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Solutions of the Fractional Combined KdV-mKdV Equation Using the Special Generalized Hyperbolic and Trigonometric Functions

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Abstract −The combined KdV-mKdV equation is one of the essential equations used in soliton physics. In this study, the analytical solutions of the space-time fractional combined KdV-mKdV equation are gained using the Sardar sub-equation approach. In this equation, the fractional derivatives are given in a conformable sense. A clue on how we can convert the fractional partial differential equation into an ordinary differential equation to acquire analytical solutions is presented in this paper. The acquired solutions are obtained in special generalized hyperbolic and trigonometric forms. The different types of soliton solutions are also found. Some are illustrated by selecting the appropriate parameter values in 2D and 3D graphs. Finally, the suggested approach is reliable, effective, and beneficial for solving many nonlinear integer and fractional order differential equations.

Keywords − Combined KdV-mKdV equation, special generalized hyperbolic and trigonometric functions, conformable fractional derivative, soliton solutions, traveling wave solutions

1. Introduction

The KdV equation, which occurs in shallow water waves, was first discovered by Korteweg and de Vries in 1895. This equation, in its general form, is

$$\frac{\partial u}{\partial t}+mu\frac{\partial u}{\partial x}+s\frac{\partial^3 u}{\partial x^3}=0$$

where s and m are arbitrary constants. Here, $\partial u/\partial t$ states the time evolution of the wave propagating in this equation. While $u(\partial u/\partial x)$ represents nonlinearity and explains the steepening of the wave, $\partial^3 u/\partial x^3$ represents linear dispersion and explains the spreading of the wave. The KdV equation presents soliton solutions and describes solitary waves with particle-like features that decrease monotonically at infinity. Besides, the modified KdV (mKdV) equation is

$$\frac{\partial u}{\partial t}+nu^2\frac{\partial u}{\partial x}+s\frac{\partial^3 u}{\partial x^3}=0$$

where n and s are arbitrary constants. This equation emerges in multi-component plasmas and electric circuits. The mKdV and the KdV equations are the same: they are both infinitely many conservative quantities and fully integrable [1].

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In soliton theory, the mKdV and the KdV equations are notable soliton equations investigated extensively by physicists and mathematicians. However, the nonlinear terms of the mKdV and the KdV equations are frequently found contemporaneously in problems such as plasma physics, liquid physics, solid physics, quantum field theory, etc. These terms collectively form the so-called combined KdV-mKdV equation [2]

$$\frac{\partial u}{\partial t} + mu\frac{\partial u}{\partial x} + nu^2\frac{\partial u}{\partial x} + s\frac{\partial^3 u}{\partial x^3} = 0$$

Here, m, n, and s are arbitrary constants. This equation, which is utilized in quantum field theory, solid-state physics, and plasma physics, describes a model for wave propagation in a one-dimensional nonlinear lattice. Besides, heat pulse propagation through single sodium fluoride crystals has been explained using the combined KdV-mKdV equation [3]. In the literature, numerous methods have been presented to find the solutions of this equation, such as the leading order analysis [3, 4], the direct [4], the homogeneous balance [2], the modified mapping [5], a new unified algebraic [6], the Adomian decomposition [7], the Fan's direct algebraic [8], the extended tanh [9], a new improved Jacobi elliptic function [10], the exp-function [11], the new generalized Jacobi elliptic function expansion [12], the improved (G'/G)-expansion [13], the complex [14], a transformation [15], the novel (G'/G)-expansion [16, 17], the quantic B-spline collocation finite element [18] the consistent tanh expansion [19], the sine-Gordon expansion [20], the quantic B-spline differential quadrature [21], an elliptic equation [22], the bilinear [23], the reduced differential transform [24], the improved tan(Φ/2)-expansion [25], the exp-(-$\Phi(\zeta)$)-expansion [26], the Kudryashov's [26], the generalized projective Riccati equation [27], and the two variable $(G'/G, 1/G)$-expansion [27] methods.

Investigating the solutions of fractional differential equations is significant in examining physical phenomena. These phenomena can appear in numerous branches of mathematics, physics, and engineering, such as plasma physics, traveling waves, aerodynamics, chemical kinematics, fluid mechanics, optical fibers, control theory, turbulence, electromagnetism, solid state physics, quantum mechanics, signal processing, etc. [28-33]. One important fractional partial differential equation in the literature is the fractional combined KdV-mKdV equation. In recent years, researchers have applied numerous methods to solve this equation. Thus far, the Jacobi elliptic function expansion [34], $(1/G')$-expansion [35], the collocation [35], and the unified [36] methods are used to acquire the solutions of the time-fractional combined KdV-mKdV equation. Besides, the sub-equation [37] and the mapping [38] methods are also utilized to get the solutions of the space-time fractional combined KdV-mKdV equation. When the literature is examined, it can be seen that the solutions of the space-time fractional combined KdV-mKdV equation have not yet been found by the Sardar sub-equation approach. Therefore, our research aims to apply the Sardar sub-equation approach to obtain the analytic solutions of the space-time fractional combined KdV-mKdV equation in the general form

$$D_t^\alpha u + muD_x^\beta u + nu^2 D_x^\beta u + sD_x^\beta D_x^\beta D_x^\beta u = 0, \qquad 0 < \alpha, \beta \leq 1 \tag{1.1}$$

where m, n, and s are the coefficients of nonlinear and dispersion terms. This equation arises in various fields of physics and engineering, such as ion acoustic waves in negative ion plasmas, internal solitary waves in shallow seas, and dust acoustic solitary waves in the atmosphere [38]. Here, D_t^α and D_x^β are the fractional derivative operators in the conformable sense. Several fractional partial differential equations have been studied in recent years by the proposed approach [39-45].

The structure of this study is given as follows. In Section 2, the definition and the theorems of the conformable fractional derivative are expressed. In Section 3, the general structure of the Sardar sub-equation approach is mentioned. In Section 4, the proposed approach gains the solutions seen from (1.1). In Section 5, 2D and 3D graphs of some solutions are exhibited to demonstrate the effectiveness of the introduced approach. In Section 6, the conclusions of the paper are given.

2. Conformable Fractional Derivative

In the literature, scientists and mathematicians have utilized various fractional derivative definitions such as Caputo, Riemann-Liouville, and Grünwald-Letnikov in their studies. While the linearity feature is satisfied by each of these fractional derivatives, this is not always true for the other features. In the definition of the Riemann-Liouville derivative from these fractional derivatives, the derivative of a constant function is not zero when the order of the derivative is not a natural number. Many fractional derivative definitions do not also satisfy the derivative formula of the product and quotient of two functions and the chain rule. Besides, in the Caputo derivative definition, the function is required to be differentiable. In 2014, to overcome these difficulties, a new definition of the fractional derivative named the conformable fractional derivative was presented by Khalil et al. [46]. Due to its familiarity with the definition of the usual derivative, this definition makes it easiest to recognize the fractional derivative. Therefore, (1.1) is investigated in a conformable sense in this study. When the literature is reviewed, the conformable fractional derivatives have been used in many studies [46-50]. The definition and the features of the conformable fractional derivative are given below.

Definition 2.1. Suppose that $h: [0,\infty) \to \mathbb{R}$ is a function, then the α-th order conformable fractional derivative of h is given by [51]

$$T_\alpha(h)(t) = \lim_{\varepsilon\to 0} \frac{h(t+\varepsilon t^{1-\alpha}) - h(t)}{\varepsilon}, \quad t > 0, \quad \alpha \in (0,1]$$

When h is α-differentiable in some $(0,\alpha)$, $\alpha > 0$ and $\lim_{t\to 0^+} h^{(\alpha)}(t)$ occurs, then $h^{(\alpha)}(0) = \lim_{t\to 0^+} h^{(\alpha)}(t)$.

Theorem 2.1. Suppose that $\alpha \in (0,1]$ and h, w are α-differentiable at $t > 0$. Then, the following features are expressed [51]:

i. $T_\alpha(a_1 h + a_2 w) = a_1 T_\alpha(h) + a_2 T_\alpha(w), \quad \forall\, a_1, a_2 \in \mathbb{R}$

ii. $T_\alpha(\eta) = 0$ for all constant functions $h(t) = \eta$

iii. $T_\alpha(t^z) = z t^{z-\alpha} \quad \forall\, z \in \mathbb{R}$

iv. $T_\alpha(hw) = hT_\alpha(w) + wT_\alpha(h)$

v. $T_\alpha\left(\frac{h}{w}\right) = \frac{wT_\alpha(h) - hT_\alpha(w)}{w^2}$

vi. If h is α-differentiable, then $T_\alpha(h)(t) = t^{1-\alpha}\frac{dh}{dt}$

Theorem 2.2. [52] Suppose that $h, w : (\alpha,\infty) \to \mathbb{R}$ be α-differentiable functions, where $0 < \alpha \leq 1$ and $y(t) = h(w(t))$. Then, $y(t)$ is α-differentiable and for all t with $t \neq 0$ and $w(t) \neq 0$, we get

$$T_\alpha(y)(t) = T_\alpha(h)\big(w(t)\big)T_\alpha(w)(t)w^{\alpha-1}(t)$$

If $t = 0$, we have

$$T_\alpha(y)(0) = \lim_{t\to 0} T_\alpha(h)\big(w(t)\big)T_\alpha(w)(t)w^{\alpha-1}(t)$$

It is seen from this theorem that the chain rule for the fractional derivative of conformable type is provided.

3. General Structure of the Sardar Sub-Equation Approach

This section briefly describes the Sardar sub-equation approach [53]. Consider the nonlinear fractional partial differential equation

$$H\left(u, u_t, u_x, D_t^{\alpha}u, D_x^{\beta}u, \ldots\right) = 0, \quad 0 < \alpha, \beta \leq 1 \tag{3.1}$$

where D_t^{α} and D_x^{β} represent the conformable fractional derivative of u. Utilizing the traveling wave transformation

$$u = u(\xi), \qquad \xi = l\frac{x^{\beta}}{\beta} - k\frac{t^{\alpha}}{\alpha} \tag{3.2}$$

such that l (wavelength) and k (wave velocity) are arbitrary nonzero constants, a nonlinear fractional partial differential equation (3.1) turns into an integer order ordinary differential equation

$$G(u, u', u'', u''', \ldots) = 0 \tag{3.3}$$

The goal of the suggested approach is to gain the solution seen from (3.1) given as

$$u(\xi) = \sum_{i=0}^{N} c_i F^i(\xi) \tag{3.4}$$

Here, c_i $(i \in \{1, 2, 3, \ldots, N\})$ are the coefficients to be determined later. The function $F(\xi)$ satisfies the ordinary differential equation

$$(dF/d\xi)^2 = \mu + \varphi F^2(\xi) + F^4(\xi) \tag{3.5}$$

where μ and φ are real constants, the solutions of (3.5) are as follows:

Case I. When $\varphi > 0$ and $\mu = 0$, then

$$F_1^{\pm}(\xi) = \pm\sqrt{-pq\varphi}\,\mathrm{sech}_{pq}\left(\sqrt{\varphi}\xi\right)$$

$$F_2^{\pm}(\xi) = \pm\sqrt{pq\varphi}\,\mathrm{csch}_{pq}\left(\sqrt{\varphi}\xi\right)$$

Case II. When $\varphi < 0$ and $\mu = 0$, then

$$F_3^{\pm}(\xi) = \pm\sqrt{-pq\varphi}\,\mathrm{sec}_{pq}\left(\sqrt{-\varphi}\xi\right)$$

$$F_4^{\pm}(\xi) = \pm\sqrt{-pq\varphi}\,\mathrm{csc}_{pq}\left(\sqrt{-\varphi}\xi\right)$$

Case III. When $\varphi < 0$ and $\mu = \frac{\varphi^2}{4}$, then

$$F_5^{\pm}(\xi) = \pm\sqrt{\frac{-\varphi}{2}}\,\mathrm{tanh}_{pq}\left(\sqrt{\frac{-\varphi}{2}}\xi\right)$$

$$F_6^{\pm}(\xi) = \pm\sqrt{\frac{-\varphi}{2}}\,\mathrm{coth}_{pq}\left(\sqrt{\frac{-\varphi}{2}}\xi\right)$$

$$F_7^{\pm}(\xi) = \pm\sqrt{\frac{-\varphi}{2}}\left(\tanh_{pq}\left(\sqrt{-2\varphi}\xi\right) \pm \sqrt{-pq}\,\mathrm{sech}_{pq}\left(\sqrt{-2\varphi}\xi\right)\right)$$

$$F_8^{\pm}(\xi) = \pm\sqrt{\frac{-\varphi}{2}}\left(\coth_{pq}\left(\sqrt{-2\varphi}\xi\right) \pm \sqrt{pq}\,\mathrm{csch}_{pq}\left(\sqrt{-2\varphi}\xi\right)\right)$$

$$F_9^{\pm}(\xi) = \pm\sqrt{\frac{-\varphi}{8}}\left(\tanh_{pq}\left(\sqrt{\frac{-\varphi}{8}}\xi\right) + \coth_{pq}\left(\sqrt{\frac{-\varphi}{8}}\xi\right)\right)$$

Case IV. When $\varphi > 0$ and $\mu = \frac{\varphi^2}{4}$, then

$$F_{10}^{\pm}(\xi) = \pm\sqrt{\frac{\varphi}{2}}\tan_{pq}\left(\sqrt{\frac{\varphi}{2}}\xi\right)$$

$$F_{11}^{\pm}(\xi) = \pm\sqrt{\frac{\varphi}{2}}\cot_{pq}\left(\sqrt{\frac{\varphi}{2}}\xi\right)$$

$$F_{12}^{\pm}(\xi) = \pm\sqrt{\frac{\varphi}{2}}\left(\tan_{pq}\left(\sqrt{2\varphi}\xi\right) \pm \sqrt{pq}\sec_{pq}\left(\sqrt{2\varphi}\xi\right)\right)$$

$$F_{13}^{\pm}(\xi) = \pm\sqrt{\frac{\varphi}{2}}\left(\cot_{pq}\left(\sqrt{2\varphi}\xi\right) \pm \sqrt{pq}\csc_{pq}\left(\sqrt{2\varphi}\xi\right)\right)$$

$$F_{14}^{\pm}(\xi) = \pm\sqrt{\frac{\varphi}{8}}\left(\tan_{pq}\left(\sqrt{\frac{\varphi}{8}}\xi\right) + \cot_{pq}\left(\sqrt{\frac{\varphi}{8}}\xi\right)\right)$$

where

$$\mathrm{sech}_{pq}(\xi) = \frac{2}{pe^{\xi} + qe^{-\xi}}, \quad \mathrm{csch}_{pq}(\xi) = \frac{2}{pe^{\xi} - qe^{-\xi}}$$

$$\sec_{pq}(\xi) = \frac{2}{pe^{i\xi} + qe^{-i\xi}}, \quad \csc_{pq}(\xi) = \frac{2i}{pe^{i\xi} - qe^{-i\xi}}$$

$$\tanh_{pq}(\xi) = \frac{pe^{\xi} - qe^{-\xi}}{pe^{\xi} + qe^{-\xi}}, \quad \coth_{pq}(\xi) = \frac{pe^{\xi} + qe^{-\xi}}{pe^{\xi} - qe^{-\xi}}$$

$$\tan_{pq}(\xi) = -i\frac{pe^{i\xi} - qe^{-i\xi}}{pe^{i\xi} + qe^{-i\xi}}, \quad \cot_{pq}(\xi) = i\frac{pe^{i\xi} + qe^{-i\xi}}{pe^{i\xi} - qe^{-i\xi}}$$

are special generalized hyperbolic and trigonometric functions. Details of them can be found in [54-58].

We begin the process by determining N with the aid of the classical balance rule. Substituting (3.4) and (3.5) into (3.3) and adjusting all the coefficients of powers F to be zero, then the system of algebraic equations is gained. By solving this system, the constants c_i and φ are found. Then, substituting these constants with the solutions of (3.5), we can acquire the solutions of (3.1).

4. Solutions of the Space-Time Fractional Combined KdV-mKdV Equation

In this part of the study, the Sardar sub-equation approach is applied to find the solutions of equation (1.1). Substituting the wave transformation (3.2) into equation (1.1), we have

$$-k\frac{\partial u}{\partial \xi}+mlu\frac{\partial u}{\partial \xi}+nlu^2\frac{\partial u}{\partial \xi}+sl^3\frac{d^3u}{d\xi^3}=0 \tag{4.1}$$

According to the suggested approach, by balancing to the highest order of linear and nonlinear terms in equation (4.1), we acquire $N=1$. Thus, the solution form of the equation (4.1) is expressed as

$$u(\xi)=c_0+c_1F(\xi) \tag{4.2}$$

Differentiating the above solution three times and using equation (3.5), the required derivatives become

$$u'=c_1F'$$

$$u'''=(c_1\varphi+6c_1F^2)F'$$

After that, substituting these derivatives into equation (4.1) and adjusting all the coefficients of F to zero, a set of algebraic equations is gained as follows:

$$-kc_1+mlc_0c_1+nlc_0^2c_1+sl^3\varphi c_1=0$$

$$mlc_1^2+2nlc_0c_1^2=0$$

$$nlc_1^3+6sl^3c_1=0$$

Solving this algebraic equation system, we get the following values:

$$\varphi=\frac{4kn+lm^2}{4l^3ns},\qquad c_0=-\frac{m}{2n},\qquad c_1=\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}$$

Substituting these values and the function F in equation (4.2), the following solutions are found:

Case I. When $\frac{4kn+lm^2}{4l^3ns}>0$ and $\mu=0$, then

$$u_1^{\pm}(\xi)=-\frac{m}{2n}\pm\sqrt{-pq\frac{4kn+lm^2}{4l^3ns}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\mathrm{sech}_{pq}\left(\sqrt{\frac{4kn+lm^2}{4l^3ns}}\xi\right)$$

$$u_2^{\pm}(\xi)=-\frac{m}{2n}\pm\sqrt{pq\frac{4kn+lm^2}{4l^3ns}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\mathrm{csch}_{pq}\left(\sqrt{\frac{4kn+lm^2}{4l^3ns}}\xi\right)$$

Case II. When $\frac{4kn+lm^2}{4l^3ns}<0$ and $\mu=0$, then

$$u_3^{\pm}(\xi) = -\frac{m}{2n} \pm \sqrt{-pq\frac{4kn+lm^2}{4l^3ns}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\operatorname{sec}_{pq}\left(\sqrt{-\frac{4kn+lm^2}{4l^3ns}}\,\xi\right)$$

$$u_4^{\pm}(\xi) = -\frac{m}{2n} \pm \sqrt{-pq\frac{4kn+lm^2}{4l^3ns}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\operatorname{csc}_{pq}\left(\sqrt{-\frac{4kn+lm^2}{4l^3ns}}\,\xi\right)$$

Case III. When $\frac{4kn+lm^2}{4l^3ns} < 0$ and $\mu = \frac{1}{4}\left(\frac{4kn+lm^2}{4l^3ns}\right)^2$, then

$$u_5^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\tanh_{pq}\left(\frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\xi\right)$$

$$u_6^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\coth_{pq}\left(\frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\xi\right)$$

$$u_7^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\left(\tanh_{pq}\left(\sqrt{2}\sqrt{-\frac{4kn+lm^2}{4l^3ns}}\,\xi\right) \pm \sqrt{-pq}\operatorname{sech}_{pq}\left(\sqrt{2}\sqrt{-\frac{4kn+lm^2}{4l^3ns}}\,\xi\right)\right)$$

$$u_8^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\left(\coth_{pq}\left(\sqrt{2}\sqrt{-\frac{4kn+lm^2}{4l^3ns}}\,\xi\right) \pm \sqrt{pq}\operatorname{csch}_{pq}\left(\sqrt{2}\sqrt{-\frac{4kn+lm^2}{4l^3ns}}\,\xi\right)\right)$$

$$u_9^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{2\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\left(\tanh_{pq}\left(\frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{2\sqrt{2}}\xi\right) + \coth_{pq}\left(\frac{\sqrt{-\frac{4kn+lm^2}{4l^3ns}}}{2\sqrt{2}}\xi\right)\right)$$

Case IV. When $\frac{4kn+lm^2}{4l^3ns} > 0$ and $\mu = \frac{1}{4}\left(\frac{4kn+lm^2}{4l^3ns}\right)^2$, then

$$u_{10}^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\tan_{pq}\left(\frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\xi\right)$$

$$u_{11}^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\cot_{pq}\left(\frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\xi\right)$$

$$u_{12}^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\left(\tan_{pq}\left(\sqrt{2}\sqrt{\frac{4kn+lm^2}{4l^3ns}}\,\xi\right) \pm \sqrt{pq}\operatorname{sec}_{pq}\left(\sqrt{2}\sqrt{\frac{4kn+lm^2}{4l^3ns}}\,\xi\right)\right)$$

$$u_{13}^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\left(\cot_{pq}\left(\sqrt{2}\sqrt{\frac{4kn+lm^2}{4l^3ns}}\xi\right) \pm \sqrt{pq}\csc_{pq}\left(\sqrt{2}\sqrt{\frac{4kn+lm^2}{4l^3ns}}\xi\right)\right)$$

$$u_{14}^{\pm}(\xi) = -\frac{m}{2n} \pm \frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{2\sqrt{2}}\left(\pm\frac{\sqrt{6}\,l\sqrt{-s}}{\sqrt{n}}\right)\left(\tan_{pq}\left(\frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{2\sqrt{2}}\xi\right) + \cot_{pq}\left(\frac{\sqrt{\frac{4kn+lm^2}{4l^3ns}}}{2\sqrt{2}}\xi\right)\right)$$

5. Graphical Representation

In this part of the paper, suitable values for unknown parameters in some solutions of (1.1) have been demonstrated by 2D and 3D graphics. From these graphics, the physical meaning of the obtained solutions is understood, and the behavior of the solitons is observed. Besides, all graphics are illustrated with the help of the Mathematica software.

In Figure 1, the solution u_3^- is illustrated for the values of $k = 0.6, l = 0.5, m = -4, n = 1, s = -2, p = q = 1.2$, $\alpha = 0.2, \beta = 0.4$ within interval $0 \le x \le 60$ and $1 \le t \le 5$. The same solution is also demonstrated with a 2D plot for $0 \le x \le 60$ and $t = 4$ in Figure 2. This figure shows that the wave frequency increases, and the wave width decreases as *x* approaches zero.

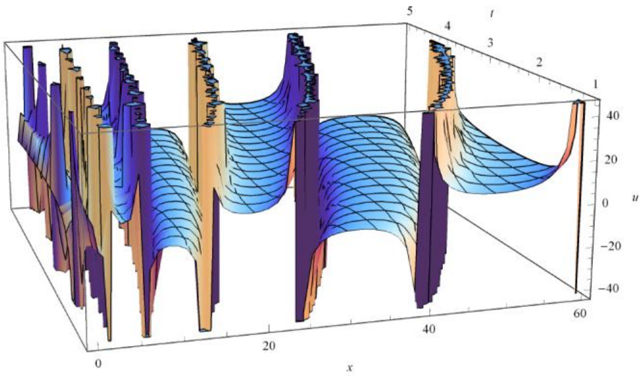

Figure 1. The 3D plot of $u_3^-(x,t)$

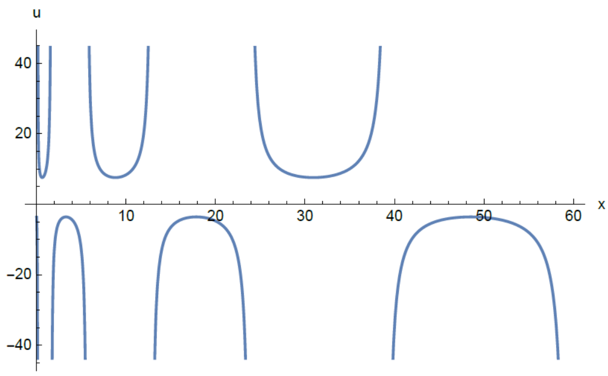

Figure 2. The 2D plot of $u_3^-(x,t)$ for $t = 4$

In Figure 3, the solution u_5^+ represents the kink soliton solution for the values of $k = 1, l = 0.6, m = 0.1, n = 0.3$, $s = -0.5$, $p = 1.5$, $q = 1.2$, $\alpha = 0.3$, $\beta = 0.5$ within interval $0 \le x \le 30$ and $1 \le t \le 4$. The same solution is also illustrated with a 2D plot for $0 \le x \le 30$ and $t = 3$ in Figure 4. Kink soliton, known as a topological soliton, has a permanent shape. Besides, the amplitude of this soliton is constant, independent of its velocity, and remains the same for zero velocity. When the kink soliton has a different screw direction, it is called the antikink soliton. Details of the kink soliton and its properties can be seen in [59].

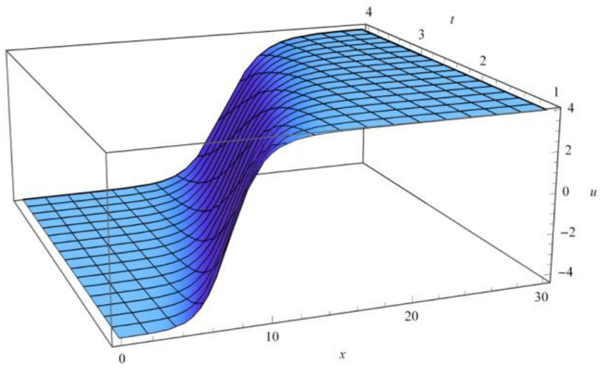

Figure 3. The 3D plot of $u_5^+(x,t)$

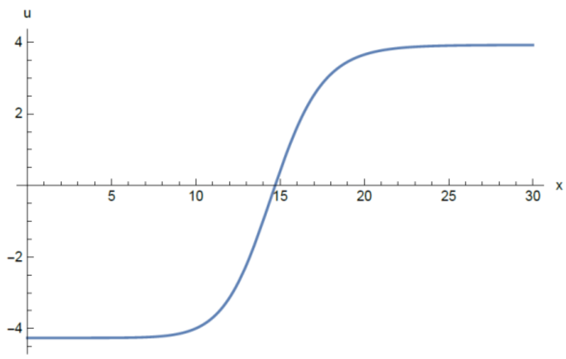

Figure 4. The 2D plot of $u_5^+(x,t)$, for $t = 3$

In Figure 5, $u_2^+(x,t)$ is demonstrated for the values of $k = -1, l = 0.7, m = 0.1, n = 0.5, s = -0.5, p = 1.1$, $q = 1.2$ $\alpha = 0.2$, $\beta = 0.3$ within interval $0 \leq x \leq 10$ and $1 \leq t \leq 5$. This figure shows that the wave loses energy as time increases and reaches a stationary state. Besides, the solution $u_2^+(x,t)$ is also illustrated with 2D plot for $0 \leq x \leq 10$ and $t = 1$ in Figure 6.

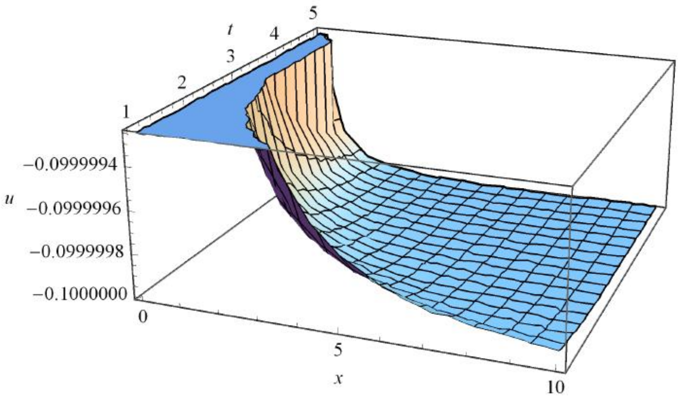

Figure 5. The 3D plot of $u_2^+(x,t)$

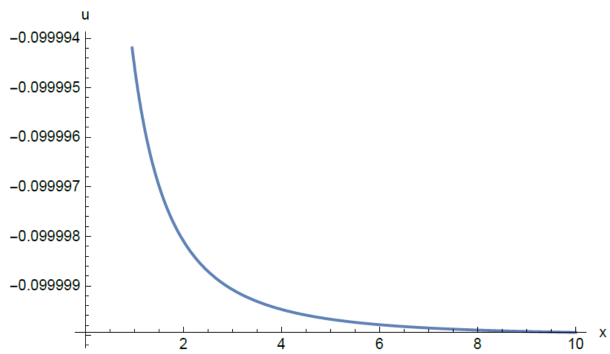

Figure 6. The 2D plot of $u_2^+(x,t)$ for $t = 1$

6. Conclusion

This study obtains the analytical solutions of the space-time fractional combined KdV-mKdV equation (1.1) using the Sardar sub-equation approach. The fractional derivatives are examined in a conformable sense. Applying the suggested approach, the fractional partial differential equation is covered to the ordinary differential equation with the help of wave transformation. The obtained solutions are expressed by special generalized trigonometric and hyperbolic function forms. The proposed approach also gains the different types of soliton solutions. It can be emphasized that all solutions obtained in this study satisfy the presented equation with the help of Mathematica software. Besides selecting suitable parameter values, some are demonstrated with 2D and 3D graphs to analyze physically. Moreover, when compared with the sub-equation [37] and the mapping [38] methods used to find solutions of (1.1), it can be observed that the Sardar sub-equation approach provides many different types of solutions. Five solutions, which are rational and hyperbolic, are found by the sub-equation method, and five hyperbolic solutions are found by the mapping method. However, 28 solutions are gained by utilizing the suggested approach. Furthermore, there is no need to use linearization, perturbation, and initial and boundary conditions with the Sardar sub-equation approach. From all the findings, it can be concluded that the approach presented is convincing, reliable, and satisfying. As a result, lots of solitons can be generated by this approach and can be effectively applied for nonlinear differential equations modeling various natural phenomena. For future studies, the suggested approach can be utilized as an alternative to getting the analytical solutions of many different types of nonlinear integer order and fractional order differential equations found in mathematics, engineering, and physics.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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