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Picture Fuzzy Topological Spaces with Picture Fuzzy Prevalence Effect Method for Group Decision-Making

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Abstract. The concept of picture fuzzy sets (*pf*-sets) expands intuitionistic fuzzy sets to model uncertain information, mainly when handling the feedbacks "Yes", "No", and "Abstain". Recently, the definitions of *pf*-sets and their elemental operations have been revised to address some inconsistencies in Cuong's original definitions. Building on these two studies, our first objective is to redefine the concept of picture fuzzy topology and investigate its properties, such as limit points and compactness. We then propose a group decision-making technique called the Picture Fuzzy Prevalence Effect Method (PFPEM) that leverages the properties of picture fuzzy topological spaces. Finally, we discuss the need for further research in this area.

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1. Introduction

The concept of fuzzy sets [48] has been proposed to handle imperfect and ambiguous information. The underlying universe's components can partially belong to a particular set in fuzzy sets. Each element of a fuzzy set is classified by its membership level in the range $[0, 1]$. The membership degrees are determined independently in a fuzzy set, while non-membership degrees are computed by subtracting the membership degrees from 1. Thus, choosing non-membership degrees individually is not an option in a fuzzy set. [5–7] has presented the concept of intuitionistic fuzzy sets (*if*-sets) to address this choicelessness. The *if*-sets enable the independent determination of membership and non-membership degrees, with their overall sum and individual values falling within the range $[0, 1]$. Subsequently, [46] has enhanced the concept of *if*-sets by introducing Pythagorean fuzzy subsets, modifying the limitations on the degrees of *if*-sets.

The concept of picture fuzzy sets (*pf*-sets) and their various fundamental operations have recently been introduced to address challenges in computational intelligence [9, 11]. In the electoral process, participants can be classified into four categories: those who endorse, those who oppose, those who withhold their vote, and those who choose not to participate in the voting. In the cases of "yes", "no", and "abstain", *pf*-sets can represent ambiguous information, while fuzzy sets and *if*-sets cannot. Additionally, [26] has identified inconsistencies in Cuong's *pf*-sets and proposed solutions to rectify them. Recently, [27] has redefined picture fuzzy soft sets (*pfs*-sets) and specific operations. Furthermore, [31] has also studied picture fuzzy soft σ-algebras and picture fuzzy soft measures.

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Utilization of picture fuzzy soft relations toward multi-attribute decision making (MADM) has been studied by [22]. [3] has made use of picture fuzzy nano topological spaces for decision making. [24] has studied generalized *pfs*-sets accompanied by their utilities in decision support procedures. Earlier, [47] has introduced an adaptable soft distinction matrix founded on *pfs*-sets, and its utilization in decision making. [30] has employed picture cubic fuzzy aggregation information for the purpose of multi-criteria group decision making (MCGDM). [23] has come up with distance and similarity measures for generalized picture fuzzy environment. A study under picture fuzzy set environment on novel point operators and multiple-rounds voting process has been presented by [25]. Recently, [35] has studied vector similarity measures of picture type-2 hesitant fuzzy sets based multi-criteria decision-making (MCDM). [42] has rendered applications to medical diagnosis and pattern recognition by defining similarity measures between picture fuzzy sets. [21] has utilized picture fuzzy information for the purpose of domination analysis. Under 2-tuple linguistic q-rung picture fuzzy environment, [1] has presented an extended MARCOS method for MCGDM. Based on picture fuzzy information, [18] has presented the Floyd-Warshall algorithm. In picture fuzzy setting, [2] has devised a novel technique to tackle linear programming problems.

Moreover, the combination of rough and fuzzy set theories has produced numerous fascinating outcomes. [43] has proposed a comprehensive framework for studying fuzzy rough sets, utilizing both axiomatic and constructive approaches. Similarly, [44] has analyzed fuzzy topological structures on rough fuzzy sets, using both constructive and axiomatic approaches. In 2012, the investigation of rough intuitionistic fuzzy sets and intuitionistic fuzzy topologies in crisp approximation spaces was carried out by [45]. Theory and applications of topological spaces under different frameworks of expansions of fuzzy sets have recently been explored by [19, 32, 33, 38, 40].

In 2015, the concept of rough picture fuzzy sets was first introduced by [41]. Their study has focused on the approximation of picture fuzzy sets within crisp approximation spaces and explored various properties related to rough picture fuzzy sets. In recent times, picture fuzzy topological spaces(PFTSs) have been a subject of exploration among researchers. They have introduced innovative concepts like rank, picture fuzzy base, and picture fuzzy sub-base [37]. Moreover, continuity in PFTSs has been investigated, and the necessary and sufficient conditions for a picture fuzzy continuous function between two spaces have been presented. Following these studies, a new type of open and closed sets has been introduced in PFTSs [4]. Furthermore, picture fuzzy continuous functions and some of their properties have been introduced and studied. [36] has presented a short while ago picture fuzzy complex proportional assessment approach with step-wise weight assessment ratio analysis and criteria importance through inter-criteria correlation.

In this study, the concept of picture fuzzy topology (PFT) has been redefined and some of its basic properties have been investigated. The major contributions of the study can be summed up as follows:

- The limit point of *pf*-sets is presented.
- Basis for a PFTS is studied.
- The main categories of PFTSs with picture fuzzy open or closed sets that meet specific requirements related to their picture fuzzy points are covered.
- Compactness in PFTS is investigated.
- A group decision-making implementation of PFT is offered.

The rest of the present study is organized as follows: Section 2 offers a clear and comprehensive discussion of the introductory concepts related to *pf*-sets, which are essential for further exploration. Section 3 redefines PFTSs and investigates the limit point of *pf*-sets, basis for a PFTS, main categories of PFTSs with picture fuzzy open and closed sets, and compactness in PFTS. Section 4 proposes a group decision-making methodology based on picture fuzzy prevalence effect method (PFPEM) utilizing properties of PFTSs. The final section discusses the paper's results and offers ideas for future research.

2. Preliminaries

Definition 2.1 ([9–11]). A picture fuzzy set (*pf*-set) $\mathbb{P}$ on the universal set S is defined as

$$\mathbb{P} = \{< x, \mu_{\mathbb{P}}(x), \eta_{\mathbb{P}}(x), \nu_{\mathbb{P}}(x) >: x \in S\},$$

where $\mu_{\mathbb{P}}$, $\eta_{\mathbb{P}}$, and $\nu_{\mathbb{P}}$ are called a degree of positive coherence, degree of neutral coherence, and degree of negative coherence of x, respectively, to $\mathbb{P}$ such that $\mu_{\mathbb{P}}(x), \eta_{\mathbb{P}}(x), \nu_{\mathbb{P}}(x) \in [0, 1]$ and $\mu_{\mathbb{P}}(x) + \eta_{\mathbb{P}}(x) + \nu_{\mathbb{P}}(x) \leq 1$, for all $x \in S$. The number $\xi_{\mathbb{P}}(x) = 1 - \mu_{\mathbb{P}}(x) - \eta_{\mathbb{P}}(x) - \nu_{\mathbb{P}}(x)$ is termed as the degree of refusal coherence of x to $\mathbb{P}$.

To address inconsistencies in the definition of *pf*-sets presented by [9, 11, 27], the definition has been revised to some extent, with new restrictions imposed on the three main functions of the *pf*-sets to $\mu(x)+\nu(x)\in[0,1]$ and $\mu(x)+\eta(x)+\nu(x)\in[0,2]$. Taking into account the picture fuzzy value $(0.23, 0.17, 0.40)$, the degree of indeterminacy according to [9, 11] comes out to be $1.20\notin[0,1]$. Thus, there was a need for redefining the degree of indeterminacy $\xi(x)$ as $\xi(x)=1-\mu(x)-\nu(x)$ instead of $2-\mu(x)-\eta(x)-\nu(x)$.

In order to simplify the notations, [26] has utilized the notation $\left\langle\begin{smallmatrix}0.23\\0.17\\0.40\end{smallmatrix}\right\rangle$ to represent the picture fuzzy value $(0.23, 0.17, 0.40)$.

Definition 2.2 ([26]). A picture fuzzy set (*pf*-set) $\mathbb{P}$ on the universal set S is defined as

$$\mathbb{P}=\left\{\left(x,\left\langle\begin{smallmatrix}\mu_{\mathbb{P}}(x)\\\eta_{\mathbb{P}}(x)\\\nu_{\mathbb{P}}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\},$$

where $\mu_{\mathbb{P}}$, $\eta_{\mathbb{P}}$, and $\nu_{\mathbb{P}}$ are called a degree of positive coherence, degree of neutral coherence, and degree of negative coherence of x, respectively, to $\mathbb{P}$ such that $\mu_{\mathbb{P}}(x),\eta_{\mathbb{P}}(x),\nu_{\mathbb{P}}(x)\in[0,1]$, $\mu_{\mathbb{P}}(x)+\nu_{\mathbb{P}}(x)\le 1$, and $\mu_{\mathbb{P}}(x)+\eta_{\mathbb{P}}(x)+\nu_{\mathbb{P}}(x)\le 2$, for all $x\in S$. The number $\xi_{\mathbb{P}}(x)=2-\mu_{\mathbb{P}}(x)-\eta_{\mathbb{P}}(x)-\nu_{\mathbb{P}}(x)$ is termed as the degree of refusal coherence of x to $\mathbb{P}$.

Hereinafter, Definition 2.2 is employed as definition of *pf*-sets and the collection of all the *pf*-sets defined on S will be designated as $PF(S)$.

Definition 2.3 ([26]). A *pf*-set in which $\mu_{ij}=0$ and $\eta_{ij}=\nu_{ij}=1$ for all i and j, is called empty or null *pf*-set and is denominated by $\widetilde{\Phi}$ or $\widehat{0}$.

Definition 2.4 ([26]). A *pf*-set in which $\mu_{ij}=1$ and $\eta_{ij}=\nu_{ij}=0$ for every i and j, is called universal or absolute *pf*-set and is symbolized by $\widetilde{S}$ or $\widehat{1}$.

Definition 2.5 ([26]). A *pf*-set $\mathbb{P}_1$ defined over S is labeled as a picture fuzzy subset of the *pf*-set $\mathbb{P}_2$ if $\mu_{\mathbb{P}_1}(x)\le\mu_{\mathbb{P}_2}(x)$, $\eta_{\mathbb{P}_1}(x)\ge\eta_{\mathbb{P}_2}(x)$, and $\nu_{\mathbb{P}_1}(x)\ge\nu_{\mathbb{P}_2}(x)$, for all $x\in S$. We designate it as $\mathbb{P}_1\subseteq\mathbb{P}_2$.

Definition 2.6 ([26]). The complement of a *pf*-set over S

$$\mathbb{P}=\left\{\left(x,\left\langle\begin{smallmatrix}\mu_{\mathbb{P}}(x)\\\eta_{\mathbb{P}}(x)\\\nu_{\mathbb{P}}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\}$$

is defined as

$$\mathbb{P}^c=\left\{\left(x,\left\langle\begin{smallmatrix}\nu_{\mathbb{P}}(x)\\1-\eta_{\mathbb{P}}(x)\\\mu_{\mathbb{P}}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\}.$$

Definition 2.7 ([26]). The union of two *pf*-sets over S

$$\mathbb{P}_1=\left\{\left(x,\left\langle\begin{smallmatrix}\mu_{\mathbb{P}_1}(x)\\\eta_{\mathbb{P}_1}(x)\\\nu_{\mathbb{P}_1}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\}$$

and

$$\mathbb{P}_2=\left\{\left(x,\left\langle\begin{smallmatrix}\mu_{\mathbb{P}_2}(x)\\\eta_{\mathbb{P}_2}(x)\\\nu_{\mathbb{P}_2}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\}$$

is defined as

$$\mathbb{P}_2\cup\mathbb{P}_2=\left\{\left(x,\left\langle\begin{smallmatrix}\max\{\mu_{\mathbb{P}_1}(x),\mu_{\mathbb{P}_2}(x)\}\\\min\{\eta_{\mathbb{P}_1}(x),\eta_{\mathbb{P}_2}(x)\}\\\min\{\nu_{\mathbb{P}_1}(x),\nu_{\mathbb{P}_2}(x)\}\end{smallmatrix}\right\rangle\right):x\in S\right\}.$$

Definition 2.8 ([26]). The intersection of two *pf*-sets over S

$$\mathbb{P}_1=\left\{\left(x,\left\langle\begin{smallmatrix}\mu_{\mathbb{P}_1}(x)\\\eta_{\mathbb{P}_1}(x)\\\nu_{\mathbb{P}_1}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\}$$

and

$$\mathbb{P}_2=\left\{\left(x,\left\langle\begin{smallmatrix}\mu_{\mathbb{P}_2}(x)\\\eta_{\mathbb{P}_2}(x)\\\nu_{\mathbb{P}_2}(x)\end{smallmatrix}\right\rangle\right):x\in S\right\}$$

is defined as

$$\mathbb{P}_2\cap\mathbb{P}_2=\left\{\left(x,\left\langle\begin{smallmatrix}\min\{\mu_{\mathbb{P}_1}(x),\mu_{\mathbb{P}_2}(x)\}\\\max\{\eta_{\mathbb{P}_1}(x),\eta_{\mathbb{P}_2}(x)\}\\\max\{\nu_{\mathbb{P}_1}(x),\nu_{\mathbb{P}_2}(x)\}\end{smallmatrix}\right\rangle\right):x\in S\right\}.$$

Definition 2.9 ([26]). The difference of two *pf*-sets over S

$$\mathbb{P}_1 = \left\{ \left(x, \left\langle \begin{smallmatrix} \mu_{\mathbb{P}_1}(x) \\ \eta_{\mathbb{P}_1}(x) \\ \nu_{\mathbb{P}_1}(x) \end{smallmatrix} \right\rangle \right) : x \in S \right\}$$

and

$$\mathbb{P}_2 = \left\{ \left(x, \left\langle \begin{smallmatrix} \mu_{\mathbb{P}_2}(x) \\ \eta_{\mathbb{P}_2}(x) \\ \nu_{\mathbb{P}_2}(x) \end{smallmatrix} \right\rangle \right) : x \in S \right\}$$

is defined as

$$\mathbb{P}_1 \setminus \mathbb{P}_2 = \left\{ \left(x, \left\langle \begin{smallmatrix} \min\{\mu_{\mathbb{P}_1}(x), \nu_{\mathbb{P}_2}(x)\} \\ \max\{\eta_{\mathbb{P}_1}(x), 1-\eta_{\mathbb{P}_2}(x)\} \\ \max\{\nu_{\mathbb{P}_1}(x), \mu_{\mathbb{P}_2}(x)\} \end{smallmatrix} \right\rangle \right) : x \in S \right\}.$$

Definition 2.10 ([39]). Let $\mathbb{P} \in PF(S)$. Then, for a fixed $x \in S$, a picture fuzzy number (PFN) is defined by

$$(\mu_{\mathbb{P}}(x), \eta_{\mathbb{P}}(x), \nu_{\mathbb{P}}(x), \xi_{\mathbb{P}}(x))$$

such that $\mu_{\mathbb{P}}(x), \eta_{\mathbb{P}}(x), \nu_{\mathbb{P}}(x), \xi_{\mathbb{P}}(x) \in [0,1]$ and $\mu_{\mathbb{P}}(x) + \eta_{\mathbb{P}}(x) + \nu_{\mathbb{P}}(x) + \xi_{\mathbb{P}}(x) = 1$. Briefly, PFN is represented as $(\mu_{\mathbb{P}}(x), \eta_{\mathbb{P}}(x), \nu_{\mathbb{P}}(x))$.

Definition 2.11. Let $\mu_A, \eta_A, \nu_A \in [0,1]$, $\mu_A + \nu_A \leq 1$, and $\mu_A + \eta_A + \nu_A \leq 2$. Then, a picture fuzzy number (PFN) $\tilde{\mathbb{P}}$ is defined by $\tilde{A} = \left\langle \begin{smallmatrix} \mu_A \\ \eta_A \\ \nu_A \end{smallmatrix} \right\rangle$.

Throughout this paper, Definition 2.11 is considered as definition of PFN.

Definition 2.12. Let $\tilde{A}$ be a PFN. A picture fuzzy point $\mho^{\zeta}_{\tilde{A}}$ is a *pf*-set of S given by $\mho^{\zeta}_{\tilde{A}} = \left\{ \left(y, \left\langle \begin{smallmatrix} \mu_\zeta(y) \\ \eta_\zeta(y) \\ \nu_\zeta(y) \end{smallmatrix} \right\rangle \right) : y \in S \right\}$, where

$$\mu_\zeta(y) = \begin{cases} \mu_A, & \text{if } y = \zeta \\ 0, & \text{otherwise} \end{cases},$$

$$\eta_\zeta(y) = \begin{cases} \eta_A, & \text{if } y = \zeta \\ 1, & \text{otherwise} \end{cases},$$

and

$$\nu_\zeta(y) = \begin{cases} \nu_A, & \text{if } y = \zeta \\ 1, & \text{otherwise} \end{cases}.$$

In this case, ζ is characterized as the support of $\mho^{\zeta}_{\tilde{A}}$. Besides, $\mho^{\zeta}_{\tilde{A}}$ is said to belong to the *pf*-set $\mathbb{P} = \left\{ \left(x, \left\langle \begin{smallmatrix} \mu_{\mathbb{P}}(x) \\ \eta_{\mathbb{P}}(x) \\ \nu_{\mathbb{P}}(x) \end{smallmatrix} \right\rangle \right) : x \in S \right\}$, written $\mho^{\zeta}_{\tilde{A}} \in \mathbb{P}$, if $\mu_{\mathbb{P}}(\zeta) \geq \mu_A$, $\eta_{\mathbb{P}}(\zeta) \leq \eta_A$, and $\nu_{\mathbb{P}}(\zeta) \leq \nu_A$. In order to be brief, we shall write $\mho$ to communicate $\mho^{\zeta}_{\tilde{A}}$.

Definition 2.13 ([11]). Let $d : PF(S) \times PF(S) \to \mathbb{R}$ be a mapping. Then, for all $\mathbb{P}_1$,$\mathbb{P}_2$,$\mathbb{P}_3 \in PF(S)$, d is metric over $PF(S)$ if and only if (iff) d satisfies the following properties:

i) $d(\mathbb{P}_1, \mathbb{P}_2) = 0 \Leftrightarrow \mathbb{P}_1 = \mathbb{P}_2$,
ii) $d(\mathbb{P}_1, \mathbb{P}_2) = d(\mathbb{P}_2, \mathbb{P}_1)$,
iii) $d(\mathbb{P}_1, \mathbb{P}_2) \leq d(\mathbb{P}_1, \mathbb{P}_3) + d(\mathbb{P}_3, \mathbb{P}_2)$.

Proposition 2.14 ([11]). *The mapping d defined by*

$$d(\mathbb{P}_1, \mathbb{P}_2) := \sqrt{\sum_{i=1}^{n} (\mu_{\mathbb{P}_1}(x_i) - \mu_{\mathbb{P}_2}(x_i))^2 + (\eta_{\mathbb{P}_1}(x_i) - \eta_{\mathbb{P}_2}(x_i))^2 + (\nu_{\mathbb{P}_1}(x_i) - \nu_{\mathbb{P}_2}(x_i))^2}$$

is a metric over $PF(S)$ and is called Euclidean metric.

3. Picture Fuzzy Topological Spaces (PFTSs)

In this section, we focus on key concepts of PFT, including its main characteristics.

Definition 3.1. A subcollection τ of the *pf*-subsets of S is called a PFT on S if following conditions are satisfied:

(i) $\widetilde{\Phi}, \widetilde{S} \in \tau$,

(ii) $\mathbb{P}_1 \cap \mathbb{P}_2 \in \tau$ whenever $\mathbb{P}_1, \mathbb{P}_2 \in \tau$, and

(iii) If $\mathbb{P}_i \in \tau, \forall\, i \in \Omega$, then $\cup_{i\in\Omega} \mathbb{P}_i \in \tau$.

The ordered pair (S, τ) or simply τ is delineated as PFTS. The constituents of τ are named picture fuzzy open sets, while their complements are called picture fuzzy closed sets.

Example 3.2. Let

$$\mathbb{P}_1 = \left\{ \left(h, \left\langle \begin{smallmatrix} 0.37 \\ 0.46 \\ 0.13 \end{smallmatrix} \right\rangle \right), \left(c, \left\langle \begin{smallmatrix} 0.52 \\ 0.11 \\ 0.06 \end{smallmatrix} \right\rangle \right) \right\},$$

$$\mathbb{P}_2 = \left\{ \left(h, \left\langle \begin{smallmatrix} 0.69 \\ 0.27 \\ 0.08 \end{smallmatrix} \right\rangle \right), \left(c, \left\langle \begin{smallmatrix} 0.71 \\ 0.08 \\ 0.04 \end{smallmatrix} \right\rangle \right) \right\},$$

and

$$\mathbb{P}_3 = \left\{ \left(h, \left\langle \begin{smallmatrix} 0.46 \\ 0.31 \\ 0.11 \end{smallmatrix} \right\rangle \right), \left(c, \left\langle \begin{smallmatrix} 0.59 \\ 0.08 \\ 0.05 \end{smallmatrix} \right\rangle \right) \right\}$$

be three *pf*-sets defined on the universal set $S = \{h, c\}$. Then, $\tau_1 = \{\widetilde{\Phi}, \widetilde{S}\}$, $\tau_2 = \{\widetilde{\Phi}, \mathbb{P}_1, \widetilde{S}\}$, $\tau_3 = \{\widetilde{\Phi}, \mathbb{P}_2, \widetilde{S}\}$, $\tau_4 = \{\widetilde{\Phi}, \mathbb{P}_3, \widetilde{S}\}$, $\tau_5 = \{\widetilde{\Phi}, \mathbb{P}_1, \mathbb{P}_2, \widetilde{S}\}$, $\tau_6 = \{\widetilde{\Phi}, \mathbb{P}_1, \mathbb{P}_3, \widetilde{S}\}$, $\tau_7 = \{\widetilde{\Phi}, \mathbb{P}_2, \mathbb{P}_3, \widetilde{S}\}$, and $\tau_8 = \{\widetilde{\Phi}, \mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \widetilde{S}\}$ are PFTSs over S. The constituents $\widetilde{\Phi}$ and $\widetilde{S}$ occurring in all these topologies enjoy the characteristic of being picture fuzzy open as well as picture fuzzy closed.

Definition 3.3. For some positive real number ε, the collection

$$B_d(\mathbb{P}, \varepsilon) = \{\mathbb{P}_i : d(\mathbb{P}, \mathbb{P}_i) < \varepsilon\}$$

is called ε-ball centered at $\mathbb{P}$. The collection of all such ε-balls is called a metric topology induced by d. A PFTS (S, τ) is called metrizable if there exists a metric d on S that induces the topology of S.

Definition 3.4. Let (S, τ_1) and (S, τ_2) be PFTSs. We say that τ_2 contains τ_1 denoted by $\tau_2 \supseteq \tau_1$, if $\mathbb{P} \in \tau_2$ whenever $\mathbb{P} \in \tau_1$. In such a case, τ_2 is said to be stronger, larger, or finer than τ_1, and τ_1 is called weaker, smaller, or coarser than τ_2. Moreover, τ_1 and τ_2 are called comparable. If $\tau_2 \supsetneqq \tau_1$, then τ_2 is called strictly finer than τ_1 or equivalently τ_1 is strictly weaker than τ_2.

For example, τ_6 is (strictly) weaker than τ_2 in Example 3.2 and hence τ_2 and τ_6 are comparable.

Definition 3.5. The topology $\tau_{\text{indiscrete}} = \{\widetilde{\Phi}, \widetilde{S}\}$ is known as indiscrete PFT over S and $\tau_{\text{discrete}} = PF(S)$ is referred to as discrete PFT over S. $\tau_{\text{indiscrete}}$ and τ_{discrete} are respectively the smallest and the largest PFTs on S. The pair $(S, \tau_{\text{indiscrete}})$ is called indiscrete PFTS whereas $(S, \tau_{\text{discrete}})$ is entitled discrete PFTS. Indeed, $\tau_{\text{indiscrete}}$ is the coarsest and τ_{discrete} is the finest topology on S.

Remark 3.6. The intersection of two PFTSs always results in a PFTS, whereas their union may not necessarily be a PFTS. The forthcoming example illustrates this observation.

Example 3.7. Assuming

$$\mathbb{P}_1 = \left\{ \left(a, \left\langle \begin{smallmatrix} 0.29 \\ 0.87 \\ 0.48 \end{smallmatrix} \right\rangle \right), \left(k, \left\langle \begin{smallmatrix} 0.82 \\ 0.34 \\ 0.17 \end{smallmatrix} \right\rangle \right) \right\}$$

and

$$\mathbb{P}_2 = \left\{ \left(a, \left\langle \begin{smallmatrix} 0.40 \\ 0.89 \\ 0.12 \end{smallmatrix} \right\rangle \right), \left(k, \left\langle \begin{smallmatrix} 0.55 \\ 0.17 \\ 0.02 \end{smallmatrix} \right\rangle \right) \right\}$$

as two *pf*-sets over $S = \{a, k\}$, then

$$\tau_1 = \left\{\widetilde{\Phi}, \mathbb{P}_1, \widetilde{S}\right\}$$

and

$$\tau_2 = \left\{\widetilde{\Phi}, \mathbb{P}_2, \widetilde{S}\right\}$$

are two PFTs over S, nevertheless

$$\tau_1 \cup \tau_2 = \left\{\widetilde{\Phi}, \mathbb{P}_1, \mathbb{P}_2, \widetilde{S}\right\}$$

fails to be a PFT on S.

Definition 3.8. Assume that (S, τ_S) is a PFTS. For some $S' \subseteq S$, $\tau_{S'}$ is a PFT on S' whose picture fuzzy open sets are $\mathbb{P}_{S'} = \mathbb{P}_S \cap \widetilde{S'}$, where $\mathbb{P}_S$ are picture fuzzy open sets of τ_S, $\mathbb{P}_{S'}$ are picture fuzzy open sets of $\tau_{S'}$, and $\widetilde{S'}$ is the universal *pf*-set on S'. Then, $\tau_{S'}$ is called a picture fuzzy subspace of τ_S, i.e.,

$$\tau_{S'} = \{\mathbb{P}_{S'} : \mathbb{P}_{S'} = \mathbb{P}_S \cap \widetilde{S'}, \mathbb{P}_S \in \tau_S\}$$

$\tau_{S'}$ is also entitled as an induced PFT or relative PFT on S'.

Example 3.9. Let

$$\mathbb{P}_1 = \left\{\left(v, \left\langle\begin{matrix}0.37\\0.46\\0.49\end{matrix}\right\rangle\right), \left(m, \left\langle\begin{matrix}0.62\\0.29\\0.22\end{matrix}\right\rangle\right), \left(d, \left\langle\begin{matrix}0.43\\0.17\\0.31\end{matrix}\right\rangle\right)\right\}$$

and

$$\mathbb{P}_2 = \left\{\left(v, \left\langle\begin{matrix}0.48\\0.31\\0.28\end{matrix}\right\rangle\right), \left(m, \left\langle\begin{matrix}0.65\\0.12\\0.07\end{matrix}\right\rangle\right), \left(d, \left\langle\begin{matrix}0.57\\0.10\\0.29\end{matrix}\right\rangle\right)\right\}$$

be two *pf*-sets over $S = \{v, m, d\}$. Then,

$$\tau_S = \{\widetilde{\Phi}, \mathbb{P}_1, \mathbb{P}_2, \widetilde{S}\}$$

is a PFT on S.

For the universal *pf*-sets over $S' = \{v, d\} \subseteq S$,

$$\widetilde{S'} = \left\{\left(v, \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle\right), \left(d, \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle\right)\right\}.$$

Since

$$\begin{aligned}
\widetilde{S'} \cap \widetilde{\Phi} &= \widetilde{\Phi};\\
\widetilde{S'} \cap \mathbb{P}_1 &= \left\{\left(v, \left\langle\begin{matrix}0.37\\0.46\\0.49\end{matrix}\right\rangle\right), \left(d, \left\langle\begin{matrix}0.43\\0.17\\0.31\end{matrix}\right\rangle\right)\right\} = \mathbb{P}_1'\\
\widetilde{S'} \cap \mathbb{P}_2 &= \left\{\left(v, \left\langle\begin{matrix}0.48\\0.31\\0.28\end{matrix}\right\rangle\right), \left(d, \left\langle\begin{matrix}0.57\\0.10\\0.29\end{matrix}\right\rangle\right)\right\} = \mathbb{P}_2'\\
\widetilde{S'} \cap \widetilde{S} &= \widetilde{S'};
\end{aligned}$$

then

$$\tau_{S'} = \{\widetilde{\Phi}, \mathbb{P}_1', \mathbb{P}_2', \widetilde{S'}\}$$

is a picture fuzzy subspace of τ_S.

Definition 3.10. Let (S, τ) be a PFTS and $\mathbb{P}$ be a *pf*-set on S.

(1) The picture fuzzy interior $\mathbb{P}^\circ$ of $\mathbb{P}$ is the union of all picture fuzzy open subsets of $\mathbb{P}$ and is the largest picture fuzzy open subset of $\mathbb{P}$.
(2) The picture fuzzy closure $\overline{\mathbb{P}}$ of $\mathbb{P}$ is the intersection of all picture fuzzy closed supersets of $\mathbb{P}$. Moreover, $\overline{\mathbb{P}}$ is the smallest picture fuzzy closed superset of $\mathbb{P}$.
(3) The picture fuzzy frontier or picture fuzzy boundary $\mathrm{Fr}(\mathbb{P})$ of $\mathbb{P}$ is defined by

$$\mathrm{Fr}(\mathbb{P}) = \overline{\mathbb{P}} \cap \overline{\mathbb{P}^c}.$$

(4) The picture fuzzy exterior $\mathrm{Ext}(\mathbb{P})$ of $\mathbb{P}$ is given as

$$\mathrm{Ext}(\mathbb{P}) = (\mathbb{P}^c)^\circ.$$

The forthcoming example (i.e., Example 3.11) will assist in clarifying the conceptions presented in Definition 3.10.

Example 3.11. Consider the PFTS (S, τ_S) provided in Example 3.9. Since constituents of this PFTS are picture fuzzy open, then

$$\begin{aligned}\widetilde{\Phi}^c &= \widetilde{S},\\ \mathbb{P}_1^c &= \left\{\left(v, \begin{pmatrix}0.49\\0.54\\0.37\end{pmatrix}\right), \left(m, \begin{pmatrix}0.22\\0.71\\0.62\end{pmatrix}\right), \left(d, \begin{pmatrix}0.31\\0.83\\0.43\end{pmatrix}\right)\right\},\\ \mathbb{P}_2^c &= \left\{\left(v, \begin{pmatrix}0.28\\0.69\\0.48\end{pmatrix}\right), \left(m, \begin{pmatrix}0.07\\0.88\\0.65\end{pmatrix}\right), \left(d, \begin{pmatrix}0.29\\0.90\\0.57\end{pmatrix}\right)\right\}\end{aligned}$$

and

$$\widetilde{S}^c = \widetilde{\Phi}$$

are the corresponding picture fuzzy closed sets. Take

$$\mathbb{P} = \left\{\left(v, \begin{pmatrix}0.41\\0.39\\0.16\end{pmatrix}\right), \left(m, \begin{pmatrix}0.19\\0.71\\0.54\end{pmatrix}\right), \left(d, \begin{pmatrix}0.39\\0.58\\0.26\end{pmatrix}\right)\right\}$$

so that

$$\mathbb{P}^c = \left\{\left(v, \begin{pmatrix}0.16\\0.61\\0.41\end{pmatrix}\right), \left(m, \begin{pmatrix}0.54\\0.29\\0.19\end{pmatrix}\right), \left(d, \begin{pmatrix}0.26\\0.42\\0.39\end{pmatrix}\right)\right\}.$$

Consequently, the picture fuzzy interior of $\mathbb{P}$ is

$$\mathbb{P}^\circ = \widetilde{\Phi} \cup \mathbb{P}_1 = \mathbb{P}_1.$$

The picture fuzzy closure of $\mathbb{P}$ is

$$\overline{\mathbb{P}} = \mathbb{P}_2 \cap \widetilde{S} = \mathbb{P}_2.$$

Moreover,

$$\overline{\mathbb{P}^c} = \mathbb{P}_2^c \cup \widetilde{\Phi} = \mathbb{P}_2^c.$$

Thus, the picture fuzzy frontier of $\mathbb{P}$ comes out to be

$$\begin{aligned}\mathrm{Fr}(\mathbb{P}) &= \overline{\mathbb{P}} \cap \overline{\mathbb{P}^c}\\ &= \mathbb{P}_2 \cap \mathbb{P}_2^c\\ &= \left\{\left(v, \begin{pmatrix}0.28\\0.69\\0.48\end{pmatrix}\right), \left(m, \begin{pmatrix}0.07\\0.88\\0.65\end{pmatrix}\right), \left(d, \begin{pmatrix}0.29\\0.90\\0.57\end{pmatrix}\right)\right\}\\ &= \mathbb{P}_2^c.\end{aligned}$$

Finally, the picture fuzzy exterior of $\mathbb{P}$ appears to be

$$\mathrm{Ext}(\mathbb{P}) = (\mathbb{P}^c)^\circ = \widetilde{\Phi}.$$

Proposition 3.12. *Assume that (S, τ) is a PFTS with $\mathbb{P} \in PF(S)$. Then,*

(1) $(\mathbb{P}^\circ)^c = \overline{(\mathbb{P}^c)}$,
(2) $(\overline{\mathbb{P}})^c = (\mathbb{P}^c)^\circ$.

Proof. We only demonstrate the proof of (1) here. The proof of (2) may be furnished on the analogous track.
Let

$$\mathbb{P} = \left\{\left(x, \begin{pmatrix}\mu_{\mathbb{P}}(x)\\ \eta_{\mathbb{P}}(x)\\ \nu_{\mathbb{P}}(x)\end{pmatrix}\right) : x \in S\right\}$$

and that the collection

$$\mathbb{Q}_i = \left\{\left(x, \begin{pmatrix}\mu_{\mathbb{Q}_i}(x)\\ \eta_{\mathbb{Q}_i}(x)\\ \nu_{\mathbb{Q}_i}(x)\end{pmatrix}\right) : x \in S\right\}, \quad i \in \Omega$$

represent the picture fuzzy open sets contained by $\mathbb{P}$. By definition,

$$\mathbb{P}^\circ = \left\{\left(x, \begin{pmatrix}\max_{i\in\Omega}(\mu_{\mathbb{Q}_i}(x))\\ \min_{i\in\Omega}(\eta_{\mathbb{Q}_i}(x))\\ \min_{i\in\Omega}(\nu_{\mathbb{Q}_i}(x))\end{pmatrix}\right) : x \in S\right\}$$

which yields

$$(\mathbb{P}^{\circ})^{c} = \left\{ \left(x, \left\langle \begin{array}{c} \min_{i\in\Omega} (\nu_{\mathbb{Q}_i}(x)) \\ 1 - \min_{i\in\Omega} (\eta_{\mathbb{Q}_i}(x)) \\ \max_{i\in\Omega} (\mu_{\mathbb{Q}_i}(x)) \end{array} \right\rangle \right) : x \in S \right\}. \tag{3.1}$$

Since

$$\mathbb{P}^{c} = \left\{ \left(x, \left\langle \begin{array}{c} \nu_{\mathbb{P}}(x) \\ 1 - \eta_{\mathbb{P}}(x) \\ \mu_{\mathbb{P}}(x) \end{array} \right\rangle \right) : x \in S \right\}$$

and $\mu_{\mathbb{Q}_i}(\zeta) \leq \mu_{\mathbb{P}}(\zeta)$, $\eta_{\mathbb{Q}_j}(\zeta) \geq \eta_{\mathbb{P}}(\zeta)$, and $\nu_{\mathbb{Q}_j}(\zeta) \geq \nu_{\mathbb{P}}(\zeta)$ for all admissible values of $i \in \Omega$, so undoubtedly

$$\left\{ \left(x, \left\langle \begin{array}{c} \nu_{\mathbb{P}}(x) \\ 1 - \eta_{\mathbb{P}}(x) \\ \mu_{\mathbb{P}}(x) \end{array} \right\rangle \right) : x \in S \right\}$$

is the assemblage of picture fuzzy closed sets containing $\mathbb{P}^c$, i.e.,

$$(\mathbb{P}^{c})^{\circ} = \left\{ \left(x, \left\langle \begin{array}{c} \min (\nu_{\mathbb{P}}(x)) \\ \max (1 - \eta_{\mathbb{P}}(x)) \\ \max (\mu_{\mathbb{P}}(x)) \end{array} \right\rangle \right) : x \in S \right\}.$$

But

$$\max (1 - \eta_{\mathbb{P}}(x)) = 1 - \min (\eta_{\mathbb{P}}(x)).$$

Hence,

$$(\mathbb{P}^{c})^{\circ} = \left\{ \left(x, \left\langle \begin{array}{c} \min (\nu_{\mathbb{P}}(x)) \\ 1 - \min (\eta_{\mathbb{P}}(x)) \\ \max (\mu_{\mathbb{P}}(x)) \end{array} \right\rangle \right) : x \in S \right\}. \tag{3.2}$$

The required result now follows immediately from (3.1) and (3.2). □

Proposition 3.13. *Let (S, τ) be a PFTS with $\mathbb{P} \in PF(S)$, then $Fr(\mathbb{P}^c) = Fr(\mathbb{P})$.*

Proof. By Definition 3.10, we have

$$\mathrm{Fr}(\mathbb{P}) = \overline{\mathbb{P}} \cap \overline{\mathbb{P}^c} = \overline{\mathbb{P}^c} \cap \overline{\mathbb{P}} = \overline{\mathbb{P}^c} \cap \overline{(\mathbb{P}^c)^c} = \mathrm{Fr}(\mathbb{P}^c).$$

□

Proposition 3.14. *Let (S, τ) be a PFTS with $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2 \in PF(S)$, then*

(i) $\mathbb{P}^{\circ} \subseteq \mathbb{P} \subseteq \overline{\mathbb{P}}$,
(ii) $(\mathbb{P}^{\circ})^{\circ} = \mathbb{P}^{\circ}$,
(iii) $\overline{(\overline{\mathbb{P}})} = \overline{\mathbb{P}}$,
(iv) $(\widetilde{S})^{\circ} = \widetilde{S}$,
(v) $\overline{(\widetilde{\Phi})} = \widetilde{\Phi}$,
(vi) $\mathbb{P}_1 \subseteq \mathbb{P}_2 \Rightarrow \mathbb{P}_1^{\circ} \subseteq \mathbb{P}_2^{\circ} \wedge \overline{\mathbb{P}_1} \subseteq \overline{\mathbb{P}_2}$,
(vii) $\overline{\mathbb{P}_1 \cup \mathbb{P}_2} = \overline{\mathbb{P}_1} \cup \overline{\mathbb{P}_2}$,
(viii) $(\mathbb{P}_1 \cap \mathbb{P}_2)^{\circ} = \mathbb{P}_1^{\circ} \cap \mathbb{P}_2^{\circ}$.

Theorem 3.15. *Let $(S', \tau_{S'})$ be a subspace of (S, τ_S) and $\mathbb{P}$ be a picture fuzzy subset of $(S', \tau_{S'})$. Then, the closure $\overline{\mathbb{P}}$ of $\mathbb{P}$ in (S, τ_S) equals $\overline{\mathbb{P}} \cap S'$.*

Proof. Let B denote the closure of $\mathbb{P}$ in $(S', \tau_{S'})$. Since $\overline{\mathbb{P}}$ is picture fuzzy closed in (S, τ_S), so $\overline{\mathbb{P}} \cap S'$ is closed in $(S', \tau_{S'})$. Since $\mathbb{P} \in \overline{\mathbb{P}} \cap S'$ and B equals the intersection of all picture fuzzy closed subsets of S' containing P, so $B \subseteq \overline{\mathbb{P}} \cap S'$. On the other hand, B is closed in S'. Therefore, $B = C \cap S'$ for some picture fuzzy closed set C in $\mathbb{P}$. Consequently, C is picture fuzzy closed set of S containing $\mathbb{P}$. Since $\overline{\mathbb{P}}$ is picture fuzzy intersection of all such picture fuzzy closed sets, so it may be concluded that $\overline{\mathbb{P}} \subseteq C$. As a consequence, we have $\overline{\mathbb{P}} \cap S' \subseteq C \cap S' = S'$. □

3.1. **Limit Points of *pf*-set.** In this subsection, we present the notion of limit points in the context of *pf*-sets.

Definition 3.16. Supposing (S, τ) a PFTS with $\mathbb{P} \in PF(S)$. A point $\mho \in PF(S)$ is called a picture fuzzy limit point, picture fuzzy accumulation point, or picture fuzzy cluster point of $\mathbb{P}$ if every picture fuzzy open set containing $\mho$ also contains a picture fuzzy point of $\mathbb{P}$ different from $\mho$. Said differently, $\mho$ is a picture fuzzy limit point of $\mathbb{P}$ if every neighborhood of $\mho$ intersects $\mathbb{P}$ in some picture fuzzy point other than $\mho$ itself.

The aggregate of all picture fuzzy limit points of $\mathbb{P}$ is termed as the derived picture fuzzy set of $\mathbb{P}$ and is designated as $der(\mathbb{P})$.

Example 3.17. Let $S = \{e, t, c\}$ and consider the following *pf*-sets on S:

$$\begin{aligned}
\mathbb{P} &= \left\{ \left(e, \begin{pmatrix} 0.46 \\ 0.39 \\ 0.52 \end{pmatrix} \right), \left(t, \begin{pmatrix} 0.18 \\ 0.03 \\ 0.31 \end{pmatrix} \right), \left(c, \begin{pmatrix} 0.81 \\ 0.51 \\ 0.01 \end{pmatrix} \right) \right\}, \\
\mathbb{P}_1 &= \left\{ \left(e, \begin{pmatrix} 0.59 \\ 0.37 \\ 0.11 \end{pmatrix} \right), \left(t, \begin{pmatrix} 0.23 \\ 0.02 \\ 0.14 \end{pmatrix} \right) \right\}, \text{ and} \\
\mathbb{P}_2 &= \left\{ \left(e, \begin{pmatrix} 0.58 \\ 0.31 \\ 0.14 \end{pmatrix} \right) \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{P}_1 \setminus \mathbb{P}_2 &= \left\{ \left(e, \begin{pmatrix} 0.14 \\ 0.69 \\ 0.59 \end{pmatrix} \right), \left(t, \begin{pmatrix} 0.23 \\ 0.02 \\ 0.14 \end{pmatrix} \right) \right\} \\
\therefore (\mathbb{P}_1 \setminus \mathbb{P}_2) \cap \mathbb{P} &= \left\{ \left(e, \begin{pmatrix} 0.14 \\ 0.69 \\ 0.59 \end{pmatrix} \right), \left(t, \begin{pmatrix} 0.18 \\ 0.03 \\ 0.31 \end{pmatrix} \right) \right\} \\
&\neq \widetilde{\Phi}.
\end{aligned}$$

Hence, $\mathbb{P}_2$ is the picture fuzzy limit point of $\mathbb{P}$.

Using the notion of picture fuzzy limit points, we can redescribe the closure of a *pf*-set. This phenomenon is paraphrased in Proposition 3.18 given below.

Proposition 3.18. *Imagine that $\mathbb{P}$ is a picture fuzzy subset of a PFTS (S, τ). If $der(\mathbb{P})$ is the collection of picture fuzzy accumulation points of $\mathbb{P}$, then the closure of $\mathbb{P}$ is given by $\overline{\mathbb{P}} = \mathbb{P} \cup der(\mathbb{P})$.*

Proof. Assume that $\mho \in der(\mathbb{P})$. Then, every neighborhood of $\mho$ intersects $\mathbb{P}$ in a point other than $\mho$. Therefore, $\mho \in \overline{\mathbb{P}}$ and hence $der(\mathbb{P}) \subseteq \overline{\mathbb{P}}$. Also, $\mathbb{P} \subseteq \overline{\mathbb{P}}$. So, we must have $\mathbb{P} \cup der(\mathbb{P}) \subseteq \overline{\mathbb{P}}$.
For the reverse inclusion, we let $\mho \in \overline{\mathbb{P}}$. If $\mho \in \mathbb{P}$, then clearly $\mho \in \mathbb{P} \cup der(\mathbb{P})$. Suppose that $\mho \notin \mathbb{P}$. Since $\mho \in \overline{\mathbb{P}}$, and every neighborhood of $\mho$ intersects $\mathbb{P}$, because $\mho \notin \mathbb{P}$, so the picture fuzzy collection of all such neighborhoods of $\mho$ must intersect $\mathbb{P}$ in a point other than $\mho$. Then $\mho \in der(\mathbb{P})$ leading to $\mho \in \mathbb{P} \cup der(\mathbb{P})$. This, in turn, yields $\overline{\mathbb{P}} \subseteq \mathbb{P} \cup der(\mathbb{P})$. □

Corollary 3.19. *A picture fuzzy subset of a PFTS is closed iff it contains all of its picture fuzzy accumulation points.*

3.2. **Basis for a PFTS.** In this subsection, we investigate basis for a PFTS and provide their some properties.

Definition 3.20. Let (S, τ) be a PFTS. If for every $\mathbb{P} \in \tau$, there is $\mathcal{B} \in \mathbb{B}$ in such a way that $\mathbb{P} = \cup\mathcal{B}$, then $\mathbb{B} \subseteq \tau$ is reckoned as picture fuzzy base or picture fuzzy basis for τ.

Example 3.21. Consider the PFTS (S, τ_S) cited in Example 3.9. The assemblage

$$\mathbb{B} = \{\mathbb{P}_1, \mathbb{P}_2\}$$

serves as a picture fuzzy basis for the PFT τ_S rendered in Example 3.9.

Lemma 3.22. *If $\mathbb{B}$ is a picture fuzzy basis for PFTS (S, τ_S) and $(S', \tau_{S'})$ is a picture fuzzy subspace of (S, τ_S), then the aggregate*

$$\mathbb{B}_{S'} = \{\mathcal{B} \cap \mathbb{P}_{S'} : \mathcal{B} \in \mathbb{B} \ \& \ \mathbb{P}_{S'} \in \tau_{S'}\}$$

serves as a picture fuzzy basis for $(S', \tau_{S'})$.

Proof. For each given picture fuzzy open set $\mathbb{P}_S$ in (S, τ_S) and $\mathbb{Q}_{S'} \in \mathbb{P}_S \cap \mathbb{P}_{S'}$, we can choose an element $\mathcal{B}$ of $\mathbb{B}$ such that $\mathbb{Q}_{S'} \in \mathcal{B} \subseteq \mathbb{P}_S$. But then $\mathbb{Q}_{S'} \in \mathcal{B} \cap \mathbb{P}_{S'} \subseteq \mathbb{P}_S \cap \mathbb{P}_{S'}$. From this inclusion, it follows that $\mathbb{B}_{S'}$ is a picture fuzzy basis for $(S', \tau_{S'})$. □

Proposition 3.23. *If (S, τ) is a PFTS, then the necessary and sufficient condition for the collection*

$$\mathbb{B} = \{\mathbb{P}_i : i \in \Omega\} \subseteq \tau$$

to be a picture fuzzy basis for τ is that for any picture fuzzy element $\mho \in \mathbb{P}'$ ($\mathbb{P}'$ being a pf*-set), there exists $\mathbb{P}'' \in \mathbb{B}$ such that $\mho \in \mathbb{P}'' \subseteq \mathbb{P}'$.*

Proof. Assume that $\mathbb{B}$ is a picture fuzzy base for τ. Then, there exist *pf*-sets $\mathbb{P}'' \in \mathbb{B}$, for any picture fuzzy open set $\mathbb{P}' \in \tau$, bearing the quality that $\mathbb{P}' = \cup\mathbb{P}''$. Therefore, there is a $\mathbb{P}''$ such that

$$\mho \in \mathbb{P}'' \subseteq \mathbb{P}'$$

for any $\mho \in \mathbb{P}'$.

Conversely, for any $\mho \in \mathbb{P}'$, there exists $\mathbb{P}'' \in \mathbb{B}$ in such a way that $\mho \in \mathbb{P}'' \subseteq \mathbb{P}'$. At that moment,

$$\mathbb{P}' = \cup_{\mho\in\mathbb{P}'}\{\mho\} \subseteq \cup_{\mho\in\mathbb{P}'}\{\mathbb{P}''\} \subseteq \mathbb{P}'$$

such that

$$\mathbb{P}' = \cup_{\mho\in\mathbb{P}'}\{\mathbb{P}''\}$$

which is the union of *pf*-sets in $\mathbb{B}$. □

Proposition 3.24. *Assume that $\mathbb{B}_1$ and $\mathbb{B}_2$ are two picture fuzzy bases for τ_1 and τ_2 over S, respectively. Then, the necessary and sufficient condition for τ_2 to be finer than τ_1 is that for each $\zeta \in S$ and each basis element $\mathbb{P}_1 \in \mathbb{B}_1$ containing ζ, there exists a basis element $\mathbb{P}_2 \in \mathbb{B}_2$ such that $\zeta \in \mathbb{P}_2 \subseteq \mathbb{P}_1$.*

Proof. Suppose that τ_2 be finer than τ_1. Assume further that $\zeta \in S$ and $\mathbb{P}_1 \in \mathbb{B}_1$ contains ζ. Since $\mathbb{B}_1$ is a base for τ_1, so τ_1 is generated by $\mathbb{B}_1$ and hence $\mathbb{P}_1 \in \tau_1$. By assumption, $\mathbb{P}_1 \in \tau_2$ also holds true. Because $\mathbb{B}_2$ is a picture fuzzy base for τ_2 and $\mathbb{P}_1$ is a picture fuzzy open set in τ_2, so there exists $\mathbb{P}_2 \in \mathbb{B}_2$ such that $\zeta \in \mathbb{P}_2 \subseteq \mathbb{P}_1$.

Conversely, assume that for every $\zeta \in S$ and every picture fuzzy base element $\mathbb{P}_1 \in \mathbb{B}_1$ containing ζ, there is a picture fuzzy base element $\mathbb{P}_2 \in \mathbb{B}_2$ such that $\zeta \in \mathbb{P}_2 \subseteq \mathbb{P}_1$. Assume arbitrarily that $\zeta \in \mathbb{P} \subseteq \tau_1$. Since $\mathbb{B}_1$ generates τ_1, so $\exists$ $\mathbb{P}_1 \in \mathbb{B}_1$ such that $\zeta \in \mathbb{P}_1 \subseteq \mathbb{P}$. Thus, by assumption, there is $\mathbb{P}_2 \in \mathbb{B}_2$ such that $\zeta \in \mathbb{P}_2 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}$. Therefore, $\mathbb{P} \subseteq \tau_2$ and hence $\tau_1 \subseteq \tau_2$. □

3.3. **Classification of PFTSs.** The definition of PFTSs presented in Definition 3.1 does not specify the characteristics that may differentiate points of the PFTS. We need some further information about the topology for this purpose. In this section, we shall discuss some major classes of PFTSs that have picture fuzzy open or closed sets satisfying certain requirements in connection to their picture fuzzy points. These characteristics are categorized as T_0, T_1, T_2, T_3, and T_4.

Definition 3.25. A PFTS (S, τ) is called a picture fuzzy Kolmogorov space or picture fuzzy T_0 space (PFT_0S) if for each couple of picture fuzzy points $\mho^{(1)}$ and $\mho^{(2)}$ with $\mho^{(1)} \neq \mho^{(2)}$, there stands at the minimum one picture fuzzy open set $\mathbb{P}$ that contains one and only one of these picture fuzzy points.

Example 3.26. Each discrete PFTS is a PFT_0S for the reason that there lies a picture fuzzy open set $\{\mho^{(1)}\}$ that undoubtedly holds $\mho^{(1)}$ leaving $\mho^{(2)}$.

The next proposition proposes necessary and sufficient requirement for PFTS to be a picture fuzzy Kolmogorov space.

Proposition 3.27. *A PFTS (S, τ) is a PFT_0S iff for any pair of distinct points $\mho^{(1)}$, $\mho^{(2)}$ in (S, τ), we have $\overline{\{\mho^{(1)}\}} \neq \overline{\{\mho^{(2)}\}}$.*

Proof. Let (S, τ) be a PFT_0S such that $\mho^{(1)}, \mho^{(2)} \in (S, \tau)$ with $\mho^{(1)}$ and $\mho^{(2)}$ being distinct. Then, there no less than one picture fuzzy open set which contains precisely one of $\mho^{(1)}$ and $\mho^{(2)}$. To be specific, let this point be $\mho^{(1)}$. But then $\mho^{(1)}$ can't be a picture fuzzy cluster point of $\{\mho^{(2)}\}$. Since $\mho^{(1)} \notin \overline{\{\mho^{(2)}\}}$, so definitely $\overline{\{\mho^{(1)}\}} \neq \overline{\{\mho^{(2)}\}}$.

Conversely, hypothesize that for any couple of distinctive points $\mho^{(1)}, \mho^{(2)} \in (S, \tau)$, $\overline{\{\mho^{(1)}\}} \neq \overline{\{\mho^{(2)}\}}$. For establishing that (S, τ) is a PFT_0S, we assume on the contrary. Then, every picture fuzzy open set that contains $\mho^{(1)}$ also contains $\mho^{(2)}$. By properties of picture fuzzy accumulation points, $\mho^{(1)} \in \overline{\{\mho^{(2)}\}}$ so that $\overline{\{\mho^{(1)}\}} \subseteq \overline{\{\mho^{(2)}\}}$. Following the same logic, it may be demonstrated that $\overline{\{\mho^{(2)}\}} \subseteq \overline{\{\mho^{(1)}\}}$. Thus, $\overline{\{\mho^{(1)}\}} = \overline{\{\mho^{(2)}\}}$, which leads to a contradiction. Hence, there must be no less than one picture fuzzy open set that contains precisely one of the points. Thus, (S, τ) is a PFT_0S. □

Remark 3.28. The feature of being a PFT_0S of any PFTS (S, τ) is transmissible. That is, the quality of being a PFT_0S is transformed to all of its subspaces.

Definition 3.29. A PFTS is called picture fuzzy Tychonoff space, picture fuzzy accessible space or a picture fuzzy T_1 space (PFT_1S) provided for any couple of dissimilar picture fuzzy points $\mho^{(1)}$, $\mho^{(2)}$ of (S, τ), there prevail two picture fuzzy open sets $\mathbb{P}_1$ and $\mathbb{P}_2$ bearing the quality that if $\mho^{(1)} \in \mathbb{P}_1$ then $\mho^{(2)} \notin \mathbb{P}_2$, or if $\mho^{(1)} \notin \mathbb{P}_1$ then $\mho^{(2)} \in \mathbb{P}_2$.

A PFT_1S is also identified as a space with picture fuzzy Fréchet topology.

Example 3.30. Each discrete PFTS (S, τ) is also a PFT_1S. Imagine that there is a couple of distinctive picture fuzzy points $\mho^{(1)}$ and $\mho^{(2)}$ in (S, τ). On that occasion, the existence of picture fuzzy open sets $\{\mho^{(1)}\}$ and $\{\mho^{(2)}\}$ is a must, bearing the quality that $\mho^{(1)} \in \{\mho^{(1)}\}$ although $\mho^{(2)} \notin \{\mho^{(1)}\}$.

Proposition 3.31. *The following assertions regarding a PFTS (S, τ) are analogous:*

(i) *(S, τ) is a PFT_1S.*

(ii) *Each picture fuzzy single-member subset of S is picture fuzzy closed.*

(iii) *Each picture fuzzy subset $\mathbb{P}$ of S is equal to the picture fuzzy intersection of all of its picture fuzzy open supersets.*

Proof. $(i) \Rightarrow (ii)$: For this, Let $\mho^{(1)} \in (S, \tau)$. We demonstrate that $(S, \tau) \setminus \{\mho^{(1)}\}$ is picture fuzzy closed. For this, let $\mho^{(2)} \in (S, \tau) \setminus \{\mho^{(1)}\}$. Then, $\mho^{(1)} \neq \mho^{(2)}$. Since (S, τ) is a PFT_1S, so there are picture fuzzy open sets $\mathbb{P}_1$ and $\mathbb{P}_2$ in the manner that $\mho^{(1)} \in \mathbb{P}_1$, $\mho^{(2)} \notin \mathbb{P}_1$ and $\mho^{(1)} \notin \mathbb{P}_2$, $\mho^{(2)} \in \mathbb{P}_2$. Thus, $\mho^{(2)} \in \mathbb{P}_2 \subseteq (S, \tau) \setminus \{\mho^{(1)}\}$. Therefore, $(S, \tau) \setminus \{\mho^{(1)}\}$ is picture fuzzy open and hence $\{\mho^{(1)}\}$ is picture fuzzy closed.

$(ii) \Rightarrow (iii)$: Suppose that each singleton picture fuzzy subset of (S, τ_S) and choose S' as any picture fuzzy subset of S. Subsequently, $S' = \cup_{\mho^{(1)} \in S'} \{\mho^{(1)}\}$. For each $\mho^{(2)} \notin S'$, $S \setminus \{\mho^{(2)}\}$ is picture fuzzy open and $S' \subseteq S \setminus \{\mho^{(2)}\}$, so that $S \setminus \{\mho^{(2)}\}$ is a picture fuzzy open superset of S'. Since, for each $\mho^{(1)} \in S'$, $\mho^{(1)} \in S \setminus \{\mho^{(2)}\}$, we have $S' = \cap_{\mho^{(1)} \in S'} S \setminus \{\mho^{(2)}\}$. Evidently, picture fuzzy open supersets of S' are included in the picture fuzzy intersection of $S \setminus \{\mho^{(2)}\}$.

$(iii) \Rightarrow (i)$: Suppose that each picture fuzzy subset S' of S is the picture fuzzy intersection of its picture fuzzy open supersets. Assume that $\mho^{(1)}$ and $\mho^{(2)}$ are two distinct points in S. In that case, $\{\mho^{(1)}\}$ is the picture fuzzy intersection of its picture fuzzy open supersets and $\{\mho^{(2)}\}$ is the picture fuzzy intersection of its picture fuzzy open supersets. Thus, there is a picture fuzzy open set $\{\mho^{(1)}\}$ which holds $\mho^{(1)}$ leaving $\mho^{(2)}$ (because, otherwise $\mho^{(1)}$ and $\mho^{(2)}$ both will be in picture fuzzy intersection of picture fuzzy open supersets of $\{\mho^{(1)}\}$). Likewise, there is a picture fuzzy open open set $\{\mho^{(2)}\}$ containing $\mho^{(2)}$ but not $\mho^{(1)}$. Hence, (S, τ_S) is a PFT_1S. □

Corollary 3.32. *A PFTS (S, τ) is PFT_1S iff each finite picture fuzzy subset of S is picture fuzzy closed.*

Proof. Take (S, τ) as a PFT_1S. By virtue of Proposition 3.31, picture fuzzy singleton subsets of S are closed. Thus, being a picture fuzzy union of a finite number of picture fuzzy closed sets, every finite picture fuzzy subset of S is picture fuzzy closed.

Conversely, suppose that every finite picture fuzzy subset of S is picture fuzzy closed. Then, every singleton picture fuzzy subset of S must also be picture fuzzy closed and hence (S, τ) is a PFT_1S. □

Corollary 3.33. *Every finite picture fuzzy T_1 space is discrete.*

Proof. Take (S, τ) to be a finite picture fuzzy T_1 space. Consequently, each picture fuzzy subset of (S, τ) is picture fuzzy closed and open as well (being picture fuzzy complement of a picture fuzzy closed set). Hence, (S, τ) may be declared as discrete. □

Remark 3.34. Each subspace of a PFT_1S is again a PFT_1S.

Definition 3.35. A PFTS (S, τ) is called a picture fuzzy Hausdorff space, picture fuzzy separated space or picture fuzzy T_2 space (PFT_2S) if corresponding to any pair of dissimilar picture fuzzy points $\mho^{(1)}$ and $\mho^{(2)}$ of (S, τ), there exist two picture fuzzy open sets $\mathbb{P}_1$ and $\mathbb{P}_2$ in such a way that $\mho^{(1)} \in \mathbb{P}_1$, $\mho^{(2)} \in \mathbb{P}_2$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \widetilde{\Phi}$.

Example 3.36. Let d be a metric in the PFTS (s, τ). If $\mho^{(1)}$ and $\mho^{(2)}$ is a pair of two dissimilar picture fuzzy points in (S, τ) bearing the quality that $d(\mho^{(1)}, \mho^{(2)}) = \varepsilon$, then $B(\mho^{(1)}, \frac{\varepsilon}{2})$ and $B(\mho^{(2)}, \frac{\varepsilon}{2})$ are two distinct picture fuzzy open sets containing $\mho^{(1)}$ and $\mho^{(2)}$ respectively. Thus, it follows that every picture fuzzy metric space is a picture fuzzy Hausdorff space.

Example 3.37. Consider the discrete PFTS (S, τ). If $\mho^{(1)}$ and $\mho^{(2)}$ is a pair of two dissimilar picture fuzzy points in (S, τ), then evidently $\{\mho^{(1)}\}$ and $\{\mho^{(2)}\}$ are disjoint picture fuzzy open sets obeying $\mho^{(1)} \in \{\mho^{(1)}\}$ and $\mho^{(2)} \in \{\mho^{(2)}\}$. Consequently, (S, τ) is a PFT_2S.

Proposition 3.38. *The necessary and sufficient condition for a PFTS (S, τ) to be a picture fuzzy Hausdorff space is that for any pair of dissimilar picture fuzzy points $\mho^{(1)}$ and $\mho^{(2)}$, there exist picture fuzzy closed sets $\mathbb{P}_1$ and $\mathbb{P}_2$ obeying $\mho^{(1)} \in \mathbb{P}_1$, $\mho^{(2)} \notin \mathbb{P}_1$, $\mho^{(1)} \notin \mathbb{P}_2$, $\mho^{(2)} \in \mathbb{P}_2$ and $\mathbb{P}_1 \cup \mathbb{P}_2 = \widetilde{S}$.*

Proof. Hypothesize that (S, τ) is a picture fuzzy Hausdorff space. Pick two distinct picture fuzzy points $\mho^{(1)}$ and $\mho^{(2)}$ of (S, τ). Then, there must exist two picture fuzzy open sets $\mathbb{Q}_1$ and $\mathbb{Q}_2$ satisfying the requirements that if $\mho^{(1)} \in \mathbb{Q}_1$ then $\mho^{(2)} \notin \mathbb{Q}_1$, or if $\mho^{(1)} \notin \mathbb{Q}_2$ then $\mho^{(2)} \in \mathbb{Q}_2$ and $\mathbb{Q}_1 \cap \mathbb{Q}_2 = \widetilde{\Phi}$. But then, $\mho^{(1)} \notin \mathbb{Q}_1^c$, $\mho^{(2)} \in \mathbb{Q}_1^c$; $\mho^{(1)} \in \mathbb{Q}_2^c$, $\mho^{(2)} \notin \mathbb{Q}_2^c$ and $\mathbb{Q}_1^c \cup \mathbb{Q}_2^c = \widetilde{S}$. Now, replacing $\mathbb{Q}_1^c$ by $\mathbb{P}_1$ and $\mathbb{Q}_2^c$ by $\mathbb{P}_2$, we obtain the desired result.

The converse follows by reverting the steps. □

Remark 3.39. The quality of being a PFT_2S of any PFTS (S, τ) is transformable i.e. each subspace of a PFT_2S is again a PFT_2S.

Definition 3.40. We entitle a PFTS (S, τ) a picture fuzzy regular space (PF regular space) if for some picture fuzzy closed set $\mathbb{P}$ and some picture fuzzy point $\mho$ that is not contained in $\mathbb{P}$, there exist picture fuzzy open sets $\mathbb{P}_1$ and $\mathbb{P}_2$ obeying $\mho \in \mathbb{P}_1$, $\mathbb{P} \subseteq \mathbb{P}_2$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \widetilde{\Phi}$.

Example 3.41. The picture fuzzy indiscrete space is trivially a picture fuzzy regular space for there is no picture fuzzy closed set $\mathbb{P}$ and a picture fuzzy point $\mho \notin \mathbb{P}$, because the only non-empty picture fuzzy closed closed set in $(S, \tau_{indiscrete})$ is $\widetilde{S}$.

This example also advocates that a picture fuzzy regular space need not a picture fuzzy Hausdorff space.

Definition 3.42. A picture fuzzy regular T_1 space is entitled as a picture fuzzy regular Hausdorff space or a picture fuzzy T_3 space (PFT_3S).

Definition 3.43. If corresponding to any two picture fuzzy closed disjoint subsets τ' and τ'' of the PFTS (S, τ), there exist picture fuzzy open sets $\mathbb{P}_1$ and $\mathbb{P}_2$ in such a way that $\tau' \subseteq \mathbb{P}_1$, $\tau'' \subseteq \mathbb{P}_2$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \widetilde{\Phi}$, then (S, τ) is called a picture fuzzy normal space.

Definition 3.44. A picture fuzzy normal T_1 space is called a picture fuzzy T_4 space (PFT_4S).

Remark 3.45. The following chain is valid for PFTSs discussed above:

$$T_k \supseteq T_{k+1}$$

for $0 \le k \le 3$. On the other hand, the counter inclusion may not be true. For instance, (S, τ) with $S = \{p, g\}$, $\tau = \{\widetilde{\Phi}, \mathbb{P}, \widetilde{S}\}$ and

$$\mathbb{P} = \left\{ \left(p, \left\langle \begin{matrix} 0.12 \\ 0.46 \\ 0.37 \end{matrix} \right\rangle \right), \left(g, \left\langle \begin{matrix} 0.39 \\ 0.14 \\ 0.51 \end{matrix} \right\rangle \right) \right\}$$

is a PFT_0S but fails to be a PFT_1S.

Proposition 3.46. *Every PFT_4S is picture fuzzy regular i.e. every picture fuzzy normal T_1 space is picture fuzzy regular.*

Proof. Assume that (S, τ) is a picture fuzzy T_4 space. Pick a picture fuzzy point $\mho$ from (S, τ). Consequently, by virtue of Proposition (3.31), $\{\mho\}$ is a picture fuzzy closed set in (S, τ). Suppose that $\mathbb{P}$ is a picture fuzzy closed set which does not include $\mho$. For the reason that (S, τ) is also picture fuzzy normal, the existence of picture fuzzy open sets $\mathbb{P}_1$ and $\mathbb{P}_2$ in such a manner that $\{\mho\} \subseteq \mathbb{P}_1$, $\mathbb{P} \subseteq \mathbb{P}_2$, and $\mathbb{P}_1 \cap \mathbb{P}_2 = \widetilde{\Phi}$ is certain. But then, $\mho \in \mathbb{P}_1$, $\mathbb{P} \subseteq \mathbb{P}_2$ and $\mathbb{P}_1 \cap \mathbb{P}_2 = \widetilde{\Phi}$. Thus, (S, τ) is picture fuzzy regular. □

3.4. **Compactness in PFTS.** In this subsection, we search the concept of compactness in a PFTS.

Definition 3.47. An assemblage $\mathbb{P}$ of picture fuzzy subsets of a PFTS (S, τ) is called a picture fuzzy cover of τ if $\cup\mathbb{P} = \tau$. If every such $\mathbb{P}$ is in τ, then $\mathbb{P}$ is delineated as a picture fuzzy open cover of τ. A finite subcollection $\mathbb{P}_1$ of $\mathbb{P}$ is termed a finite picture fuzzy sub-cover of τ if $\cup\mathbb{P}_1 = \tau$.

Definition 3.48. If every picture fuzzy open cover of a PFTS (S, τ) carries a finite picture fuzzy sub-cover, then (S, τ) is called picture fuzzy compact topological space.

Every PFTS (S, τ) comprising a finite number of elements is a picture fuzzy compact topological space.

Proposition 3.49. *Every picture fuzzy closed subset of a picture fuzzy compact space carries the same characteristic.*

Proof. Take $\mathbb{P}$ as a picture fuzzy closed subset of a picture fuzzy compact space (S, τ). If $\{U_\alpha : \alpha \in \Omega\}$ is any picture fuzzy open cover for $\mathbb{P}$, then $\exists$ a picture fuzzy open set V_α in (S, τ) in order that $U_\alpha = V_\alpha \cap \mathbb{P}$, $\alpha \in \Omega$. The collection $\{\mathbb{P}^c, V_\alpha : \alpha \in \Omega\}$ serves as a picture fuzzy open cover for (S, τ). Since (S, τ) is a picture fuzzy compact space, so there ought to be a finite picture fuzzy sub-cover $\{\mathbb{P}^c, V_{\alpha_i} : i = 1, \ldots, n\}$ of (S, τ) i.e.

$$\begin{aligned} \tau &= \mathbb{P}^c \cup_{i=1}^n V_{\alpha_i} \\ \therefore \mathbb{P} &= \tau \cap \mathbb{P} \\ &= \cup_{i=1}^n (V_{\alpha_i} \cap \mathbb{P}) \\ &= \cup_{i=1}^n U_{\alpha_i}. \end{aligned}$$

Hence, $\mathbb{P}$ is picture fuzzy compact. □

Proposition 3.50. *Assume that C is a picture fuzzy compact subset of a picture fuzzy Hausdorff space (S, τ) and $\mho \in S \setminus C$, then there are picture fuzzy disjoint open sets $U_\mho$ and $V_\mho$ such that $\mho \in U_\mho$ and $C \subseteq V_\mho$.*

Proof. Suppose that C is a picture fuzzy compact subset of a picture fuzzy Hausdorff space (S, τ) and $\mho \in S \setminus C$. Since (S, τ) is picture fuzzy Hausdorff, so for every $\mho' \in C \setminus \{\mho\}$, $\exists$ picture fuzzy open sets $U_{\mho\mho'}$ and $V_{\mho'}$ such that $\mho \in U_{\mho\mho'}$, $\mho' \in V_{\mho'}$ and $U_{\mho\mho'} \cap V_{\mho'} = \widetilde{\Phi}$.

Now, $\{V_{\mho'} \cap C : \mho' \in C\}$ is a picture fuzzy open cover for C. Due to the reason that C is picture fuzzy compact, this picture fuzzy open cover has a finite picture fuzzy sub-cover

$$V_{\mho_1'} \cap C, V_{\mho_2'} \cap C, \ldots, V_{\mho_n'} \cap C.$$

Let $U_{\mho\mho_1'}, U_{\mho\mho_2'}, \ldots, U_{\mho\mho_n'}$ be the corresponding picture fuzzy open sets in (S, τ) carrying $\mho$. Take

$$U_\mho = \cap_{i=1}^n U_{\mho\mho_i'}$$

and

$$V_\mho = \cup_{i=1}^n V_{\mho_i'}.$$

Then, $\mho \in U_\mho$ and $C \subseteq V_\mho$ and

$$\begin{aligned} U_\mho \cap V_\mho &= U_\mho \cap (\cup_{i=1}^n V_{\mho_i'}) \\ &= \cup_{i=1}^n (U_\mho \cap V_{\mho_i'}) \\ &= \widetilde{\Phi} \end{aligned}$$

as desired. □

Proposition 3.51. *Every picture fuzzy compact Hausdorff space is picture fuzzy normal.*

Proof. Suppose that C_1 and C_2 are two picture fuzzy disjoint subsets of a picture fuzzy compact Hausdorff space (S, τ). By Proposition 3.50, for any $\mho \in C_1$, $\exists$ picture fuzzy open sets $U_\mho$ and $V_\mho$ such that $\mho \in U_\mho$, $C_2 \subseteq V_\mho$ and $U_\mho \cap V_\mho = \widetilde{\Phi}$.

The collections $\{U_\mho : \mho \in C_1\}$ form a picture fuzzy open covering for C_1. Since C_1 is picture fuzzy closed, C_1 is picture fuzzy compact, so there is a finite number of elements $\mho_1, \mho_2, \ldots, \mho_n$ such that $C_1 \subseteq \cup_{i=1}^n U_{\mho_i}$. Let $\cup_{i=1}^n U_{\mho_i} = U$, $\cap_{i=1}^n V_{\mho_i} = V$, then $C_1 \subseteq U$, $C_2 \subseteq V$ and $U \cap V = \widetilde{\Phi}$.

Hence, (S, τ) is picture fuzzy normal. □

4. Picture Fuzzy Prevalence Effect Method (PFPEM) in Group Decision-making and Its Comparison

4.1. **Implementation.** Within this part, we present the utility of PFT in group decision-making. To this end, we modify the Prevalence Effect Method (PEM) [15] employing pf-sets and PFT. Therefore, we suggest Picture Fuzzy Prevalence Effect Method (PFPEM) for the group decision-making process.

Algorithm: PFPEM

Step 1. Construct the pf-sets according to each decision-maker $DM = \{DM_1, \ldots, DM_p\}$ and each attribute such that $C = \{\chi_1, \ldots, \chi_m\}$ is the collection of choices evaluated under the family of attributes $Q = \{q_1, \ldots, q_n\}$. Here, the collection of pf-sets of DM can be determined as a PFT according the each attribute. In addition, *pfs*-sets/matrices can be constructed by DM, C, and Q without PFT.

Step 2. Construct information matrix D_i $(i = 1, \ldots, p)$, as accorded by decision makers. Then, obtain the aggregated matrix

$$D = \frac{\sum_{i=1}^{p} DM_i}{p} = \left[\left\langle \begin{matrix} \mu_{ij} \\ \eta_{ij} \\ \nu_{ij} \end{matrix} \right\rangle\right]_{m\times n}$$

Step 3. Obtain the score matrix $[s_{i1}]$ defined by

$$s_{i1} = \mu^s - \eta^s \nu^s$$

such that

$$\mu^s = \sum_{j=1}^{n}\left[\left(\frac{1}{m}\sum_{k=1}^{m}\mu_{kj}\right)\frac{1}{n}\left(\sum_{t=1}^{n}\mu_{it}\right)\mu_{ij}\right]$$

$$\eta^s = \sum_{j=1}^{n}\left[\left(\frac{1}{m}\sum_{k=1}^{m}\eta_{kj}\right)\frac{1}{n}\left(\sum_{t=1}^{n}\eta_{it}\right)\eta_{ij}\right]$$

$$\nu^s = \sum_{j=1}^{n}\left[\left(\frac{1}{m}\sum_{k=1}^{m}\nu_{kj}\right)\frac{1}{n}\left(\sum_{t=1}^{n}\nu_{it}\right)\nu_{ij}\right]$$

Step 4. Obtain the ranking orders according to score matrix.

The flowchart of the algorithm is presented in Figure 1. An illustrative example of PFPEM is provided as follows:

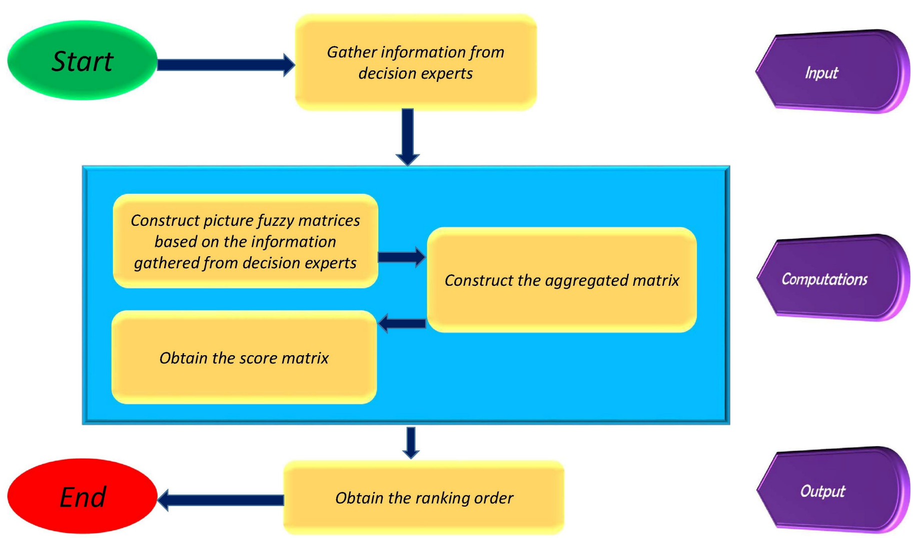

Figure 1. Flowchart of the algorithm

As an illustration of the algorithm, we address a state managerial problem by following the procedural steps outlined in the algorithm.

Example 4.1. Assume that an investment firm wants to put money into the best option (taken from [20,47]). Consider the *pf*-sets according to each decision-maker $DM = \{DM_1, DM_2, DM_3, DM_4\}$, which refers to the "attractiveness of projects" that the firm is considering for investment. Assume there are six potential projects, i.e. $C = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6\}$, and four criteria to consider: q_1 ="Risk Analysis," q_2 ="Growth Analysis," q_3 ="Social-Political Impact Analysis," and q_4 ="Environmental Analysis". The firm assesses the possibilities based on the parameters and construct DM_1, DM_2, DM_3, and DM_4 according to decision-makers as follows:

$$D_1 = \begin{bmatrix} \left\langle\begin{matrix}0.31\\0.22\\0.41\end{matrix}\right\rangle & \left\langle\begin{matrix}0.54\\0.21\\0.15\end{matrix}\right\rangle & \left\langle\begin{matrix}0.60\\0.14\\0.26\end{matrix}\right\rangle & \left\langle\begin{matrix}0.38\\0.21\\0.40\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.12\\0.41\\0.33\end{matrix}\right\rangle & \left\langle\begin{matrix}0.81\\0.11\\0.02\end{matrix}\right\rangle & \left\langle\begin{matrix}0.26\\0.51\\0.20\end{matrix}\right\rangle & \left\langle\begin{matrix}0.65\\0.15\\0.18\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.23\\0.52\\0.21\end{matrix}\right\rangle & \left\langle\begin{matrix}0.13\\0.48\\0.37\end{matrix}\right\rangle & \left\langle\begin{matrix}0.72\\0.15\\0.03\end{matrix}\right\rangle & \left\langle\begin{matrix}0.29\\0.58\\0.12\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.12\\0.46\\0.37\end{matrix}\right\rangle & \left\langle\begin{matrix}0.23\\0.59\\0.18\end{matrix}\right\rangle & \left\langle\begin{matrix}0.32\\0.49\\0.15\end{matrix}\right\rangle & \left\langle\begin{matrix}0.14\\0.32\\0.45\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.45\\0.09\\0.36\end{matrix}\right\rangle & \left\langle\begin{matrix}0.60\\0.23\\0.14\end{matrix}\right\rangle & \left\langle\begin{matrix}0.81\\0.11\\0.06\end{matrix}\right\rangle & \left\langle\begin{matrix}0.43\\0.18\\0.35\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.44\\0.40\\0.13\end{matrix}\right\rangle & \left\langle\begin{matrix}0.42\\0.36\\0.22\end{matrix}\right\rangle & \left\langle\begin{matrix}0.43\\0.27\\0.13\end{matrix}\right\rangle & \left\langle\begin{matrix}0.35\\0.29\\0.34\end{matrix}\right\rangle \end{bmatrix}, \quad D_2 = \begin{bmatrix} \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle & \left\langle\begin{matrix}0\\1\\1\end{matrix}\right\rangle \end{bmatrix},$$

$$D_3 = \begin{bmatrix} \left\langle\begin{matrix}0.26\\0.27\\0.46\end{matrix}\right\rangle & \left\langle\begin{matrix}0.49\\0.26\\0.20\end{matrix}\right\rangle & \left\langle\begin{matrix}0.55\\0.19\\0.31\end{matrix}\right\rangle & \left\langle\begin{matrix}0.33\\0.26\\0.45\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.07\\0.46\\0.38\end{matrix}\right\rangle & \left\langle\begin{matrix}0.76\\0.16\\0.07\end{matrix}\right\rangle & \left\langle\begin{matrix}0.21\\0.56\\0.25\end{matrix}\right\rangle & \left\langle\begin{matrix}0.60\\0.20\\0.23\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.18\\0.57\\0.26\end{matrix}\right\rangle & \left\langle\begin{matrix}0.08\\0.53\\0.42\end{matrix}\right\rangle & \left\langle\begin{matrix}0.67\\0.20\\0.08\end{matrix}\right\rangle & \left\langle\begin{matrix}0.24\\0.63\\0.17\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.07\\0.51\\0.42\end{matrix}\right\rangle & \left\langle\begin{matrix}0.18\\0.64\\0.23\end{matrix}\right\rangle & \left\langle\begin{matrix}0.27\\0.54\\0.20\end{matrix}\right\rangle & \left\langle\begin{matrix}0.09\\0.37\\0.50\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.40\\0.14\\0.41\end{matrix}\right\rangle & \left\langle\begin{matrix}0.55\\0.28\\0.19\end{matrix}\right\rangle & \left\langle\begin{matrix}0.76\\0.16\\0.11\end{matrix}\right\rangle & \left\langle\begin{matrix}0.38\\0.23\\0.40\end{matrix}\right\rangle \\ \left\langle\begin{matrix}0.39\\0.45\\0.18\end{matrix}\right\rangle & \left\langle\begin{matrix}0.37\\0.41\\0.26\end{matrix}\right\rangle & \left\langle\begin{matrix}0.38\\0.32\\0.18\end{matrix}\right\rangle & \left\langle\begin{matrix}0.30\\0.34\\0.39\end{matrix}\right\rangle \end{bmatrix}, and \quad D_4 = \begin{bmatrix} \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle \\ \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle & \left\langle\begin{matrix}1\\0\\0\end{matrix}\right\rangle \end{bmatrix}.$$

Each decision maker's picture fuzzy matrix D_i is provided, where projects are represented in rows and attributes are indicated in columns (For more details for the matrix representation, see [28]). Thus, the aggregated matrix is as follows:

$$D = \begin{bmatrix}
\left\langle\begin{matrix}0.3925\\0.3950\\0.4425\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5075\\0.4500\\0.4350\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5375\\0.4475\\0.4775\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4275\\0.4100\\0.4575\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.2975\\0.3950\\0.3750\end{matrix}\right\rangle & \left\langle\begin{matrix}0.6425\\0.4925\\0.4700\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3675\\0.4550\\0.3775\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5625\\0.4625\\0.4700\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.3525\\0.4500\\0.3725\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3025\\0.4150\\0.3875\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5975\\0.4800\\0.4500\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3825\\0.4800\\0.3650\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.4625\\0.3975\\0.4650\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3525\\0.4675\\0.3650\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3975\\0.4650\\0.3800\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3075\\0.3775\\0.4100\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.5225\\0.4800\\0.4175\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5375\\0.4700\\0.4475\end{matrix}\right\rangle & \left\langle\begin{matrix}0.6725\\0.4925\\0.4800\end{matrix}\right\rangle & \left\langle\begin{matrix}0.7525\\0.4150\\0.4575\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.4575\\0.4725\\0.4050\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4475\\0.4575\\0.4225\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4525\\0.4375\\0.4025\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4125\\0.4225\\0.4350\end{matrix}\right\rangle
\end{bmatrix}.$$

Thereafter,the scores and ranking order of PFPEM are as follows:

$$\left\{{}^{0.3125}\chi_1, {}^{0.3147}\chi_2, {}^{0.2262}\chi_3, {}^{0.1818}\chi_4, {}^{0.4422}\chi_5, {}^{0.2731}\chi_6\right\} \text{ and } \chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_1 \prec \chi_2 \prec \chi_5.$$

4.2. **Comparison.** For the fair comparison of PFPEM with the Method 1 [47], Method 2 (PEM) [15], and Method 3 [27], the single matrix is needed. Therefore, consider that the original data from [20,47] as the matrix D and it is as follows:

$$D = \begin{bmatrix}
\left\langle\begin{matrix}0.31\\0.22\\0.41\end{matrix}\right\rangle & \left\langle\begin{matrix}0.54\\0.21\\0.15\end{matrix}\right\rangle & \left\langle\begin{matrix}0.60\\0.14\\0.26\end{matrix}\right\rangle & \left\langle\begin{matrix}0.38\\0.21\\0.40\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.12\\0.41\\0.33\end{matrix}\right\rangle & \left\langle\begin{matrix}0.81\\0.11\\0.02\end{matrix}\right\rangle & \left\langle\begin{matrix}0.26\\0.51\\0.20\end{matrix}\right\rangle & \left\langle\begin{matrix}0.65\\0.15\\0.18\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.23\\0.52\\0.21\end{matrix}\right\rangle & \left\langle\begin{matrix}0.13\\0.48\\0.37\end{matrix}\right\rangle & \left\langle\begin{matrix}0.72\\0.15\\0.03\end{matrix}\right\rangle & \left\langle\begin{matrix}0.29\\0.58\\0.12\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.12\\0.46\\0.37\end{matrix}\right\rangle & \left\langle\begin{matrix}0.23\\0.59\\0.18\end{matrix}\right\rangle & \left\langle\begin{matrix}0.32\\0.49\\0.15\end{matrix}\right\rangle & \left\langle\begin{matrix}0.14\\0.32\\0.45\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.45\\0.09\\0.36\end{matrix}\right\rangle & \left\langle\begin{matrix}0.60\\0.23\\0.14\end{matrix}\right\rangle & \left\langle\begin{matrix}0.81\\0.11\\0.06\end{matrix}\right\rangle & \left\langle\begin{matrix}0.43\\0.18\\0.35\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.44\\0.40\\0.13\end{matrix}\right\rangle & \left\langle\begin{matrix}0.42\\0.36\\0.22\end{matrix}\right\rangle & \left\langle\begin{matrix}0.43\\0.27\\0.13\end{matrix}\right\rangle & \left\langle\begin{matrix}0.35\\0.29\\0.34\end{matrix}\right\rangle
\end{bmatrix}.$$

The proposed PFPEM and Method 1-3 are applied to the D and the score values and ranking orders are presented in Tables 1 and 2, respectively. Moreover, the visual results of the Table 2 can be observed in Figure 2.

Table 1. The comparison of the score values

	Method 1	Method 2	Method 3	Method 4	Method 5	Method 6	Method 7	Method 8	Proposed PFPEM
χ_1	0.2500	0.4453	0.4188	0.5476	0.3100	0.5266	0.6305	0.7509	0.4394
χ_2	0.5000	0.4359	0.4200	0.5447	0.1200	0.5252	0.2966	0.7738	0.4313
χ_3	0.2500	0.2571	0.2993	0.3697	0.1300	0.3207	0.0626	0.5486	0.2574
χ_4	0	0.1653	0.1625	0.2750	0.2300	0.2511	0.0548	0.4911	0.1459
χ_5	0.7500	0.7655	0.5938	0.7474	0.5700	0.8067	1.0000	1.0000	0.7638
χ_6	0	0.3455	0.3415	0.4451	0.4200	0.4696	0.5539	0.6974	0.3378

Method 1 cannot rank Project-4 (χ_4) and Project-6 (χ_6) as well as the Project-1 (χ_1) and Project-3 (χ_3). Except for the alternatives Project-1 (χ_1) and Project-2 (χ_2), the aforesaid ranking orders show that the proposed approach, i.e.

TABLE 2. The comparison of the ranking orders

Methods	Reference	Environment	Operation/Concepts	Ranking Orders
Method 1	[47]	*pfs*-sets	Adjustable soft discernibility matrix	$\chi_4 = \chi_6 \prec \chi_1 = \chi_3 \prec \chi_2 \prec \chi_5$
Method 2	[15]	*fpfs*-matrices	Weighted aggregation operator	$\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_2 \prec \chi_1 \prec \chi_5$
Method 3	[27]	*pfs*-sets	Weighted aggregation operator	$\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_1 \prec \chi_2 \prec \chi_5$
Method 4	[17]	*fpfs*-matrices	TOPSIS-based concept	$\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_1 \prec \chi_2 \prec \chi_5$
Method 5	[14]	*ifpifs*-matrices	min-max operator	$\chi_2 \prec \chi_3 \prec \chi_4 \prec \chi_1 \prec \chi_6 \prec \chi_5$
Method 6	[8]	fuzzy sets	Fuzzy TOPSIS	$\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_2 \prec \chi_1 \prec \chi_5$
Method 7	[34]	fuzzy sets	Fuzzy VIKOR	$\chi_4 \prec \chi_3 \prec \chi_2 \prec \chi_6 \prec \chi_1 \prec \chi_5$
Method 8	[49]	fuzzy sets	Fuzzy COPRAS	$\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_1 \prec \chi_2 \prec \chi_5$
PFPEM	In this study	*pf*-sets/matrices	PFT and weighted aggregation operator	$\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_2 \prec \chi_1 \prec \chi_5$

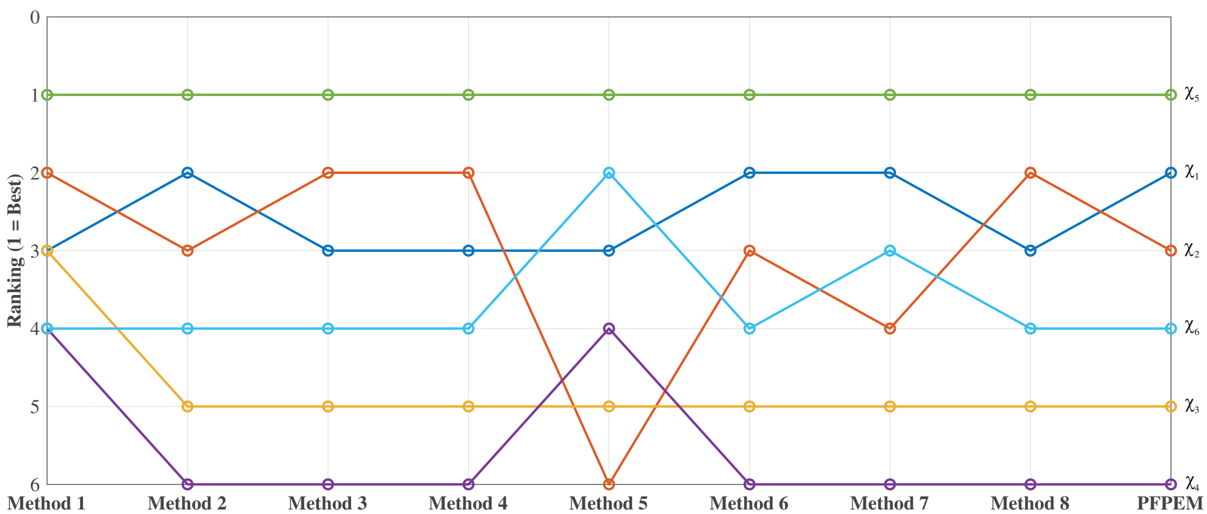

FIGURE 2. Slope chart for the ranking orders of the methods herein

PFPEM, and Method 3 provide the same ranking orders. Besides, PFPEM and Method 2 provide the same ranking orders. Furthermore, all the methods confirm that Project-5 (χ_5) is the most appropriate project for the firm, while Project-4 (χ_4) is not.

According to the findings presented in Table 2 and Figure 2, χ_4 is consistently ranked lowest across all methods, indicating that the project is less attractive compared to other alternatives regarding risk, growth, or environmental criteria. Similarly, χ_5 ranks last in nearly all methods, suggesting high risk or low return potential. This consistency establishes a strong consensus that the firm should not prioritize these projects.

The proposed PFPEM method entirely aligns with Method 6 (Fuzzy TOPSIS) and Method 8 (Fuzzy COPRAS), with the ranking of $\chi_4 \prec \chi_3 \prec \chi_6 \prec \chi_2 \prec \chi_1 \prec \chi_5$. This indicates that PFPEM can produce results similar to fuzzy set-based methods and is consistent with certain approaches in the literature. However, the ranking differences between χ_1, χ_2, and χ_6 observed in Methods 2, 3, 4, and 7 may arise from the operational concepts employed (TOPSIS, VIKOR, weighted aggregation) or the mathematical structures of the environments (*pfs*-sets, *fpfs*-matrices). For example, while the VIKOR method concentrates on compromise solutions, TOPSIS is based on proximity to the ideal solution, which influences the rankings.

PFPEM's integration of PFT and a weighted aggregation operator on *pf*-sets provided a balanced approach in ranking intermediate projects (χ_1, χ_2, χ_3, and χ_6) by modeling uncertainties more flexibly. Notably, χ_2 drops to fourth place in PFPEM, despite being ranked third in Method 7 (Fuzzy VIKOR), suggesting that the criteria weights or risk tolerance are integrated differently. Consequently, the choice of method depends on the firm's priorities (risk aversion, social impact, or environmental sustainability). In general, PFPEM, Method 2 and 6 provide the same ranking orders. Furthermore, all the methods confirm that χ_5 is the most appropriate project for the firm, while χ_4 is not except Method 5. Consequently, this study demonstrates PFPEM's competitiveness with existing methods and its flexibility as an alternative in MCDM processes.

5. Conclusion

The idea of picture fuzzy sets (*pf*-sets) offers valuable benefits for managing a wide range of situations that individuals may encounter in their daily lives, such as when they are presented with options to approve, disapprove, remain uncertain, or choose not to respond at all. These sets can be applied in several fields, including but not limited to pattern recognition, economics, electoral systems, life sciences, artificial intelligence, marketing analysis, business, decision-making, speech recognition, neural networks, and operations research. Their versatility is one of the key reasons they have become so popular in recent years, as professionals from various industries have found them invaluable tools for analysing complex data and making more informed decisions.

This study presents a new definition of PFT and explores its fundamental properties. It discusses the limit point of *pf*-sets and examines the basis for a PFTS. Additionally, it covers the primary categories of PFTSs, which include picture fuzzy open or closed sets that satisfy specific requirements regarding their picture fuzzy points. The study also investigates compactness in PFTS and concludes by proposing a group decision-making implementation that employs PFT.

The concepts presented in this study are significant and can be further explored in various sectors beyond the academic realm. For instance, these concepts can be applied in real-world scenarios like business decision-making processes, medical image diagnosis systems, and even personal relationships. Furthermore, in light of the soft topological spaces [12, 13, 16], picture fuzzy soft topology (PFST), the foundation for the presented concepts, can be expanded and studied in greater depth. This involves exploring notions, such as picture fuzzy soft limit point, picture fuzzy soft open or closed sets, and compactness within PFST. Besides, all the concepts herein can be generalized on picture fuzzy parameterized picture fuzzy soft sets space [29]. By conducting further research in this area, the practical applications of these concepts in everyday life can be better understood.

The theoretical contributions presented in the study extend beyond decision making and topological analysis, encompassing significant application potential in the field of Machine Learning. Picture fuzzy structures, which simultaneously evaluate membership, non-membership, and neutral situations in real life, facilitate the development of more flexible and realistic models compared to classical methods in data classification problems. In this context, it is assessed that the proposed conceptual structures can form the foundation for improving algorithms such as Picture Fuzzy Soft k-Nearest Neighbor (PFS-kNN) [28]. PFS-kNN redefines the decision mechanism of the classical kNN algorithm by considering the membership, non-membership, and neutral degrees of each example, thereby enhancing classification accuracy. Such an approach can support the creation of more powerful and explainable models, particularly in areas characterized by high uncertainty, such as medical diagnosis, customer behavior analysis, and the classification of social media data. Therefore, integrating the theoretical framework presented in this study with machine learning applications will facilitate the expansion of both theoretical and practical contributions in future research.

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Conflicts of Interest

All the authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors Contribution Statement

K. Naeem: Conceptualization, Methodology, Writing - Original Draft, Writing - Review & Editing. S. Memiş: Software, Writing - Original Draft, Writing - Review & Editing, Visualization. S. Azeem: Methodology, Writing - Review & Editing. All the authors have read and agreed to the published version of the manuscript.

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