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Research Article

The m-weak group orthogonality for operators

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Abstract

The main goal is extending the concept of the core–EP orthogonality to the m-weak group orthogonality for bounded linear Drazin invertible Hilbert space operators, using the m-weak group inverse. Different properties and characterizations of m-weak group orthogonal operators are proved as well as their operator matrix forms. The connection between the m-weak group binary relation and the m-weak group orthogonality is given. We also study additive properties for the m-weak group inverse. Consequently, we study the weak group orthogonality for operators.

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1. Introduction

In this paper, X and Y are infinite-dimensional complex Hilbert spaces and $B(X,Y)$ represents the set of all bounded linear operators from X to Y. In the case that $X=Y$, we set $B(X)=B(X,X)$. For $A\in B(X,Y)$, denote that A^* is the adjoint of A, $N(A)$ is the null space of A and $R(A)$ is the range of A.

For $A\in B(X,Y)$, where $R(A)$ is closed in Y, there is unique $B\in B(Y,X)$, which is called the Moore-Penrose inverse of A, denoted by $A^\dagger$, like in [2], satisfying $ABA=A$, $BAB=B$, $(AB)^*=AB$ and $(BA)^*=BA$. An inner inverse of A is an operator B which satisfies the condition $ABA=A$. Let $A\{1\}$ denote the set of all inner inverses of A.

The Drazin inverse A^D of $A\in B(X)$ is unique solution to the next four equations $AB=BA$, $BAB=B$ and $A^{k+1}B=A^k$, for some non-negative integer k [2]. The smallest such k is the index $\operatorname{ind}(A)=k$ of A. For $\operatorname{ind}(A)=1$, A^D reduces to the group inverse of A. $B(X)^D$ represents the set of Drazin invertible operators of $B(X)$. The core–EP inverse of $A\in B(X)^D$, denoted by $A^{\textcircled{D}}$ is unique solution to $BAB=B$ and $R(B)=R(A^*)=R(A^k)$, where $\operatorname{ind}(A)=k$. From [6,24], we know:

$$A^{\textcircled{D}}=A^DA^k(A^k)^\dagger.$$

Many other important results of the core–EP inverse can be found in [1,3,4,12,13,21].

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One of the most important generalized inverses is the group inverse, which has been applied in solving differential equations and many other problems, for example Markov chains [2]. The weak group inverse (or WGI) for square matrices with arbitrary index was defined as a generalization of the group inverse [22]. By [20, 22], for $A \in B(X)^D$, we have the expression for the weak group inverse as: $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$. Then, in the case when $\operatorname{ind}(A) = 1$, the WGI reduces to the group inverse. Many properties of the WGI inverse are introduced in [23, 25].

The m-weak group inverse is the extension of the weak group inverse, which is introduced in [26]. For an arbitrary $m \in \mathbb{N}$ and $A \in B(X)^D$, the m-weak group inverse of A is the unique operator $A^{\textcircled{w},m}$ satisfying the system of equations: $AB = (A^{\textcircled{\dagger}})^m A^m$ and $AB^2 = B$. Notice that the m-weak group inverse can be expressed by:

$$A^{\textcircled{w},m} = (A^{\textcircled{\dagger}})^{m+1} A^m$$

and it is an outer inverse of A, i.e. $A^{\textcircled{w},m} A A^{\textcircled{w},m} = A^{\textcircled{w},m}$. Also, $A(A^{\textcircled{w},m})^2 = A^{\textcircled{w},m}$. If $m = 1$, the m-weak group inverse reduces to weak group inverse. For the application and properties of the m-weak group inverse see [8, 10, 11, 18, 19].

Various pre-orders and partial orders are explained in terms of various generalized inverse. For $A, B \in B(X)^D$ the core–EP pre order [15] is defined as $A \leq^{\textcircled{\dagger}} B$ when the following is satisfied: $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$ and $A^{\textcircled{\dagger}} A = A^{\textcircled{\dagger}} B$.

The m-weak group binary relation is introduced for operators in [9] as an extension of the core–EP pre-order for operators. For $A, B \in B(X)^D$ and $m \in N$, we say that A is below to B with respect to the m-weak group relation (denoted by $A \leq_{\textcircled{w},m} B$) if

$$A^{\textcircled{w},m} B = A^{\textcircled{w},m} A \qquad \text{and} \qquad BA^{\textcircled{w},m} = AA^{\textcircled{w},m}.$$

Also, by [9], we say that A is below to B with respect to the weak group relation (denoted by $A \leq_{\textcircled{w}} B$) if $A^{\textcircled{w}} B = A^{\textcircled{w}} A$ and $BA^{\textcircled{w}} = AA^{\textcircled{w}}$.

Let us remind the definition of orthogonality for $A, B \in B(X)$. If $AB = 0$ and $BA = 0$, the operators A and B are orthogonal which is denoted by $A \perp B$. Further, A and B are $*$-orthogonal (denoted by $A \perp_* B$) if $A^* B = 0$ and $BA^* = 0$ (range and domain orthogonality) [7]. Also, A and B of index 1 are the core orthogonal (denoted by $A \perp_{\textcircled{\#}} B$) [5] if $A^{\textcircled{\#}} B = 0$ and $BA^{\textcircled{\#}} = 0$, which is equivalent to $A^* B = 0$ and $BA = 0$.

In [16], the concept of the core–EP orthogonality is defined for a pair of Drazin invertible bounded linear operators on a Hilbert space. Let us remind the definition of core–EP orthogonality for two operators. Let $A, B \in B(X)^D$. Then A is core–EP orthogonal to B, denoted by $A \perp_{\textcircled{\dagger}} B$, if $A^{\textcircled{\dagger}} B = 0$ and $BA^{\textcircled{\dagger}} = 0$. Thus, the core–EP orthogonality is a generalization of the core orthogonality. The relation between the core–EP orthogonality and the core–EP additivity $(A + B)^{\textcircled{\dagger}} = A^{\textcircled{\dagger}} + B^{\textcircled{\dagger}}$ is investigated in [16].

The main goal of this paper is to explore the orthogonality of bounded linear Drazin invertible Hilbert space operators and extend earlier results for core–EP orthogonality. Based on the m-weak group inverse as a generalization of the core–EP inverse, the notion of the m-weak group orthogonality is introduced extending the core–EP orthogonality. Different properties and characterizations of the m-weak group orthogonality are given. The operator matrix forms of m-weak group orthogonal operators are developed. The m-weak group binary relation is connected with the m-weak group orthogonality. For core–EP orthogonal operators, we present equivalent conditions for additivity $(A + B)^{\textcircled{w},m} = A^{\textcircled{w},m} + B^{\textcircled{w},m}$ to be satisfied. As consequences, we obtain results related to the weak group orthogonality, the weak group relation and the weak group additivity.

Our paper contains the following two sections. Exactly, the Section 2 contains all new results. Here, we begin with the definition and characterizations of the m-weak orthogonality for operators. Further, we consider the m-weak group relation and the m-weak group additivity $(A + B)^{\textcircled{w},m} = A^{\textcircled{w},m} + B^{\textcircled{w},m}$. We conclude the paper in Section 3 with some final remarks.

2. m-weak group orthogonality

The m-weak group orthogonality is introduced in this section for operators as an extension of the core–EP orthogonality for operators.

Definition 2.1. For $A, B \in B(X)$ and $m \in \mathbb{N}$, we say that A is m-weak group orthogonal to B (denoted by $A \perp_{\textcircled{w},m} B$) if

$$A^{\textcircled{w},m} B = 0 \qquad \text{and} \qquad BA^{\textcircled{w},m} = 0.$$

In the case that $m = 1$ in Definition 2.1, we define the weak group orthogonality.

Definition 2.2. For $A, B \in B(X)$, we say that A is weak group orthogonal to B (denoted by $A \perp_{\textcircled{w}} B$) if

$$A^{\textcircled{w}} B = 0 \qquad \text{and} \qquad BA^{\textcircled{w}} = 0.$$

We start with the following characterizations of the m-weak group orthogonality.

Theorem 2.3. *Let $A, B \in B(X)^D$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $A \perp_{\textcircled{w},m} B$;
(ii) $A^{\textcircled{w},m} B$ *and* $BA^{\textcircled{w},m}$ *are idempotents and* $A^{\textcircled{w},m}(A + B)A^{\textcircled{w},m} = A^{\textcircled{w},m}$;
(iii) $A^{\textcircled{w},m} B = BA^{\textcircled{w},m}$ *and* $A^{\textcircled{w},m}(A + B)A^{\textcircled{w},m} = A^{\textcircled{w},m}$;
(iv) $B = (I - AA^{\textcircled{w},m})G(I - A^{\textcircled{w},m} A)$, *for arbitrary* $G \in B(X)$.

Proof. (i) $\Longrightarrow$ (ii) $\wedge$ (iii): From $A \perp_{\textcircled{w},m} B$, we have that $A^{\textcircled{w},m} B = 0$ and $BA^{\textcircled{w},m} = 0$, which implies that $A^{\textcircled{w},m} B$ and $BA^{\textcircled{w},m}$ are idempotents. Also, by $A^{\textcircled{w},m} AA^{\textcircled{w},m} = A^{\textcircled{w},m}$, we get

$$A^{\textcircled{w},m}(A + B)A^{\textcircled{w},m} = A^{\textcircled{w},m} AA^{\textcircled{w},m} + A^{\textcircled{w},m} BA^{\textcircled{w},m} = A^{\textcircled{w},m}.$$

(ii) $\Longrightarrow$ (i): Since $A^{\textcircled{w},m} AA^{\textcircled{w},m} = A^{\textcircled{w},m}$, it follows $A^{\textcircled{w},m} = A^{\textcircled{w},m}(A + B)A^{\textcircled{w},m} = A^{\textcircled{w},m} + A^{\textcircled{w},m} BA^{\textcircled{w},m}$. Hence, $A^{\textcircled{w},m} BA^{\textcircled{w},m} = 0$, which gives $A^{\textcircled{w},m} B = (A^{\textcircled{w},m} B)^2 = 0$ and $BA^{\textcircled{w},m} = (BA^{\textcircled{w},m})^2 = 0$. So, $A \perp_{\textcircled{w},m} B$.

(iii) $\Longrightarrow$ (i): By the conditions in (iii), we have $A^{\textcircled{w},m} = A^{\textcircled{w},m}(A + B)A^{\textcircled{w},m} = A^{\textcircled{w},m} + (A^{\textcircled{w},m})^2 B$, which yields $(A^{\textcircled{w},m})^2 B = 0$. Therefore, by $A(A^{\textcircled{w},m})^2 = A^{\textcircled{w},m}$, $A^{\textcircled{w},m} B = A(A^{\textcircled{w},m})^2 B = 0$ and $BA^{\textcircled{w},m} = A^{\textcircled{w},m} B = 0$.

(i) $\Longrightarrow$ (iv): The equation $A^{\textcircled{w},m} B = 0$ has a solution, by $A \in A^{\textcircled{w},m}\{1\}$ and [2, p. 52], in the form

$$B = (I - AA^{\textcircled{w},m})H, \tag{2.1}$$

for arbitrary $H \in B(X)$. When (2.1) is substituted in $BA^{\textcircled{w},m} = 0$, it follows

$$(I - AA^{\textcircled{w},m})HA^{\textcircled{w},m} = 0. \tag{2.2}$$

Now, by $I - AA^{\textcircled{w},m} \in (I - AA^{\textcircled{w},m})\{1\}$ and [2, p. 52],

$$H = G - (I - AA^{\textcircled{w},m})GA^{\textcircled{w},m} A, \tag{2.3}$$

for arbitrary $G \in B(X)$. The equlities (2.1) and (2.3) give $B = (I - AA^{\textcircled{w},m})G(I - A^{\textcircled{w},m} A)$.

(iv) $\Longrightarrow$ (i): If $B = (I - AA^{\textcircled{w},m})M(I - A^{\textcircled{w},m} A)$, for arbitrary $M \in B(X)$, we calculate that $A^{\textcircled{w},m} B = 0$ and $BA^{\textcircled{w},m} = 0$. □

By Theorem 2.3, we obtain characterizations of the weak group orthogonality.

Corollary 2.4. *Let $A, B \in B(X)^D$. Then the following statements are equivalent:*

(i) $A \perp_{\textcircled{w}} B$;
(ii) $A^{\textcircled{w}} B$ *and* $BA^{\textcircled{w}}$ *are idempotents and* $A^{\textcircled{w}}(A + B)A^{\textcircled{w}} = A^{\textcircled{w}}$;
(iii) $A^{\textcircled{w}} B = BA^{\textcircled{w}}$ *and* $A^{\textcircled{w}}(A + B)A^{\textcircled{w}} = A^{\textcircled{w}}$;
(iv) $B = (I - AA^{\textcircled{w}})G(I - A^{\textcircled{w}} A)$, *for arbitrary* $G \in B(X)$.

Necessary and sufficient conditions for $A^{\textcircled{w},m} B = 0$ are presented now.

Lemma 2.5. *For $A, B \in B(X)^D$, $m \in \mathbb{N}$ and $\operatorname{ind}(A) = k$, the following statements are equivalent:*

(i) $A^{\textcircled{w},m}B = 0$;
(ii) $A^{\textcircled{\dagger}}A^m B = 0$;
(iii) $(A^k)^\dagger A^m B = 0$;
(iv) $(A^k)^* A^m B = 0$;
(v) $R(B) \subseteq N(A^{\textcircled{w},m})$;
(vi) $R(B) \subseteq N((A^k)^* A^m)$.

Proof. (i) $\iff$ (ii): According to [19, Lemma 2.1], $A^{\textcircled{w},m} = (A^D)^{m+1}A^k(A^k)^\dagger A^m$. Using $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$, we have the following consequence:

$$\begin{aligned} A^{\textcircled{w},m}B = 0 &\iff (A^D)^{m+1}A^k(A^k)^\dagger A^m B = 0 \\ &\iff A^D A^k (A^k)^\dagger A^m B = 0 \\ &\iff A^{\textcircled{\dagger}} A^m B = 0. \end{aligned}$$

(ii) $\iff$ (iii): By the properties of the core–EP inverse, we have the next equivalences:

$$\begin{aligned} A^{\textcircled{\dagger}}A^m B = 0 &\iff AA^D A^k (A^k)^\dagger A^m B = 0 \\ &\iff A^k (A^k)^\dagger A^m B = 0 \\ &\iff (A^k)^\dagger A^m B = 0. \end{aligned}$$

(iii) $\iff$ (iv): It is clear by properties of the Moore-Penrose inverse.
(i) $\iff$ (v) $\iff$ (vi): Obviously because $N(A^{\textcircled{w},m}) = N((A^k)^* A^m)$ by [8]. ☐

We also study equivalent conditions for $BA^{\textcircled{w},m} = 0$.

Lemma 2.6. *For $A, B \in B(X)^D$, $m \in \mathbb{N}$ and $\operatorname{ind}(A) = k$, the following statements are equivalent:*

(i) $BA^{\textcircled{w},m} = 0$;
(ii) $BA^{\textcircled{\dagger}} = 0$;
(iii) $BA^D = 0$;
(iv) $BA^k = 0$;
(v) $R(A^{\textcircled{w},m}) \subseteq N(B)$;
(vi) $R(A^k) \subseteq N(B)$.

Proof. (i) $\Longrightarrow$ (ii): Applying $A^{\textcircled{w},m} = (A^D)^{m+1}A^k(A^k)^\dagger A^m$, $BA^{\textcircled{w},m} = 0$ is equivalent to $B(A^D)^{m+1}A^k(A^k)^\dagger A^m = 0$, which gives

$$BA^D A^k = B(A^D)^{m+1}A^k A^m = B(A^D)^{m+1}A^k(A^k)^\dagger A^m A^k = 0.$$

Since $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$, it follows $BA^{\textcircled{\dagger}} = BA^D A^k (A^k)^\dagger = 0$.
(ii) $\Longrightarrow$ (i): Note that $BA^{\textcircled{\dagger}} = 0$ implies $BA^{\textcircled{w},m} = B(A^{\textcircled{\dagger}})^{m+1}A^m = 0$.
(ii) $\iff$ (iii): These equivalence follows by $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$.
The rest is clear. ☐

If we combine the conditions of Lemma 2.5 and Lemma 2.6, we can characterize the m-weak group orthogonality.

Theorem 2.7. *Let $A, B \in B(X)^D$, $m \in N$ and $\operatorname{ind}(A) = k$. Then the following statements are equivalent:*

(i) $A \perp_{\textcircled{w},m} B$;
(ii) $A^{\textcircled{\dagger}}A^m B = 0$ *and* $BA^{\textcircled{\dagger}} = 0$;
(iii) $(A^k)^\dagger A^m B = 0$ *and* $BA^D = 0$;
(iv) $(A^k)^* A^m B = 0$ *and* $BA^k = 0$;
(v) $R(B) \subseteq N(A^{\textcircled{w},m})$ *and* $R(A^{\textcircled{w},m}) \subseteq N(B)$;

(vi) $R(B) \subseteq N((A^k)^*A^m)$ *and* $R(A^k) \subseteq N(B)$.

Consequently, we get characterizations for the weak group orthogonality.

Corollary 2.8. *Let* $A, B \in B(X)^D$ *and* $\operatorname{ind}(A) = k$. *Then the following statements are equivalent:*

(i) $A \perp_{\textcircled{w}} B$;
(ii) $A^{\textcircled{\dagger}} AB = 0$ *and* $BA^{\textcircled{\dagger}} = 0$;
(iii) $(A^k)^\dagger AB = 0$ *and* $BA^D = 0$;
(iv) $(A^k)^*AB = 0$ *and* $BA^k = 0$;
(v) $R(B) \subseteq N(A^{\textcircled{w}})$ *and* $R(A^{\textcircled{w}}) \subseteq N(B)$;
(vi) $R(B) \subseteq N((A^k)^*A)$ *and* $R(A^k) \subseteq N(B)$.

The assumption $A \perp_{\textcircled{w},m} B$ gives the next equalities related to products of some idempotents.

Lemma 2.9. *Let* $A, B \in B(X)^D$, $m \in \mathbb{N}$ *and* $A \perp_{\textcircled{w},m} B$. *Then the following statements are valid:*

(i) $B^D BA^D A = 0$;
(ii) $AA^{\textcircled{w},m} BB^{\textcircled{w},m} = 0$;
(iii) $B^{\textcircled{w},m} BA^{\textcircled{w},m} A = 0$.

Proof. (i) $BA^D BA^D A = 0$ follows from $BA^D = 0$, which is proved in Lemma 2.6.
(ii) $AA^{\textcircled{w},m} BB^{\textcircled{w},m} = 0$ which is following from $A^{\textcircled{w},m} B = 0$.
(iii) $B^{\textcircled{w},m} BA^{\textcircled{w},m} A = 0$ because of the statement $BA^{\textcircled{w},m} = 0$. □

The following operator matrix form of a Drazin invertible operator was presented in [15], and its m-weak group inverse in [8].

Lemma 2.10. *If* $A \in B(X)^D$, $m \in \mathbb{N}$ *and* $\operatorname{ind}(A) = k$, *there is the orthogonal sum* $X = R(A^k) \oplus N((A^k)^*)$ *such that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \tag{2.4}$$

where $A_1 \in B(R(A^k))$ *is invertible and* $A_3 \in B(N((A^k)^*))$ *is nilpotent. In addition,*

(i) $A^{\textcircled{w},m} = \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & 0 \end{bmatrix}$;

(ii) $AA^{\textcircled{w},m} = \begin{bmatrix} I & A_1^{-m} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & 0 \end{bmatrix}$;

(iii) $A^{\textcircled{w},m} A = \begin{bmatrix} I & A_1^{-(m+1)} \sum_{j=0}^{m} A_1^j A_2 A_3^{m-j} \\ 0 & 0 \end{bmatrix}$.

Proof. The equality (2.4) holds by [15, Corollary 2.2]. The statements (i), (ii) and (iii) are presented and proved in [8]. □

The operator matrix forms of A and B which satisfy $A \perp_{\textcircled{w},m} B$ are given.

Theorem 2.11. *Let* $A, B \in B(X)^D$, $m \in \mathbb{N}$ *and* $\operatorname{ind}(A) = k$. *Then the following statements are equivalent:*

(i) $A \perp_{\textcircled{w},m} B$;

(ii) *there is the orthogonal sum* $X = R(A^k) \oplus N((A^k)^*)$ *such that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -A_1^{-m} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_4 \\ 0 & B_4 \end{bmatrix},$$

where $A_1 \in B(R(A^k))$ *is invertible,* $A_3 \in B(N((A^k)^*))$ *is nilpotent and* $B_4 \in B(N((A^k)^*))^D$.

Proof. (i) $\Longrightarrow$ (ii): Let A has the form as in (2.4) with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$ and

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} : \begin{bmatrix} R(A^k) \\ N((A^k)^* \end{bmatrix} \to \begin{bmatrix} R(A^k) \\ N((A^k)^* \end{bmatrix}.$$

Then $A^{\textcircled{w},m}$ is represented by Lemma 2.10(i). When the condition $A \perp_{\textcircled{w},m} B$ is satisfied,

$$0 = BA^{\textcircled{w},m} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} A_1^{-1} & A_1^{-(m-1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_1 A_1^{-1} & B_1 A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ B_3 A_1^{-1} & B_3 A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \end{bmatrix},$$

which implies $B_1 = 0$ and $B_3 = 0$. From

$$0 = A^{\textcircled{w},m} B = \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ 0 & B_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A_1^{-1} B_2 + A_1^{-(m+1)} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_4 \\ 0 & 0 \end{bmatrix},$$

we get $B_2 = -A_1^{-m} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_4$. So, we can use matrix form of operator B as equal to

$$B = \begin{bmatrix} 0 & -A_1^{-m} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j} B_4 \\ 0 & B_4 \end{bmatrix}.$$

(ii) $\Longrightarrow$ (i): It is clear by direct calculations. □

Theorem 2.11 gives the next equivalent condition for $A \perp_{\textcircled{w}} B$.

Corollary 2.12. *Let* $A, B \in B(X)^D$ *and* $\operatorname{ind}(A) = k$. *Then the following statements are equivalent:*

(i) $A \perp_{\textcircled{w}} B$;

(ii) *there is the orthogonal sum* $X = R(A^k) \oplus N((A^k)^*)$ *such that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -A_1^{-1} A_2 B_4 \\ 0 & B_4 \end{bmatrix},$$

where $A_1 \in B(R(A^k))$ *is invertible,* $A_3 \in B(N((A^k)^*))$ *is nilpotent and* $B_4 \in B(N((A^k)^*))^D$.

The following characterizations of the m-weak group relation can be verified as Lemma 2.5 and Lemma 2.6.

Theorem 2.13. *For $A, B \in B(X)^D$, $m \in \mathbb{N}$ and $\mathrm{ind}(A) = k$, the following statements are equivalent:*

(i) $A \leq_{\textcircled{w},m} B$;
(ii) $A^{\textcircled{\dagger}} A^m B = A^{\textcircled{\dagger}} A^{m+1}$ *and* $BA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}$;
(ii) $(A^k)^{\dagger} A^m B = (A^k)^{\dagger} A^{m+1}$ *and* $BA^D = AA^D$;
(iv) $(A^k)^* A^m B = (A^k)^* A^{m+1}$ *and* $BA^k = A^{k+1}$;
(v) $R(B - A) \subseteq N(A^{\textcircled{w},m})$ *and* $R(A^{\textcircled{w},m}) \subseteq N(B - A)$;
(vi) $R(B - A) \subseteq N((A^k)^* A^m)$ *and* $R(A^k) \subseteq N(B - A)$.

Consequently, we get the result about the weak group relation.

Corollary 2.14. *For $A, B \in B(X)^D$ and $\mathrm{ind}(A) = k$, the following statements are equivalent:*

(i) $A \leq_{\textcircled{w}} B$;
(ii) $A^{\textcircled{\dagger}} AB = A^{\textcircled{\dagger}} A^2$ *and* $BA^{\textcircled{\dagger}} = AA^{\textcircled{\dagger}}$;
(ii) $(A^k)^{\dagger} AB = (A^k)^{\dagger} A^2$ *and* $BA^D = AA^D$;
(iv) $(A^k)^* AB = (A^k)^* A^2$ *and* $BA^k = A^{k+1}$;
(v) $R(B - A) \subseteq N(A^{\textcircled{w}})$ *and* $R(A^{\textcircled{w}}) \subseteq N(B - A)$;
(vi) $R(B - A) \subseteq N((A^k)^* A)$ *and* $R(A^k) \subseteq N(B - A)$.

We characterize $A \perp_{\textcircled{w},m} B$ by the m-weak group relation.

Theorem 2.15. *Let $A, B \in B(X)^D$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $A \leq_{\textcircled{w},m} A + B$;
(ii) $A \perp_{\textcircled{w},m} B$;
(iii) $A \leq_{\textcircled{w},m} A - B$.

Proof. We observe that

$$\begin{aligned} A \leq_{\textcircled{w},m} A + B &\iff (A+B)A^{\textcircled{w},m} = AA^{\textcircled{w},m} \quad \text{and} \quad A^{\textcircled{w},m}(A+B) = A^{\textcircled{w},m}A \\ &\iff BA^{\textcircled{w},m} = 0 \quad \text{and} \quad A^{\textcircled{w},m}B = 0 \iff A \perp_{\textcircled{w},m} B \\ &\iff A^{\textcircled{w},m}(A-B) = A^{\textcircled{w},m}A \quad \text{and} \quad (A-B)A^{\textcircled{w},m} = AA^{\textcircled{w},m} \\ &\iff A \leq_{\textcircled{w},m} (A - B) \end{aligned}$$

□

In the case that $A \leq_{\textcircled{w},m} B$, by [9], we have operator matrix forms of A and B and now we investigate the form of $B^{\textcircled{w},m}$.

Theorem 2.16. *Let $A, B \in B(X)^D$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $A \leq_{\textcircled{w},m} B$;
(ii) *there is the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$ such that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & A_2 + A_1^{-m} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j}(A_3 - B_4) \\ 0 & B_4 \end{bmatrix},$$

where $A_1 \in B(R(A^k))$ is invertible, $A_3 \in B(N((A^k)^*))$ is nilpotent and $B_4 \in B(N((A^k)^*))^D$.

Moreover,

$$B^{\textcircled{w},m} = \begin{bmatrix} A_1^{-1} & \sum_{i=0}^{m-1} A_1^{-i-2} B_2 B_4^i - \sum_{i=0}^{m} A_1^{-(m+1)+i} B_2 (B_4^{\textcircled{\dagger}})^{i+1} B_4^m \\ 0 & B_4^{\textcircled{w},m} \end{bmatrix}.$$

Proof. (i) $\Longrightarrow$ (ii): The operator A can be written by (2.4) with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$ and $A^{\textcircled{w},m}$ is represented by Lemma 2.10(i). By [9], we can use the matrix form of operator B as equal to

$$B = \begin{bmatrix} A_1 & B_2 \\ 0 & B_4 \end{bmatrix},$$

where $B_2 = A_2 + A_1^{-m} \sum_{j=0}^{m-1} A_1^j A_2 A_3^{m-1-j}(A_3 - B_4)$. The hypothesis $B \in B(X)^D$ gives the Drazin invertibility of B_4.

Now, we calculate $B^{\textcircled{w},m}$. Applying [14, Lemma 2.3], we have that:

$$B^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & -A_1^{-1} B_2 B_4^{\textcircled{D}} \\ 0 & B_4^{\textcircled{D}} \end{bmatrix}.$$

Then we calculate:

$$(B^{\textcircled{D}})^{m+1} = \begin{bmatrix} A_1^{-(m+1)} & -\sum_{i=0}^{m} A_1^{-(m+1)+i} B_2 (B_4^{\textcircled{D}})^{i+1} \\ 0 & (B_4^{\textcircled{D}})^{m+1} \end{bmatrix}$$

and

$$B^m = \begin{bmatrix} A_1^m & \sum_{i=0}^{m-1} A_1^{m-i-1} B_2 B_4^i \\ 0 & B_4^m \end{bmatrix}.$$

Therefore,

$$B^{\textcircled{w},m} = (B^{\textcircled{D}})^{m+1} B^m = \begin{bmatrix} A_1^{-1} & \sum_{i=0}^{m-1} A_1^{-i-2} B_2 B_4^i - \sum_{i=0}^{m} A_1^{-(m+1)+i} B_2 (B_4^{\textcircled{D}})^{i+1} B_4^m \\ 0 & B_4^{\textcircled{w},m} \end{bmatrix}.$$

(ii) $\Longrightarrow$ (i): It follows by elementary computations. □

As a consequence of Theorem 2.16, we obtain the following result.

Corollary 2.17. *Let $A, B \in B(X)^D$. Then the following statements are equivalent:*

(i) $A \leq_{\textcircled{w}} B$;

(ii) *there is the orthogonal sum $X = R(A^k) \oplus N((A^k)^*$ such that*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & A_2 + A_1^{-1} A_2 (A_3 - B_4) \\ 0 & B_4 \end{bmatrix},$$

where $A_1 \in B(R(A^k))$ is invertible, $A_3 \in B(N((A^k)^*))$ is nilpotent and $B_4 \in B(N((A^k)^*))^D$.

Moreover,

$$B^{\textcircled{w}} = \begin{bmatrix} A_1^{-1} & A_1^{-2} B_2 - \sum_{i=0}^{1} A_1^{-2+i} B_2 (B_4^{\textcircled{D}})^{i+1} B_4 \\ 0 & B_4^{\textcircled{w}} \end{bmatrix}.$$

Under the assumption $A \perp_{\textcircled{D}} B$, we consider necessary and sufficient conditions for $(A+B)^{\textcircled{w},m} = A^{\textcircled{w},m} + B^{\textcircled{w},m}$.

Theorem 2.18. *Let $A, B \in B(X)^D$ and $m \in \mathbb{N}$. If $A \perp_{\textcircled{D}} B$, the following statements are equivalent:*

(i) $A + B \in B(X)^D$ *and* $(A+B)^{\textcircled{w},m} = A^{\textcircled{w},m} + B^{\textcircled{w},m}$;

(ii) $A + B \in B(X)^D$ *and* $(A+B)^{\textcircled{w},m}(I - AA^{\textcircled{D}}) = (A^{\textcircled{w},m} + B^{\textcircled{w},m})(I - AA^{\textcircled{D}})$;

(iii) $A + B \in B(X)^D$, $AA^{\textcircled{D}}(A+B)^{\textcircled{w},m} = A^{\textcircled{w},m}$ *and* $(I - AA^{\textcircled{D}})(A+B)^{\textcircled{w},m} = B^{\textcircled{w},m}$;

(iv) $A+B \in B(X)^D$, $A^{\textcircled{D}}(A+B)^{\textcircled{W},m} = A^{\textcircled{D}}A^{\textcircled{W},m}$ *and* $(I-AA^{\textcircled{D}})(A+B)^{\textcircled{W},m} = B^{\textcircled{W},m}$.

Proof. Using [16, Theorem 2.1] and $A\perp_{\textcircled{D}}B$, for $\text{ind}(A)=k$, we can write with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $A_1 \in B(R(A^k))$ is invertible, $A_3 \in B[N((A^k)^*)]$ is nilpotent and $B_2 \in B[N((A^k)^*)]^D$. Now,

$$A^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad A^{\textcircled{W},m} = \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)}\sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j} \\ 0 & 0 \end{bmatrix},$$

$$B^{\textcircled{W},m} = \begin{bmatrix} 0 & 0 \\ 0 & B_2^{\textcircled{W},m} \end{bmatrix} \quad \text{and} \quad A^{\textcircled{W},m}+B^{\textcircled{W},m} = \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)}\sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j} \\ 0 & B_2^{\textcircled{W},m} \end{bmatrix}.$$

(i) $\iff$ (ii): The Drazin invertibility of

$$A+B = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3+B_2 \end{bmatrix}$$

gives the Drazin invertibility of A_3+B_2. By [14, Lemma 2.3], for a corresponding operator S, we have

$$(A+B)^{\textcircled{D}} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2(A_3+B_2)^{\textcircled{D}} \\ 0 & (A_3+B_2)^{\textcircled{D}} \end{bmatrix}$$

and

$$(A+B)^{\textcircled{W},m} = [(A+B)^{\textcircled{D}}]^{m+1}(A+B)^m = \begin{bmatrix} A_1^{-1} & S \\ 0 & (A_3+B_2)^{\textcircled{W},m} \end{bmatrix}.$$

Because

$$(A+B)^{\textcircled{W},m}(I-AA^{\textcircled{D}}) = \begin{bmatrix} 0 & S \\ 0 & (A_3+B_2)^{\textcircled{W},m} \end{bmatrix}$$

and

$$(A^{\textcircled{W},m}+B^{\textcircled{W},m})(I-AA^{\textcircled{D}}) = \begin{bmatrix} 0 & A_1^{-(m+1)}\sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j} \\ 0 & B_2^{\textcircled{W},m} \end{bmatrix},$$

we deduce that $(A+B)^{\textcircled{W},m} = A^{\textcircled{W},m}+B^{\textcircled{W},m}$ if and only if $S = A_1^{-(m+1)}\sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j}$ and $(A_3+B_2)^{\textcircled{W},m} = B_2^{\textcircled{W},m}$ if and only if $(A+B)^{\textcircled{W},m}(I-AA^{\textcircled{D}}) = (A^{\textcircled{W},m}+B^{\textcircled{W},m})(I-AA^{\textcircled{D}})$.

(i) $\iff$ (iii): By the part (i) $\iff$ (ii) and $A(A^{\textcircled{D}})^2 = A^{\textcircled{D}}$, it follows

$$AA^{\textcircled{D}}(A+B)^{\textcircled{W},m} = \begin{bmatrix} A_1^{-1} & S \\ 0 & 0 \end{bmatrix}$$

and

$$A^{\textcircled{W},m} = AA^{\textcircled{D}}A^{\textcircled{W},m} = \begin{bmatrix} A_1^{-1} & A_1^{-(m+1)}\sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j} \\ 0 & 0 \end{bmatrix}.$$

So, $AA^{\textcircled{D}}(A+B)^{\textcircled{W},m} = A^{\textcircled{W},m}$ is equivalent to $S = A_1^{-(m+1)}\sum_{j=0}^{m-1} A_1^jA_2A_3^{m-1-j}$. From

$$(I-AA^{\textcircled{D}})(A+B)^{\textcircled{W},m} = \begin{bmatrix} 0 & 0 \\ 0 & (A_3+B_2)^{\textcircled{W},m} \end{bmatrix},$$

$(I-AA^{\textcircled{\dagger}})(A+B)^{\textcircled{w},m}=B^{\textcircled{w},m}$ if and only if $(A_3+B_2)^{\textcircled{w},m}=B_2^{\textcircled{w},m}$. Thus, the equivalence of statements (i) and (iii) holds.

(iii) $\Longleftrightarrow$ (iv): The properties of the core–EP inverse imply this equivalence. □

Theorem 2.18 yields additive properties for the weak group inverse.

Corollary 2.19. *Let $A, B \in B(X)^D$. If $A\perp_{\textcircled{\dagger}}B$, the following statements are equivalent:*

(i) $A+B\in B(X)^D$ *and* $(A+B)^{\textcircled{w}}=A^{\textcircled{w}}+B^{\textcircled{w}}$;

(ii) $A+B\in B(X)^D$ *and* $(A+B)^{\textcircled{w}}(I-AA^{\textcircled{\dagger}})=(A^{\textcircled{w}}+B^{\textcircled{w}})(I-AA^{\textcircled{\dagger}})$;

(iii) $A+B\in B(X)^D$, $AA^{\textcircled{\dagger}}(A+B)^{\textcircled{w}}=A^{\textcircled{w}}$ *and* $(I-AA^{\textcircled{\dagger}})(A+B)^{\textcircled{w}}=B^{\textcircled{w}}$;

(iv) $A+B\in B(X)^D$, $A^{\textcircled{\dagger}}(A+B)^{\textcircled{w}}=A^{\textcircled{\dagger}}A^{\textcircled{w}}$ *and* $(I-AA^{\textcircled{\dagger}})(A+B)^{\textcircled{w}}=B^{\textcircled{w}}$.

3. Conclusion

Our aim is to generalize the notion of the core–EP orthogonality to the m-weak group orthogonality for operators, based on the m-weak group inverse as an extension of the core–EP inverse. Many characterizations of the m-weak group orthogonal operators are proved including their operator matrix forms. Additive properties for the m-weak group inverse are considered for core–EP orthogonal operators. As consequences, we obtain the results about the weak group orthogonality for operators.

In the future, we are going to extend our results to weighted m-weak group inverse for operators [17].

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