y remains rational in y , the Finsler metric tensor components $g_{ij}(x, y)$ are rational functions in y and can be expressed in the form $g_{ij}(x, y) = \theta(x, y) a_{ij}(x, y)$ with $\theta(x, y) = 1$ and all $a_{ij}(x, y)$ are rational functions in y . That is, F is a rational Finsler metric. $\square$

Remark 5.5. The above Lemma shows that if F is a rational function in y , then F is a rational Finsler metric. However, the converse is not true. Indeed, *Example 1* has F rational Finsler metric while F is not itself a rational function in y .

Theorem 5.6. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$. If F is a rational function in y , then the generalized ϕ -Matsumoto metric $\hat{F}$ is a rational Finsler metric.*

Proof. As F is a rational function in directional argument, F is a rational Finsler metric, by Lemma 5.4, that is, $g_{ij}(x, y) = \theta(x, y) a_{ij}(x, y)$ with $\theta(x, y) = 1$ and $a_{ij}(x, y)$ are rational functions in y .

Under the ϕ -Matsumoto change (3.5), the local expression of its Finsler metric components (3.13) are given by

$$\begin{aligned} \hat{g}_{ij} &= \frac{F^2(F - 2\Phi)}{(F - \Phi)^3} g_{ij} + \frac{3F^4}{(F - \Phi)^4} \phi_i \phi_j + \frac{F^2 \Phi (4\Phi - F)}{(F - \Phi)^4} \ell_i \ell_j \\ &\quad + \frac{F^3(F - 4\Phi)}{(F - \Phi)^4} \{\phi_i \ell_j + \phi_j \ell_i\}, \end{aligned} \quad (5.2)$$

which can be written in the form

$$\begin{aligned} \hat{g}_{ij} &= \frac{F^2}{(F - \Phi)^4} \left\{ (F - \Phi)(F - 2\Phi) a_{ij} + 3F^2 \phi_i \phi_j + \Phi(4\Phi - F) \ell_i \ell_j \right\} \\ &\quad + \frac{F^2}{(F - \Phi)^4} F(F - 4\Phi) \{\phi_i \ell_j + \phi_j \ell_i\}. \end{aligned}$$

The following functions are obviously rational in y

$$\begin{aligned} \ell_i \ell_j &= \frac{g_{ri} y^r}{F} \frac{g_{kj} y^k}{F} = \frac{g_{ri} y^r g_{kj} y^k}{F^2} = \frac{a_{ri} y^r a_{kj} y^k}{a_{ms} y^m y^s} = \frac{a_{ri} y^r a_{kj} y^k}{a_{ms} y^m y^s}, \\ F \ell_i &= g_{ri} y^r = a_{ri} y^r. \end{aligned}$$

Now, setting

$$\hat{\theta} = \frac{F^2}{(F - \Phi)^4}, \quad (5.3)$$

$$\begin{aligned} \hat{a}_{ij} = & (F^2 + 2\Phi^2) a_{ij} + 3F^2 \phi_i \phi_j + 4\Phi^2 \ell_i \ell_j - 4\Phi \{ \phi_i F \ell_j + \phi_j F \ell_i \} \\ & - F\Phi(3a_{ij} + \ell_i \ell_j) + F \{ \phi_i F \ell_j + \phi_j F \ell_i \}, \end{aligned} \quad (5.4)$$

imply that $\hat{g}_{ij} = \hat{\theta} \hat{a}_{ij}$. By Remark 3.2, the functions ϕ_i , for all $i = 1, \dots, n$, are independent of y and Φ is linear in y . Thereby, as F is a rational function in y , each of the functions $\hat{\theta}$ and $\hat{a}_{ij}$ are rational in y . Hence, $\hat{F}$ is a rational Finsler metric. $\square$

The following result provides an answer to the above mentioned question.

Theorem 5.7. *Let (M, F) be a Finsler manifold admitting a concurrent π -vector field $\bar{\varphi}$. If F is a rational Finsler metric, then the following assertions are equivalent:*

- (a) F is a rational function in y . (b) $\hat{F}$ is a rational Finsler metric.

Proof. (a) $\implies$ (b): It follows from Theorem 5.6.

(b) $\implies$ (a): Now, suppose $\hat{F}$ is a rational Finsler metric. In fact, the Finsler metric tensor $\hat{g}_{ij}$, given by (5.2), can be written as $\hat{g}_{ij} = \hat{\theta} \hat{a}_{ij}$ with $\hat{\theta} = \frac{\theta F^2}{(F - \Phi)^4}$,

$$\begin{aligned} \hat{a}_{ij} = & \left\{ (F^2 + 2\Phi^2) a_{ij} + 3a_{rk} y^k y^r \phi_i \phi_j + 4\Phi^2 \frac{a_{ri} y^r a_{kj} y^k}{a_{ms} y^m y^s} - 4\Phi \{ \phi_i a_{rj} y^r + \phi_j a_{ri} y^r \} \right\} \\ & - F \left\{ \Phi(3a_{ij} + \frac{a_{ri} y^r a_{kj} y^k}{a_{ms} y^m y^s}) + \phi_i a_{rj} y^r + \phi_j a_{ri} y^r \right\}. \end{aligned}$$

Since F is a rational Finsler metric, the functions

$$\left\{ (F^2 + 2\Phi^2) a_{ij} + 3a_{rk} y^k y^r \phi_i \phi_j + 4\Phi^2 \frac{a_{ri} y^r a_{kj} y^k}{a_{ms} y^m y^s} - 4\Phi \{ \phi_i a_{rj} y^r + \phi_j a_{ri} y^r \} \right\} \quad (5.5)$$

are rational in y for all i, j . However, the functions

$$F \left\{ -\Phi(3a_{ij} + \frac{a_{ri} y^r a_{kj} y^k}{a_{ms} y^m y^s}) + \phi_i a_{rj} y^r + \phi_j a_{ri} y^r \right\}$$

are rational functions in y if and only if F is a rational function in y . That is, $\hat{a}_{ij}$ are rational functions in y if and only if F is a rational function in y . Which is equivalent to $\hat{F}$ is a rational Finsler metric if and only if F is a rational function in y . $\square$

Remark 5.8. If F is an almost rational Finsler metric, i.e., θ is not a rational function in y , we can not conclude anything about the almost rationality of $\hat{F}$ that can be seen from (5.5).

Acknowledgements

This research paper is dedicated to the memory of Professor Nabil L. Youssef. Also, the authors would like to thank Professor Erasmo Caponio for his careful reading and valuable comments, which improved this paper.

Author contributions. The authors have made substantive contributions to the article and assume full responsibility for its content. The authors read and approved the final manuscript.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. Not applicable.

Data availability. No data was used for the research described in the article.

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