Turk. J. Math. Comput. Sci.
17(1)(2025) 17–25

DOI : 10.47000/tjmcs.1620260

The Revisited $q-$Derangement Polynomials and Incomplete $q-$Derangement Polynomials

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Received: 15-01-2025 • Accepted: 14-04-2025

Abstract. In this paper, the $q-$derangement polynomials of the second type $\mathfrak{d}_{\psi,q}(\gamma)$ and their generating function are defined. Subsequently, the incomplete $q-$derangement polynomials $d_{\psi,q}(\gamma, m)$ and numbers $d_{\psi,q}(m)$ are introduced and various properties are derived. Furthermore, the relationship between the incomplete $q-$derangement numbers and $q-$derangement numbers is demonstrated.

2020 AMS Classification: 05A19, 11B65, 11B83, 11B37

Keywords: Derangement polynomials, $q-$analogue, recurrence relation.

1. Introduction

Recently, $q-$analogues have become a focal point of interest for many mathematicians and are extensively used in the generalization of classical mathematical expressions [3, 13, 18]. In particular, the $q-$analogues of the derivatives, integrals and polynomials frequently emerge not only in mathematics but also in application areas such as physics and engineering.

In this section, the fundamental $q-$notations and properties utilized are first presented for $q \in \mathbb{C}$ and $0 < |q| < 1$.

The $q-$integer is defined by

$$[\psi]_q = \frac{q^{\psi} - 1}{q - 1}$$

for any positive integer ψ [1, 13]. The $q-$analogue of $\psi!$ is

$$[\psi]_q! = \begin{cases} 1 & \text{if } \psi = 0, \\ [\psi]_q [\psi - 1]_q [\psi - 2]_q \cdots [1]_q & \text{if } \psi = 1, 2, \dots . \end{cases}$$

The definition of the $q-$binomial coefficient is

$$\binom{\psi}{\omega}_q = \frac{[\psi]_q!}{[\psi - \omega]_q! \, [\omega]_q!} \quad \omega = 0, 1, 2, \dots, \psi$$

with $\binom{\psi}{0}_q = 1$ and $\binom{\psi}{\omega}_q = 0$ for $\psi < \omega$, [1, 13].

The $q-$binomial coefficients have the following properties:

$$\binom{\psi}{\omega}_q = \binom{\psi}{\psi - \omega}_q,$$

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$$\binom{\psi}{\omega}_q\binom{\omega}{\xi}_q=\binom{\psi}{\omega}_q\binom{\psi-\xi}{\omega-\xi}_q$$

and

$$\binom{\psi+1}{\omega}_q=q^{\omega}\binom{\psi}{\omega}_q+\binom{\psi}{\omega-1}_q.$$

In [13], Gauss's binomial formula is given by:

$$(\gamma+a)_q^{\psi}=\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q q^{\binom{\omega}{2}}a^{\omega}\gamma^{\psi-\omega}$$

for $\psi\geq 1$. If $\beta\gamma=q\gamma\beta$, where q is a number that commutes with both γ and β, then we have:

$$(\gamma+\beta)_q^{\psi}=\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q\gamma^{\omega}\beta^{\psi-\omega}.$$

The definitions of the $q-$exponential functions are

$$e_q(t)=\sum_{\psi=0}^{\infty}\frac{t^{\psi}}{[\psi]_q!},\qquad |t|<\frac{1}{|1-q|}\tag{1.1}$$

and

$$E_q(t)=\sum_{\psi=0}^{\infty}q^{\frac{1}{2}\psi(\psi-1)}\frac{t^{\psi}}{[\psi]_q!},\quad t\in\mathbb{C}.\tag{1.2}$$

Clearly, we see that $e_q(t)E_q(-t)=1$ from Eq. (1.1) and Eq. (1.2). Also, if $\gamma\beta=q\beta\gamma$, then $e_q(\gamma+\beta)=e_q(\beta)e_q(\gamma)$.

The $q-$derivative is given by

$$D_qf(\gamma)=\frac{f(q\gamma)-f(\gamma)}{(q-1)\gamma}\tag{1.3}$$

in [11, 13].

The Jackson integral is defined as follows:

$$\int_a^b f(\gamma)\,d_q\gamma=\int_0^b f(\gamma)\,d_q\gamma-\int_0^a f(\gamma)\,d_q\gamma,$$

where

$$\int_0^a f(\gamma)\,d_q\gamma=a(1-q)\sum_{i=0}^{\infty}q^if\left(aq^i\right)$$

for $0<a<b$ [13]. Specifically, for $\psi>-1$

$$\int_0^1\gamma^{\psi}d_q\gamma=\frac{1}{[\psi+1]_q}.\tag{1.4}$$

2. The Revisited q−Derangement Polynomials

The first topic we cover in this area of our study is the $q-$derangement numbers and polynomials [5,9,23]. Then, by reevaluating the q-derangement polynomials in [12, 14, 17, 19, 24], we derive an alternative generating function. Thus, we define the $q-$ derangement polynomials of the second kind $\mathfrak{d}_{\psi,q}(\gamma)$. We also look at a few of these numbers and polynomials properties.

The following generating function defines the $q-$derangement polynomials $d_{\psi,q}(\gamma)$.

$$\frac{1}{1-t}e_q\left((\gamma-1)\,t\right)=\sum_{\psi=0}^{\infty}d_{\psi,q}(\gamma)\frac{t^{\psi}}{[\psi]_q!}\tag{2.1}$$

for $\psi\geq 0$. When $\gamma=0$ is written in Eq. (2.1), $d_{\psi,q}(0)=d_{\psi,q}$ are the $q-$derangement numbers [17].

From Eq. (2.1), clearly, it is seen that

$$d_{\psi,q}(\gamma) = [\psi]_q! \sum_{\omega=0}^{\psi} \frac{(\gamma-1)^{\omega}}{[\omega]_q!}$$

in [17]. Also, for $\gamma = 0$ in the above equation,

$$d_{\psi,q} = [\psi]_q! \sum_{\omega=0}^{\psi} \frac{(-1)^{\omega}}{[\omega]_q!} \tag{2.2}$$

is obtained.

First few $q-$derangement numbers $d_{\psi,q}$ are as follows:

$$\begin{aligned} d_{0,q} &= 1,\ d_{1,q} = 0,\ d_{2,q} = 1,\ d_{3,q} = [3]_q - 1, \\ d_{4,q} &= [4]_q\left([3]_q - 1\right) + 1, \\ d_{5,q} &= [5]_q!\,[4]_q!\left([3]_q - 1\right) + [5]_q! - 1. \end{aligned}$$

Definition 2.1. For $\psi \geq 0$, the following generating function defines the $q-$derangement polynomials of the second kind $\mathfrak{d}_{\psi,q}(\gamma)$,

$$\frac{e_q(-t)}{1-t} e_q(\gamma t) = \sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!}. \tag{2.3}$$

The equation (2.3) shows that the $q-$derangement values $d_{\psi,q}$ obtained in equation (2.2) are present when $\gamma = 0$. Also, we take commutative variable $(\gamma - 1)$ and t in equation Eq. (2.1), we achieve Eq. (2.3).

$$\begin{aligned} \sum_{\psi=0}^{\infty} d_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} &= \frac{1}{1-t} e_q\left((\gamma - 1)t\right) \\ &= \frac{1}{1-t} e_q(\gamma t - t) \\ &= \frac{1}{1-t} e_q(-t)\, e_q(\gamma t). \end{aligned}$$

Theorem 2.2. *The $q-$derangement polynomials $\mathfrak{d}_{\psi,q}(\gamma)$ have continuing property:*

$$\mathfrak{d}_{\psi,q}(\gamma) = \sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q d_{\omega,q} \gamma^{\psi-\omega}. \tag{2.4}$$

Proof. From the Eq. (2.3), we obtain

$$\begin{aligned} \sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} &= \frac{e_q(-t)}{1-t} e_q(\gamma t) \\ &= \sum_{\psi=0}^{\infty} d_{\psi,q} \frac{t^{\psi}}{[\psi]_q!} \sum_{\psi=0}^{\infty} \gamma^{\psi} \frac{t^{\psi}}{[\psi]_q!} \\ &= \sum_{\psi=0}^{\infty} \left(\sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q d_{\omega}, q\gamma^{\psi-\omega} \right) \frac{t^{\psi}}{[\psi]_q!}. \end{aligned}$$

We thereby achieve the intended outcome. □

Theorem 2.3. *For ψ nonnegative integer,*

$$\mathfrak{d}_{\psi,q}(\gamma + \beta) = \sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q \mathfrak{d}_{\omega,q}(\gamma) \beta^{\psi-\omega},$$

where γ and β are the $q-$commuting variables.

Proof. We have

$$\begin{aligned}\sum_{\psi=0}^{\infty}\mathfrak{d}_{\psi,q}(\gamma+\beta)\frac{t^{\psi}}{[\psi]_q!} &=\frac{e_q(-t)}{1-t}e_q((\gamma+\beta)t)\\ &=\frac{e_q(-t)}{1-t}e_q(\gamma t)e_q(\beta t)\\ &=\sum_{\psi=0}^{\infty}\mathfrak{d}_{\psi,q}(\gamma)\frac{t^{\psi}}{[\psi]_q!}\sum_{\psi=0}^{\infty}\beta^{\psi}\frac{t^{\psi}}{[\psi]_q!}.\end{aligned}$$

If we apply the Cauchy product rule in this last equation and equalize the coefficients, we obtain

$$\sum_{\psi=0}^{\infty}\mathfrak{d}_{\psi,q}(\gamma+\beta)\frac{t^{\psi}}{[\psi]_q!}=\sum_{\psi=0}^{\infty}\left(\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}\mathfrak{d}_{\omega,q}(\gamma)\beta^{\psi-\omega}\right)\frac{t^{\psi}}{[\psi]_q!}.$$

□

Theorem 2.4. *The polynomials $\mathfrak{d}_{\psi,q}(\gamma)$ have the following properties.*

$$D_q\mathfrak{d}_{\psi,q}(\gamma)=[\psi]_q\,\mathfrak{d}_{\psi-1,q}(\gamma)\quad(\textit{The derivative property})\tag{2.5}$$

and

$$\int_0^1\mathfrak{d}_{\psi,q}(\gamma)d_q(\gamma)=\frac{\mathfrak{d}_{\psi+1,q}(1)-\mathfrak{d}_{\psi+1,q}(0)}{[\psi+1]_q}\quad(\textit{The integral property}).\tag{2.6}$$

Proof. Firstly, for the proof of equation (2.5), let's apply the equation (1.3) to the equation (2.4).

$$\begin{aligned}D_q\mathfrak{d}_{\psi,q}(\gamma)&=D_q\left(\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q d_{\omega,q}\gamma^{\psi-\omega}\right)\\ &=D_q\left(\sum_{\omega=0}^{\psi-1}\binom{\psi}{\omega}_q d_{\omega,q}\gamma^{\psi s}+d_{\psi,q}\right)\\ &=\sum_{\omega=0}^{\psi-1}\binom{\psi}{\omega}_q d_{\omega,q}[\psi-\omega]_q\gamma^{\psi-1-\omega}\\ &=[\psi]_q\,\mathfrak{d}_{\psi-1,q}(\gamma).\end{aligned}$$

Now, let's show that the equation (2.6) is true. When we apply Eq. (1.4) to Eq. (2.4), we get

$$\begin{aligned}\int_0^1\mathfrak{d}_{\psi,q}(\gamma)d_q(\gamma)&=\int_0^1\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q d_{\omega,q}\gamma^{\psi-\omega}d_q(\gamma)\\ &=\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q d_{\omega,q}\int_0^1\gamma^{\psi-\omega}d_q(\gamma)\\ &=\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q d_{\omega,q}\frac{1}{[\psi-\omega+1]_q}.\end{aligned}$$

On the other hand, from the Eq. (2.4), we have

$$\mathfrak{d}_{\psi,q}(1)=\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q d_{\omega,q}$$

and

$$\mathfrak{d}_{\psi,q}(0)=d_{\omega,q}.$$

Thus,

$$\begin{aligned}\int_0^1 \mathfrak{d}_{\psi,q}(\gamma) d_q(\gamma) &= \sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q d_{\omega,q} \frac{1}{[\psi-\omega+1]_q} \\ &= \frac{1}{[\psi+1]_q} \left(\sum_{\omega=0}^{\psi+1} \binom{\psi+1}{\omega}_q d_{\omega,q} - d_{\psi+1,q} \right) \\ &= \frac{\mathfrak{d}_{\psi+1,q}(1) - \mathfrak{d}_{\psi+1,q}(0)}{[\psi+1]_q}.\end{aligned}$$

□

Theorem 2.5. *The formula for the q−derangement polynomials is as follows:*

$$\mathfrak{d}_{\psi,q}(\gamma) = [\psi]_q \, \mathfrak{d}_{\psi-1,q}(\gamma) - \sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q (-1)^{\omega} \gamma^{\psi-\omega}.$$

Proof. From Eq. (2.3), we find

$$(1-t) \sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} = e_q(-t) \, e_q(\gamma t), \tag{2.7}$$

$$\sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} - t \sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} = \sum_{\psi=0}^{\infty} (-1)^{\psi} \frac{t^{\psi}}{[\psi]_q!} \sum_{\psi=0}^{\infty} \gamma^{\psi} \frac{t^{\psi}}{[\psi]_q!}.$$

Thus,

$$\sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} = t \sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} + \sum_{\psi=0}^{\infty} \left(\sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q (-1)^{\omega} \gamma^{\psi-\omega} \right) \frac{t^{\psi}}{[\psi]_q!}. \tag{2.8}$$

Also, if we apply Eq. (1.3) to Eq. (2.7), we have

$$\begin{aligned}\sum_{\psi=0}^{\infty} D_q \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} &= \frac{t}{1-t} e_q(-t) \, e_q(\gamma t) \\ &= t \sum_{\psi=0}^{\infty} D_q \mathfrak{d}_{\psi,q}(\gamma) \frac{t^{\psi}}{[\psi]_q!} \\ &= [\psi]_q \, \mathfrak{d}_{\psi-1,q}(\gamma).\end{aligned}$$

From Eq. (2.8), we obtain

$$\mathfrak{d}_{\psi,q}(\gamma) = [\psi]_q \, \mathfrak{d}_{\psi-1,q}(\gamma) + \sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q (-1)^{\omega} \gamma^{\psi-\omega}.$$

□

Moreover, when we make the necessary adjustments to this last equation, we have

$$(\gamma-1)_q^{\psi} = \mathfrak{d}_{\psi,q}(\gamma) - [\psi]_q \, \mathfrak{d}_{\psi-1,q}(\gamma).$$

Theorem 2.6. *The polynomials $\mathfrak{d}_{\psi,q}(\gamma)$ satisfy the following property.*

$$\sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q q^{\frac{1}{2}\omega(\omega-1)} \mathfrak{d}_{\psi,q}(\gamma) = \sum_{\omega=0}^{\psi} \binom{\psi}{\omega}_q d_{\psi-\omega,q} \, (\gamma+1)_q^{\omega}.$$

Proof. Let's use Eq. (2.3) and Cauchy product rule for proof. We obtain

$$\sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma)\frac{t^{\psi}}{[\psi]_q!}E_q(t) = \sum_{\psi=0}^{\infty} \mathfrak{d}_{\psi,q}(\gamma)\frac{t^{\psi}}{[\psi]_q!}\sum_{\psi=0}^{\infty} q^{\frac{1}{2}\omega(\omega-1)}\frac{t^{\psi}}{[\psi]_q!}$$
$$= \sum_{\psi=0}^{\infty}\left(\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q q^{\frac{1}{2}\omega(\omega-1)}\mathfrak{d}_{\psi-\omega,q}(\gamma)\right)\frac{t^{\psi}}{[\psi]_q!} \tag{2.9}$$

and

$$\frac{e_q(-t)}{1-t}e_q(\gamma t)E_q(t) = \sum_{\psi=0}^{\infty} d_{\psi,q}\frac{t^{\psi}}{[\psi]_q!}\sum_{\psi=0}^{\infty}\gamma^{\psi}\frac{t^{\psi}}{[\psi]_q!}\sum_{\psi=0}^{\infty} q^{\frac{1}{2}\omega(\omega-1)}\frac{t^{\psi}}{[\psi]_q!} \tag{2.10}$$
$$= \sum_{\psi=0}^{\infty} d_{\psi,q}\frac{t^{\psi}}{[\psi]_q!}\sum_{\psi=0}^{\infty}\left(\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q q^{\frac{1}{2}\omega(\omega-1)}\gamma^{\psi-\omega}\right)\frac{t^{\psi}}{[\psi]_q!}$$
$$= \sum_{\psi=0}^{\infty} d_{\psi,q}\frac{t^{\psi}}{[\psi]_q!}\sum_{\psi=0}^{\infty}(\gamma+1)^{\psi}\frac{t^{\psi}}{[\psi]_q!}$$
$$= \sum_{\psi=0}^{\infty}\left(\sum_{\omega=0}^{\psi}\binom{\psi}{\omega}_q d_{\psi-\gamma,q}(\gamma+1)^{\omega}\right)\frac{t^{\psi}}{[\psi]_q!}.$$

From the equality of Eq. (2.9) and Eq. (2.10), we reach the desired result. □

3. Incomplete $q-$Derangement Polynomials and Numbers

When we look at the mathematical literature, incomplete polynomials first appear in Baishanski's study [2]. Subsequently, various authors have conducted studies on incomplete numbers, polynomials, and their $q-$analogues [6, 8, 10, 16]. For example, incomplete Leonardo [4], incomplete Horadam [21], incomplete Bell polynomials [15], incomplete $q-$Fibonacci and Lucas polynomials [20, 22] and incomplete $q-$Chebyshev polynomials [7].

The incomplete $q-$derangement numbers and incomplete $q-$derangement polynomials are defined in this section. We then go over some of these numbers and polynomials features. Additionally, we derive a variety of recurrence relations by expressing the incomplete $q-$derangement numbers in terms of $q-$derangement numbers.

Definition 3.1. The definition of the incomplete $q-$derangement polynomials is

$$d_{\psi,q}(\gamma,m) = \sum_{\omega=0}^{m}\binom{\psi}{\omega}_q[\psi-\omega]_q!(\gamma-1)^{\omega} \tag{3.1}$$

for $0 \leq m \leq \psi$.

From Eq. (3.1), we have

$$d_{\psi,q}(\gamma,m) = [\psi]_q!\sum_{\omega=0}^{m}\frac{(\gamma-1)^{\omega}}{[\omega]_q!}. \tag{3.2}$$

Definition 3.2. The incomplete $q-$derangement numbers are defined by

$$d_{\psi,q}(m) = [\psi]_q!\sum_{\omega=0}^{m}\frac{(-1)^{\omega}}{[\omega]_q!} \tag{3.3}$$

for $0 \leq m \leq \psi$.

Obviously, if we write $\gamma = 0$ in Eq. (3.3), we see that $d_{\psi,q}(0,m) = d_{\psi,q}(m)$.

In Table 1, we see the first few incomplete $q-$derangement numbers.

$n\backslash k$	0	1	2	3	4
0	$[0]_q!$	0	$\frac{1}{[2]_q}$	$\frac{q}{[3]_q}$	$\frac{q[4]_q[2]_q+1}{[4]_q!}$
1	$[1]_q!$	0	$\frac{1}{[2]_q}$	$\frac{q}{[3]_q}$	$\frac{q[4]_q[2]_q+1}{[4]_q!}$
2	$[2]_q!$	0	$\frac{1}{[3]_q}$	$\frac{q[2]_q}{[3]_q}$	$\frac{q[4]_q[2]_q+1}{[4]_q[3]_q}$
3	$[3]_q!$	0	$[4]_q\,[3]_q$	q	$\frac{q[4]_q[2]_q+1}{[4]_q}$
4	$[4]_q!$	0	$[5]_q\,[4]_q\,[3]_q$	$\frac{q[4]_q!}{[3]_q}$	$q\,[4]_q\,[2]_q+1$

Table 1. The first few incomplete $q-$derangement numbers

Theorem 3.3. *For $m, \psi \in \mathbb{N} \cup \{0\}$ and $0 \leq m \leq \psi$, the incomplete $q-$derangement polynomials $d_{\psi,q}(\gamma, m)$ have the following properties:*

$$d_{\psi,q}(\gamma, m) = [\psi]_q! \frac{d_{m,q}(\gamma)}{[m]_q!} \tag{3.4}$$

and

$$d_{\psi+1,q}(\gamma, m) = [\psi+1]_q\, d_{\psi,q}(\gamma, m). \tag{3.5}$$

Proof. By multiplying and dividing $[m]_q!$ by the right side of Eq. (3.2), we obtain the proof of Eq. (3.4).

For the proof of Eq. (3.5), if we utilize Eq. (3.4), we obtain

$$\begin{aligned} d_{\psi+1,q}(\gamma, m) &= [\psi+1]_q! \frac{d_{m,q}(\gamma)}{[m]_q!} \\ &= [\psi+1]_q\, [\psi]_q! \frac{d_{m,q}(\gamma)}{[m]_q!} \\ &= [\psi+1]_q\, d_{\psi,q}(\gamma, m). \end{aligned}$$

□

Writing $\gamma = 0$ in Eq. (3.4) yields

$$d_{\psi,q}(m) = [\psi]_q! \frac{d_{m,q}}{[m]_q!}.$$

Theorem 3.4. *For $m, \psi \in \mathbb{N} \cup \{0\}$ and $0 \leq m \leq \psi$,*

$$d_{\psi,q}(\gamma, m) = d_{\psi,q}(\gamma, m-1) + [\psi]_q! \frac{(\gamma-1)^m}{[m]_q!}.$$

Proof. If we use the Eq. (3.2), we obtain

$$\begin{aligned} d_{\psi,q}(\gamma, m) &= [\psi]_q! \sum_{\omega=0}^{m} \frac{(\gamma-1)^\omega}{[\omega]_q!} \\ &= [\psi]_q! \sum_{\omega=0}^{m-1} \frac{(\gamma-1)^\omega}{[\omega]_q!} + [\psi]_q! \frac{(\gamma-1)^m}{[m]_q!} \\ &= d_{\psi,q}(\gamma, m-1) + [\psi]_q! \frac{(\gamma-1)^m}{[m]_q!}. \end{aligned}$$

□

We see the first few incomplete $q-$derangement polynomials below.

$$\left.\begin{aligned} d_{0,q}(\gamma, 0) &= 1 \\ d_{0,q}(\gamma, 1) &= \gamma \\ d_{0,q}(\gamma, 2) &= \tfrac{1}{[2]_q}\gamma^2 + \left(1 - \tfrac{2}{[2]_q}\right)\gamma + \tfrac{1}{[2]_q} \\ d_{0,q}(\gamma, 3) &= \tfrac{1}{[3]_q!}\gamma^3 - \left(\tfrac{3}{[3]_q!} - \tfrac{1}{[2]_q}\right)\gamma^2 + \left(\tfrac{3}{[3]_q!} - \tfrac{2}{[2]_q} + 1\right)\gamma + \tfrac{1}{[3]_q!} + \tfrac{1}{[2]_q} \end{aligned}\right\} \text{ for } \psi = 0,$$

$$\left.\begin{aligned} d_{1,q}(\gamma,0) &= 1 \\ d_{1,q}(\gamma,1) &= \gamma \\ d_{1,q}(\gamma,2) &= \tfrac{1}{[2]_q}\gamma^2 + \left(1-\tfrac{2}{[2]_q}\right)\gamma + \tfrac{1}{[2]_q} \\ d_{1,q}(\gamma,3) &= \tfrac{1}{[3]_q!}\gamma^3 - \left(\tfrac{3}{[3]_q!}-\tfrac{1}{[2]_q}\right)\gamma^2 + \left(\tfrac{3}{[3]_q!}-\tfrac{2}{[2]_q}+1\right)\gamma + \tfrac{1}{[3]_q!} + \tfrac{1}{[2]_q} \end{aligned}\right\} \text{for } \psi = 1,$$

$$\left.\begin{aligned} d_{2,q}(\gamma,0) &= [2]_q \\ d_{2,q}(\gamma,1) &= [2]_q\,\gamma \\ d_{2,q}(\gamma,2) &= \gamma^2 + \left([2]_q-2\right)\gamma + 1 \\ d_{2,q}(\gamma,3) &= \tfrac{1}{[3]_q}\gamma^3 - \left(\tfrac{3}{[3]_q}-1\right)\gamma^2 + \left(\tfrac{3}{[3]_q}-2+[2]_q\right)\gamma + \tfrac{1}{[3]_q} + 1 \end{aligned}\right\} \text{for } \psi = 2,$$

$$\left.\begin{aligned} d_{3,q}(\gamma,0) &= [3]_q \\ d_{3,q}(\gamma,1) &= [3]_q\,\gamma \\ d_{3,q}(\gamma,2) &= [3]_q\,\gamma^2 + \left([3]_q!-2\,[3]_q\right)\gamma + [3]_q \\ d_{3,q}(\gamma,3) &= \gamma^3 - \left(3-[3]_q\right)\gamma^2 + \left(3-2\,[3]_q+[3]_q!\right)\gamma + [3]_q + 1 \end{aligned}\right\} \text{for } \psi = 3.$$

Theorem 3.5. *The incomplete $q-$derangement polynomials $d_{\psi,q}(\gamma,m)$ satisfy the following property.*

$$d_{\psi+1,q}(\gamma,m) = (\gamma-1)\,d_{\psi,q}(\gamma,m-1) + \sum_{\omega=0}^{m}\binom{\psi}{\omega}_q q^{\omega}\,[\psi-\omega+1]_q!\,(\gamma-1)^{\omega}.$$

Proof. From Eq. (3.1), we have

$$d_{\psi+1,q}(\gamma,m) = \sum_{\omega=0}^{m}\binom{\psi+1}{\omega}_q [\psi+1-\omega]_q!\,(\gamma-1)^{\omega}. \tag{3.6}$$

If we use Eq. (3.2) in Eq. (3.6), we obtain

$$\begin{aligned} d_{\psi+1,q}(\gamma,m) &= \sum_{\omega=0}^{m}\left(q^{\omega}\binom{\psi}{\omega}_q + \binom{\psi}{\omega-1}_q\right)[\psi+1-\omega]_q!\,(\gamma-1)^{\omega} \\ &= \sum_{\omega=0}^{m} q^{\omega}\binom{\psi}{\omega}_q [\psi+1-\omega]_q!\,(\gamma-1)^{\omega} + \sum_{\omega=0}^{m}\binom{\psi}{\omega-1}_q [\psi+1-\omega]_q!\,(\gamma-1)^{\omega} \\ &= \sum_{\omega=0}^{m} q^{\omega}\binom{\psi}{\omega}_q [\psi-\omega+1]_q!\,(\gamma-1)^{\omega} + \sum_{\omega=1}^{m}\binom{\psi}{\omega-1}_q [\psi+1-\omega]_q!\,(\gamma-1)^{\omega} \\ &= \sum_{\omega=0}^{m} q^{\omega}\binom{\psi}{\omega}_q [\psi-\omega+1]_q!\,(\gamma-1)^{\omega} + \sum_{\omega=0}^{m-1}\binom{\psi}{\omega}_q [\psi-\omega]_q!\,(\gamma-1)^{\omega+1} \\ &= (\gamma-1)\,d_{\psi,q}(\gamma,m-1) + \sum_{\omega=0}^{m} q^{\omega}\binom{\psi}{\omega}_q [\psi-\omega+1]_q!\,(\gamma-1)^{\omega}. \end{aligned}$$

The proof is so finished. □

4. Conclusion

In this study, $q-$derangement polynomials were reviewed and obtained with the help of a new generating function. These polynomials were named with $q-$derangement polynomials of the second type and their various summation properties and recurrence relations were shown. Moreover, incomplete $q-$derangement polynomials and numbers were defined, thus a new research that will contribute to the literature was brought to the scientific world.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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