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# Uniqueness of Solution of an Inverse Problem for the Quantum Kinetic Equation

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**Abstract**— This study focuses on an inverse problem for the quantum kinetic equations, the cornerstone of quantum mechanics. These equations describe the evolution of elementary particles under strong interactions. They are fundamental to understanding the behavior of quantum systems and play a pivotal role in describing nanostructure processes and nanodiagnostics. The main target of the problem is to determine the unknown source function on the right-hand side of the equation. This paper obtains a pointwise Carleman estimate. It then uses the Carleman estimate to show the uniqueness of the problem's solution.

***Keywords*** — *Quantum kinetic equation, inverse problem, uniqueness of the solution, pointwise Carleman estimate*

**Mathematics Subject Classification (2020)** 35Q40, 35R45

## 1. Introduction

Quantum kinetic equations play a pivotal role in understanding the fundamental dynamics of particles and systems under quantum mechanical frameworks. These equations, derived from quantum mechanics principles, provide detailed descriptions of non-equilibrium processes and the transport properties of particles. Quantum kinetic equations are foundational in exploring inverse problems, which aim to determine the cause from observed effects, especially in applications involving nanostructures and nanotechnology [1, 2]. Their mathematical framework has proven essential for investigating stability and uniqueness in quantum systems, allowing for precise control and monitoring in applied physics.

Furthermore, applying quantum kinetic equations extends to modern challenges, such as renewable energy and information technology [3]. In high-energy physics and astrophysics, quantum kinetic theory offers insights into phenomena such as chiral transport and spin polarization, which are crucial for understanding the dynamics of quark-gluon plasmas in relativistic heavy-ion collisions. The development of such models provides theoretical insights and aids in practical applications like material science and energy systems [4, 5]. Recent studies highlight the importance of quantum kinetic equations in nanotechnology, including applications in modeling nanostructure dynamics, development of nanoscale diagnostic tools, and addressing inverse problems to design and control nanodevices.

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In this study, we deal with the quantum kinetic equation

$$\begin{aligned}\frac{i}{(2\pi)^n}\int\limits_{\mathbb{R}^{2n}}[\phi(x-y)-\phi(x+y)]\exp[i(p-p')]\omega(x,p',t)dp'dy = \partial_t\omega(x,p,t)-\mu(t)\lambda(x,p)\\ +\sum_{j=1}^{n}p_j\partial_{x_j}\omega(x,p,t)+\sigma(x)\omega(x,p,t)\end{aligned} \tag{1.1}$$

with the following conditions

$$\omega(x,p,t)|_{x_1\leq 0}=\omega_1(x,p,t) \tag{1.2}$$

and

$$\omega(x,p,0)=\omega_0(x,p) \tag{1.3}$$

in the domain $Q=\{(x,p,t)\ :\ x_1>0,\ x\in\Omega,\ p\in\mathbb{R}^n,\ t\in\mathbb{R}\}$. Here, $\Omega\subset\mathbb{R}^n$, $x=(x_1,x)\in\mathbb{R}^n$, $x=(x_2,...,x_n)\in\mathbb{R}^{n-1}$. In this paper, the following notations are used: For all $1\leq j\leq n$,

$$\partial_t\omega=\frac{\partial\omega}{\partial t},\quad \partial_{x_j}\omega=\frac{\partial\omega}{\partial x_j},\quad \partial_{x_j}\partial_{y_j}\omega=\frac{\partial^2\omega}{\partial x_j\partial y_j},\quad \partial^2_{\eta_j}\omega=\frac{\partial^2\omega}{\partial\eta_j^2},\quad \text{and}\quad \Delta_\eta\omega=\sum_{j=1}^{n}\partial^2_{\eta_j}\omega$$

In physical applications, $\omega(x,p,t)$ is the quantum distribution function, $\phi$ is the mean potential, $\sigma(x)$ is the absorption, and $\lambda(x,p)$ is the unknown function. The functions $\phi(x)$ and $\sigma(x)$ satisfy $\left|D_x^\beta\phi(x)\right|<M$ and $\left|D_x^\beta\sigma(x)\right|<M$ where $0\leq\beta\leq 2$.

By applying the Fourier transform to (1.1) with respect to $p$,

$$\partial_t\hat{\omega}(x,y,t)+i\sum_{j=1}^{n}\partial_{x_j}\partial_{y_j}\hat{\omega}(x,y,t)+\sigma(x)\hat{\omega}(x,y,t)-i[\phi(x-y)-\phi(x+y)]\hat{\omega}(x,y,t)=\mu(t)\hat{\lambda}(x,y) \tag{1.4}$$

where $i=\sqrt{-1}$ is the parameter of the Fourier transform. From (1.2) and (1.3),

$$\hat{\omega}(x,y,t)|_{x_1\leq 0}=\hat{\omega}_1(x,y,t) \tag{1.5}$$

and

$$\hat{\omega}(x,y,0)=\hat{\omega}_0(x,y) \tag{1.6}$$

Changing the variables as $x-y=\zeta$ and $x+y=\tau$ and introducing the function $w(\zeta,\tau,t)=\hat{\omega}(\frac{\zeta+\tau}{2},\frac{\tau-\zeta}{2},t)$, $\sigma(\zeta,\tau)=\sigma(\frac{\zeta+\tau}{2})$, and $g(\zeta,\tau)=\hat{\lambda}(\frac{\zeta+\tau}{2},\frac{\tau-\zeta}{2})$ in (1.4),

$$\partial_t w(\zeta,\tau,t)+i(\Delta_\tau-\Delta_\zeta)w(\zeta,\tau,t)+i[\phi(\zeta)-\phi(\tau)w(\zeta,\tau,t)]+\sigma(\zeta,\tau)w(\zeta,\tau,t)=\mu(t)g(\zeta,\tau) \tag{1.7}$$

By (1.5) and (1.6),

$$w(\zeta,\tau,t)|_{\zeta_1+\tau_1\leq 0}=w_1\quad\text{and}\quad w(\zeta,\tau)=w_0$$

Consider the set

$$W=\left\{w\in C^2(Q)\cap H^2(Q)\ :\ w=0\text{ for }x_1-y_1\geq\zeta_0\right\}$$

and assume that the Fourier transform of $w$ concerning $t$ is finite. We suppose that $f\in C^2(\mathbb{R})\cap H^2(\mathbb{R})$. (1.7) is known as an ultrahyperbolic Schrödinger equation. To obtain a pointwise Carleman estimate, we consider the following equation

$$\partial_t w(\zeta,\tau,t)+i(\Delta_\tau-b^{-2}\Delta_\zeta)w(\zeta,\tau,t)+i[\phi(\tau)-\phi(\zeta)]w(\zeta,\tau,t)+\sigma(\zeta,\tau)w(\zeta,\tau,t)=\mu(t)g(\zeta,\tau) \tag{1.8}$$

where $b\in C^1(\overline{\Omega})$ and $b>0$. By $w=0$, for $x_1-y_1\geq\zeta_0$,

$$w(\zeta,\zeta_0,'\ \tau,t)=\partial_{\tau_1}w(\zeta,\zeta_0,'\ \tau,t)=0 \tag{1.9}$$

and

$$w(\zeta,\tau,0)=w(\zeta,\tau) \tag{1.10}$$

where $\tau=(\tau_1,...,\tau_2)$.

We consider an inverse problem of determining $w(\zeta,\tau,t)$ and $g(\zeta,\tau)$ in (1.8) from the Cauchy data (1.9) and additional data (1.10). We study the uniqueness of the solution of the inverse problem. Using (1.8) and the condition $w(\zeta,\tau,0)=w_0(\zeta,\tau)=0$,

$$g(\zeta,\tau)=\frac{i}{\mu(0)}\int\limits_{\mathbb{R}}\xi\hat{w}(\zeta,\tau,\xi)d\xi$$

and

$$\partial_t w(\zeta,\tau,t)+i(\Delta_\tau-b^{-2}\Delta_\zeta)w(\zeta,\tau,t)+i[\phi(\tau)-\phi(\zeta)]w(\zeta,\tau,t)+\sigma(\zeta,\tau)w(\zeta,\tau,t)=\frac{i\mu(t)}{\mu(0)}\int\limits_{\mathbb{R}}\xi\hat{w}(\zeta,\tau,\xi)d\xi$$

The solvability of some inverse problems for the quantum kinetic equations was considered in [1,2,6–9]. Anikonov [1] discussed multidimensional inverse and ill-posed problems for differential equations, including hyperbolic, parabolic, and quantum kinetic equations. The existence, uniqueness, and stability of solutions to these problems are investigated. Moreover, some methods of constructing a solution are given. Anikonov and Neshchadim [2] obtained an inequality for the quantum kinetic equation, and based on this identity, the uniqueness of the inverse problem of determining the solution and unknown right-hand side was proved by using the boundary and initial data. Anikonov [7] considered inverse problems in kinetic theory, particularly those associated with integral geometry and transport equations. The author presents new methods for solving inverse problems for kinetic equations. Different types of inverse problems for elliptic, hyperbolic, parabolic, and fractional parabolic equations were considered in [10–14]. Moreover, for ultrahyperbolic equations, see [8,15,16] and for ultrahyperbolic Schödinger equations, see [17–19]. Besides, some recent works of inverse problems involving kinetic equations can be seen in [20,21] for other types of equations. In [20], an inverse source problem for the kinetic equation was studied in an unbounded domain with Cauchy data. The uniqueness of the solution was proved by means of a pointwise Carleman estimate. In [21], an inverse source problem for the kinetic equation was considered, and a numerical solution to the problem was obtained using a hybrid method composed of finite difference approximation and Lagrange's polynomial interpolation.

The remaining organization of this paper is as follows: Section 2 presents a pointwise Carleman estimate. Section 3 is devoted to the proof of Theorem 3.1. The last section discusses the need for further research.

## 2. A Pointwise Carleman Estimate

Carleman estimates were first introduced by Carleman in 1939 to prove the uniqueness of ill-posed Cauchy problems. This method has been applied to inverse problems for partial differential equations since 1981. The works of Bukhgeim and Klibanov demonstrated that Carleman estimates are a powerful tool for providing global uniqueness of multidimensional inverse problems. Additionally, this tool has been effective in obtaining Hölder and Lipschitz stability estimates as well as in developing numerical methods [22,23].

This section presents a Carleman estimate for the quantum kinetic equation used for the proof of

Theorem 3.1. Let

$$A_0 w_k = \Delta_\tau w_k + b^{-2} |s|^2 w_k, \quad k \in \{1,2\}$$

We introduce the function

$$\psi(\tau) = \delta\tau_1 + \frac{1}{2}\sum_{j=2}^{n}(\tau_j - \tau_j^0)^2 + \alpha_0, \quad \alpha_0 > 0$$

and the set

$$Q_0 = \left\{ (s,\tau) \ : \ s \in \mathbb{R}^n, \quad \tau \in \mathbb{R}^n, \quad \tau_1 > 0, \quad 0 < \delta\tau_1 < \kappa - \sum_{j=2}^{n}(\tau_j - \tau_j^0)^2 \right\}$$

Here, the parameters $\delta$, $\tau$, and $v$ are positive numbers, $0 < \kappa < 1$, $\delta > 1$, $\tau_0 = (\tau_1^0, ..., \tau_n^0)$, and $\kappa + \alpha_0 = m < 1$. We define a weight function

$$\varphi = e^{\gamma\psi^{-\nu}}$$

We have $\tau_0 \in Q_0$ and $Q_0 \subset Q$, for sufficiently small $\kappa > 0$. The proof of Theorem 3.1 is based on the following proposition, proved in [20] for a different equation with a similar principle part.

**Proposition 2.1.** Let $b \in C^1(\overline{\Omega})$, $b > 0$, $\partial_{\tau_1} b > 0$ on $\overline{\Omega}$, and $\delta > \delta_0$. Then, the following Carleman estimate is valid for all $w_k \in C^2(Q)$:

$$\begin{aligned} \psi^{\nu+1}(A_0 w_k)^2\varphi^2 - 2n\gamma v w_k (A_0 w_k)\varphi^2 \geq {} & 2\gamma^3 v^3 \psi^{-2\nu-2} w_k^2 \varphi^2 + 2\gamma v b^{-2} |s|^2 w_k^2 \varphi^2 \\ & + 2\gamma v |\nabla_\tau w_k|^2 \varphi^2 - 2n\gamma v d_1(w_k) + d_2(w_k) \end{aligned} \tag{2.1}$$

where $\gamma$ and $\nu$ are large parameters,

$$d_1(w_k) = \sum_{j=1}^{n} \partial_{\tau_j}\left( \left( w_k \partial_{\tau_j} w_k + \psi^{-\nu-1}\gamma v \partial_{\tau_j}\psi w_k^2 \right)\varphi^2 \right) \quad \text{and} \quad d_2(w_k) = \sum_{j=1}^{5} d_{2j}(w_k)$$

such that

$$\begin{aligned} d_{21}(w_k) = {} & 4\gamma\nu \sum_{j,r=1}^{n} \partial_{\tau_r}\left( \partial_{\tau_j}\psi \left( \partial_{\tau_j} w_k - \gamma\nu\psi^{-\nu-1}\partial_{\tau_j}\psi w_k \right) \left( \partial_{\tau_r} w_k - \gamma\nu\psi^{-\nu-1}\partial_{\tau_r}\psi w_k \right)\varphi^2 \right) \\ = {} & -2\gamma\nu \sum_{j,r=1}^{n} \partial_{\tau_j}\left( \partial_{\tau_j}\psi \left( \partial_{\tau_r} w_k - \gamma\nu\psi^{-\nu-1}\partial_{\tau_r}\psi w_k \right)^2 \varphi^2 \right) + 2\gamma^2\nu^2\psi^{-\nu-1}(n-1)\sum_{j=1}^{n}\partial_{\tau_j}\left( \partial_{\tau_j}\psi w_k^2 \varphi^2 \right) \end{aligned}$$

$$d_{22}(w_k) = 2\gamma^3 v^3 \psi^{-2\nu-2}\partial_{\tau_j}\left( |\nabla_\tau \psi|^2 w_k^2 \varphi^2 \sum_{j=1}^{n} \partial_{\tau_j}\psi \right)$$

$$d_{23}(w_k) = -2\gamma^2\nu^2 (v+1)\psi^{-\nu-2}\sum_{j=1}^{n}\partial_{\tau_j}\left( \partial_{\tau_j}\psi |\nabla_\tau\psi|^2 w_k^2 \varphi^2 \right)$$

$$d_{24}(w_k) = 2\gamma^2\nu^2 (n-1)\psi^{-\nu-1}\sum_{j=1}^{n}\partial_{\tau_j}\left( \partial_{\tau_j}\psi w_k^2 \varphi^2 \right)$$

and

$$d_{25}(w_k) = 2\gamma\nu |s|^2 \sum_{j=1}^{n}\partial_{\tau_j}\left( \partial_{\tau_j}\psi b^{-2}\varphi^2 w_k^2 \right)$$

## 3. Main Results

This section proposes a theorem regarding the uniqueness of the solution to the considered inverse problem.

**Theorem 3.1.** There exists at most one solution $(w, g) \in W \times L^1(\mathbb{R}^{2n})$ of the inverse problem provided that $\partial_{\tau_1} b > 0$ and $\mu(0) \neq 0$.

Applying the Fourier transform with respect to $(\zeta, t)$ to (1.8)-(1.10),

$$-\xi\hat{w} - \Delta_\tau \hat{w} - b^{-2}|s|^2 \hat{w} - \hat{\phi}(\tau)\hat{w} - \hat{\phi}(\zeta) * \hat{w} + \hat{\sigma}(\zeta,\tau) * \hat{w} = -\frac{\hat{\mu}(\xi)}{\mu(0)} \int_{\mathbb{R}} \xi \hat{w}(s,\tau,\xi) d\xi \tag{3.1}$$

and

$$\hat{w}(s, 0,' \tau, \xi) = \partial_{\tau_1} \hat{w}(s, 0,' \tau, \xi) = 0 \tag{3.2}$$

where $*$ is the convolution with respect to $\xi$.

Proof. In order to prove Theorem 3.1, we employ the Fredholm alternative; therefore, we treat the homogeneous version of the problem. We demonstrate that this problem has only zero solution, which confirms the uniqueness. We write $\hat{w} = w_1 + iw_2$ and $\mu = \mu_1 + i\mu_2$ in (3.1), and we get the following system of equations:

$$\Delta_\tau w_k + b^{-2}|s|^2 w_k = \rho_k, \quad k \in \{1, 2\} \tag{3.3}$$

where

$$\rho_1 = \frac{1}{\mu(0)} \left( \mu_1 \int_{\mathbb{R}} \xi w_1 \, d\xi - \mu_2 \int_{\mathbb{R}} \xi w_2 \, d\xi \right) - \xi w_1 - \mathrm{Re}(\hat{\phi}(\tau) w) - \mathrm{Re}(\hat{\phi}(\zeta) * w) - \mathrm{Re}(\hat{\sigma}(\zeta,\tau) * w)$$

and

$$\rho_2 = \frac{1}{\mu(0)} \left( \mu_1 \int_{\mathbb{R}} \xi w_2 \, d\xi + \mu_2 \int_{\mathbb{R}} \xi w_1 \, d\xi \right) - \xi w_2 - \mathrm{Im}(\hat{\phi}(\tau) w) - \mathrm{Im}(\hat{\phi}(\zeta) * w) - \mathrm{Im}(\hat{\sigma}(\zeta,\tau) * w)$$

From (3.2),

$$w_1(s, 0, \tau, \xi) = 0, \quad \partial_{\tau_1}(w_1)(s, 0, \tau, \xi) = 0, \quad w_2(s, 0, \tau, \xi) = 0, \quad \text{and} \quad \partial_{\tau_1}(w_2)(s, 0, \tau, \xi) = 0$$

By (3.3),

$$\begin{aligned}(\rho_1)^2 + (\rho_2)^2 \leq\ & 6\frac{(\mu_1^2 + \mu_2^2)}{\mu^2(0)} C_1 \int_{\mathbb{R}} (1 + \xi^2)^2 (w_1^2 + w_2^2) d\xi + 6\xi^2 (w_1^2 + w_2^2) \\ & + 6\left|\hat{\phi}(\tau) w\right|^2 + 6\left|\hat{\phi}(\zeta) * w\right|^2 + 6\left|\hat{\sigma}(\zeta,\tau) * w\right|^2\end{aligned} \tag{3.4}$$

where

$$\left( \int_{\mathbb{R}} \xi w_k d\xi \right)^2 \leq C_1 \int_{\mathbb{R}} (1 + \xi^2)^2 w_k^2 d\xi \quad \text{and} \quad C_1 = \int_{\mathbb{R}} (1 + \xi^2)^{-1} d\xi$$

By (2.1),

$$\begin{aligned}((\rho_k)^2 + \gamma^2 \nu^2 n^2 w_k^2)\varphi^2 + \psi^{\nu+1}(A_0 w_k)^2 \varphi^2 &\geq -2\gamma\nu n w_k (A_0 w_k)\varphi^2 + \psi^{\nu+1}(A_0 w_k)^2 \varphi^2 \\ &\geq 2\gamma^3 \nu^3 \psi^{-2\nu-2} w_k^2 \varphi^2 + 2\gamma\nu b^{-2}|s|^2 w_k^2 \varphi^2 \\ &\quad + 2\gamma\nu \left|\nabla_\tau w_k\right|^2 \varphi^2 - 2\gamma\nu n d_1(w_k) + d_2(w_k)\end{aligned} \tag{3.5}$$

for $k \in \{1,2\}$. From (3.4) and (3.5) and the equalities $w_1^2 + w_2^2 = |\hat{w}|^2$ and $\mu_1^2 + \mu_2^2 = |\mu|^2$,

$$\begin{aligned} &2\gamma^3\nu^3\psi^{-2\nu-2}|\hat{w}|^2\varphi^2 + 2\gamma\nu b^{-2}|s|^2|\hat{w}|^2\varphi^2 + 2\gamma\nu|\nabla_\tau\hat{w}|^2\varphi^2 + \sum_{k=1}^{2}(d_2(w_k) - 2\gamma\nu n d_1(w_k)) \\ &\leq \varphi^2\left(6\tfrac{|\mu|^2}{\mu^2(0)}C_1\int_{\mathbb{R}}(1+\xi^2)^2|\hat{w}|^2d\xi + 6\xi^2|\hat{w}|^2 + 6\left|\hat{\phi}(\tau)\hat{w}\right|^2 + 6\left|\hat{\phi}(\zeta)*\hat{w}\right|^2 + 6|\hat{\sigma}(\zeta,\tau)*\hat{w}|^2\right) \end{aligned} \tag{3.6}$$

Multiplying (3.6) by $(1+\xi^2)^2$ and integrating with respect to $\xi$ over $\mathbb{R}$,

$$\begin{aligned} &\int_{\mathbb{R}}\varphi^2(|\hat{w}|^2(2\gamma^3\nu^3\psi^{-2\nu-2} + 2\gamma\nu b^{-2}|s|^2) + 2\gamma\nu|\nabla_\tau\hat{w}|^2)(1+\xi^2)^2d\xi \\ &\leq (6\overline{\mu_0}C_2\varphi^2\int_{\mathbb{R}}((1+\xi^2)^2|\hat{w}|^2)d\xi + 6\varphi^2\int_{\mathbb{R}}((1+\xi^2)^2\xi^2|\hat{w}|^2)d\xi) + 6\int_{\mathbb{R}}(1+\xi^2)^2\left|\hat{\phi}(\tau)\hat{w}\right|^2d\xi \\ &+6\int_{\mathbb{R}}(1+\xi^2)^2\left|\hat{\phi}(\zeta)*\hat{w}\right|^2d\xi + 6\int_{\mathbb{R}}(1+\xi^2)^2|\hat{\sigma}(\zeta,\tau)*\hat{w}|^2)d\xi - \sum_{k=1}^{2}\int_{\mathbb{R}}(1+\xi^2)^2(d_2(w_k) - 2\gamma\nu n d_1(w_k))d\xi \end{aligned} \tag{3.7}$$

where $\overline{\mu_0} = \max\limits_{\tau\in\overline{\Omega}}\left\{\frac{C_1}{\mu^2(0)}\right\}$ and $C_2 = \int_{\mathbb{R}}(1+\xi^2)^2|\mu|^2d\xi$. In (3.7), the big parameter $\gamma$ can be chosen as the first five terms on the right-hand side can be absorbed by the terms on the left-hand side. Thus,

$$\gamma^3\nu^3\int_{\mathbb{R}}|\hat{w}|^2\varphi^2d\xi \leq -div(\hat{w})$$

where

$$div(\hat{w}) = \sum_{k=1}^{2}\int_{\mathbb{R}}(1+\xi^2)^2(d_2(w_k) - 2\gamma\nu n d_1(w_k))d\xi$$

Since $\varphi^2 > 1$ on $Q_0$, then

$$\int_{\mathbb{R}}|\hat{w}|^2d\xi \leq \int_{\mathbb{R}}|\hat{w}|^2\varphi^2d\xi \leq -\frac{1}{\gamma^3\nu^3}div(\hat{w}) \tag{3.8}$$

Integrating (3.8) over $Q_0$,

$$\int_{Q_0}\int_{\mathbb{R}}|\hat{w}|^2d\xi ds d\tau \leq 0$$

as $\gamma \to \infty$, which implies that $\hat{w} = 0$. Hence, it can be observed that $w = 0$. Consequently, from (1.8), we have $g = 0$ and this completes the proof of Theorem 3.1. $\square$

## 4. Conclusion

In this work, we prove the uniqueness of the solution to the inverse problem of determining the source function in quantum kinetic equations. The key tool is a Carleman estimate, an essential inequality in proving the solvability of Cauchy and inverse problems for partial differential equations. Quantum kinetic equations remain a cornerstone of mathematical physics, bridging the gap between theoretical constructions and real-world applications in advanced technology and energy solutions. Therefore, it is important to investigate the solutions of inverse problems for quantum kinetic equations that occur in various areas of science and engineering.

In future studies, the well-posedness of the inverse problems for quantum kinetic equations under different potential functions and boundary conditions can be investigated, thus developing methods applicable to more general physical systems. Further research can explore various theoretical, computational, and experimental directions to enhance understanding of these problems. The following studies can be carried out on numerical solutions to inverse problems, and the theoretical knowledge obtained can be applied to different differential operators and systems with variable coefficients. In addition, the findings of this study can be further substantiated by presenting current life problems and graphical representations.

## Author Contributions

The author read and approved the final version of the paper.

## Conflicts of Interest

The author declares no conflict of interest.

## Ethical Review and Approval

No approval from the Board of Ethics is required.

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