Dumlupınar Üniversitesi Fen Bilimleri Dergisi
Sayı: 1 1999

UNDERLYING GRUPOIDS

Murat ALP[*] - Sedat PAK[*]

1991 A. M. S. C.: 13D99, 16A99, 17B99, 17D99, 18D35.
Keywords: Crossed modules, Cat^1-groups, underlying grupoids.

ABSTRACT

In this paper we describe a package XMOD (Wensley and Alp,1993) of functions for computing with crossed modules, their morphisms and derivations; cat^1-groups, their morphisms and sections, written using the sf GAP (Schönert , 1993) group theory programming language. We have also enumerated the isomorphism classes of cat^1-groups in (Alp ,1997) and (Alp, Wensley ,1997) and (Alp , 1997) We gave the application algorithms and some mathematical results on cat^1-group structures in (Alp , 1998) . We also made a computational comment on pre-crossed modules, pre-cat^1-groups and underlying grupoids in this paper.

ÖZET

Bu makalede GAP programının ortak paketi XMod ( ) tanımlamanın yanı sıra Section ların GAP programına uygulanması incelenmiştir.

[*] Dumlupınar Üniversitesi Fen-Edebiyat Fakültesi MatematikBölümü Kütahya

1. Introduction

A starting point for this paper was to consider the possibility of implementing functions for doing calculations with crossed modules, derivations, actor crossed modules, cat1-groups, sections, induced crossed modules and induced cat1-groups in GAP (Schönert , 1993).

We should first explain the importance of crossed modules. The general points are:

- *crossed modules may be thought of as 2-dimensional groups;*
- *a number of phenomena in group theory are better seen from a crossed module point of view;*
- *crossed modules occur geometrically as* $\pi_2(X,A) \rightarrow \pi_1 A$ *when* A *is a subspace of* X *or as* $\pi_1 F \rightarrow \pi_1 E$ *where* $F \rightarrow E \rightarrow B$ *is a fibration;*
- *crossed modules are usefully related to forms of double groupoids.*

Particular constructions, such as induced crossed modules, are important for the applications of the 2-dimensional Van-Kampen Theorem of Brown and Higgins (Alp, Wensley ,1997) ,and so for the computation of homotopy 2-types.

For all these reasons, the facilitation of the computations with crossed modules should be advantageous. It should help to solve specific problems, and it should make it easier to construct examples and see relations with better known theories.

The powerful computer algebra system GAP provides a high level programming language with several advantages for the coding of new mathematical structures. The GAP system has been developed over the last 15 years at RWTH in Aachen. Some of its most exciting features are:

- *it has a highly developed, easy to understand programming language incorporated;*
- *it is especially powerful for group theory;*
- *it is portable to a wide variety of operating systems on many hardware platforms.*
- *it is public domain and it has a lively forum, with open discussion. These make it increasingly used by the mathematical community.*

On the other hand, GAP has some disadvantages, too:

- *the built in programming language is an interpreted language, which makes GAP programs relatively slow compared to compiled languages such as C or Pascal. GAP source can not be compiled. This will change in version 4 to be released during 1997;*
- *the demands on system resources are quite high for the serious calculations.*

However, the advantages outweigh the disadvantages, and so GAP was chosen.

The term crossed module was introduced by J. H. C. Whitehead in (Whitehead , 1948). Most references of crossed modules state the axioms of a crossed module using left actions, but we shall use right actions since this is the convention used by most computational group packages.

In (Loday , 1982) Loday reformulated the notion of a crossed module as a cat^1 group, namely a group G with a pair of homomorphisms $t,h : G \longrightarrow G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat^1-group $C=(e;t,h : G \longrightarrow R)$ as two groups G, and R, two epimorphisms $t,h : G \longrightarrow R$ and a morphism $e : R \longrightarrow G$ satisfying these certain axioms.

In section 2 we recall the basic properties of crossed modules and cat^1-groups. we made computational comment on underlying groupoids pre-crossed modules and pre-cat^1-groups.

2. Crossed Modules and Cat^1-Groups

In this section we recall the descriptions of three equivalent categories: **XMod**, the category of crossed modules and their morphisms; **Cat1**, the category of cat^1-groups and their morphisms; and **GpGpd,** the subcategory of groups in the category **Gpd** of groupoids. We also describe functors between these categories which exhibit the equivalences.

A crossed module $X=(\partial : S \longrightarrow R)$ is a pair of groups R and S together with an action of R on S and a group homomorphism ∂, called the boundary map of X, satisfying the following axioms :

$$\textbf{CM1:} \quad \partial\left(s^{r}\right) = r^{-1}(\partial s)r$$

$$\textbf{CM2:} \quad t^{\partial s} = s^{-1}ts$$

for all $s,t \in S$ and $r \in R$

The standard constructions for crossed modules are as follows:

1. *Any homomorphism $\partial : S \longrightarrow R$ provides a crossed module is S is abelian and $\mathrm{im}\,\partial \subseteq Z(R)$ with R acting trivially on S.*
2. *A conjugation crossed module is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.*
3. *A central extension crossed module has as boundary a surjection $\partial : S \longrightarrow R$ with central kernel, where $r \in R$ acts on S by conjugation with $\partial^{-1} r$.*

4. *An automorphism crossed module has as range a subgroup R of the automorphism group Aut(S) of S which contains the inner automorphism group of S. The boundary maps $s \in S$ to the inner automorphism of S by s.*
5. *An R-Module crossed module has an R-module as source and ∂ is the zero map.*
6. *The direct product $X_1 \times X_2$ of two crossed modules has source $S_1 \times S_2$, $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with R_1, R_2 acting trivially on S_1, S_2 respectively.*

A morphism between two crossed modules $X = (\partial : S \to R)$ and $X' = (\partial' : S' \to R')$ is a pair $<\sigma, \rho>$ where $\sigma : S \to S'$ and $\rho : R' \to R'$ are homomorphisms such that $\partial' \sigma = \rho \partial$ and $\sigma(s^r) = (\sigma s)^{(\rho r)}$. When $X = X'$ and σ, ρ are automorphisms then $\langle \sigma, \rho \rangle$ is an automorphism of X. The group of automorphisms is denoted by Aut(X).

In (Loday , 1982) Loday reformulated the notion of a crossed module as a cat^1-group, namely a group G with a pair of homomorphisms $t,h : G \to G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat^1-group $C = (e;t,h : G \to R)$ as two groups G, and R, two epimorphisms $t,h : G \to R$ and a morphism $e : R \to G$ satisfying:

CAT 1 : $te = he = \mathrm{id}_R$,

CAT 2 : $[\ker t, \ker h] = \{1_G\}$.

The maps t and h are often referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of C, because em source is the GAP term for the domain of a function. Note that CAT1 emplies e is an embedding.

A morphism $C \to C'$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G \to G'$ and $\rho : \mathrm{R} \to \mathrm{R}'$ are homomorphisms satisfying

$$\mathrm{h}'\gamma = \rho\,\mathrm{h} \ , \ \mathrm{t}'\gamma = \rho\,\mathrm{t} \ , \ \mathrm{e}'\rho = \gamma\,\mathrm{e}$$

The crossed module X associated to C has $S = \ker t$ and $\partial = h|_S$. The cat^1-group associated to X has $G = R \propto S$, using the action from X, and

$$t(r,s) = r, \ \ h(r,s) = r\,(\partial\, s), \ \ er = (r, 1). \qquad (1)$$

We denote by epsilon the inclusion of S in G.

An arbitrary cat^1-group $C = (e;t,h : G \longrightarrow R)$ is isomorphic to a semidirect product cat^1-group as follows. Since G acts on $S = \ker t$ by conjugation, R acts on S by

$$s^r = s^{er} = (\mathrm{er})^{-1}\mathrm{s}(\mathrm{er}).$$

The semidirect product $R \propto S$ has composition and inverse given by:

$$(r_1,s_1)(r_2,s_2) = (r_1r_2, s_1^{r_2}s_2), \quad (r,s)^{-1} = \left(r^{-1}, (s^{-1})^{r^{-1}}\right)$$

There is an isomorphism

$$\theta = R \propto S \longrightarrow G, \quad (r,s) \mapsto (er)s \tag{2}$$

with inverse

$$\theta^{-1} \quad = \quad G \longrightarrow R \propto S, \; g \mapsto \left(tg, (etg^{-1})g\right)$$

Proposition 2.1 $C' = (e';t',h' : R \propto S \longrightarrow R)$ is a cat^1-group where $t' = t\theta$, $h' = h\theta$, $e' = \theta^{-1}e$ and $(\theta, \mathrm{id}_R) : C' \longrightarrow C$ is an isomorphism.

Proof: Since $t'e' = t\,\theta\,\theta^{-1}\,e = te$, $h'e' = h\,\theta\,\theta^{-1}\,e = he$ and $[\ker t', \ker h'] = [\ker(t\theta), \ker(h\theta)] = \{\theta^{-1}g \mid g \in [\ker t, \ker h]\} = 1$, axioms CAT1 and CAT2 are satisfied and $C' = (e';t',h' : R \propto S \longrightarrow R)$. It follows from (2) that $(\theta, \mathrm{id}_R) : C' \longrightarrow C$ is an isomorphism.

3. Pre-crossed modules and Pre-cat^1-groups

When axioms CM2 and Cat2 are not satisfied, the corresponding structures are known as pre-crossed modules and pre-cat^1-groups. The Peiffer subgroup P of S is the subgroup of $\ker(\partial)$ generated by Peiffer commutators

$$[\![s_1, s_2]\!] \quad = \quad (s_1^{-1})^{\partial s_2} s_2^{-1} s_1 s_2,$$

and $X = (\partial : S/P \longrightarrow R)$ is a crossed module. The image of $[\![s_1, s_2]\!]$ under $\varepsilon : S \longrightarrow R \propto S$ is

$$\varepsilon\,[\![s_1, s_2]\!] = [\![(1_R, s_1^{\partial s_2}), ((\partial s_2)^{-1}, s_2)]\!] \in [\ker t', \ker h'] \tag{3}$$

The image εP is the Peiffer subgroup of $R \propto S$, and $C'' = (e'';t'',h'' : R \propto S \,/\, [\ker t', \ker h'] \longrightarrow R)$ is the cat^1-group corresponding to X, where $t' = t''v$,

$h'=h''\nu$, $e''=\nu e'$ and $\nu : R\propto S \rightarrow R\propto S/[\ker t', \ker h']$ is the natural homomorphism.

$$\varepsilon P=[\ker t', \ker h'] \tag{4}$$

The following diagram illustrates the arrangement of homomorphisms.

$$\begin{tikzcd}
R\propto S \arrow[r, "\nu"] \arrow[dr, shift left, "t'"] \arrow[dr, shift right, "h'"'] & R\propto S/[\ker t', \ker h'] \arrow[d, shift right, "t''"'] \arrow[d, shift left, "h''"] \\
& R \arrow[ul, bend left=60, "e'"] \arrow[u, bend right=60, "e''"']
\end{tikzcd}$$

The underlying groupoid G of a cat^1-group C has the elements of R as the set of objects and the elements of G as arrows. The identity arrow at r is er. For each arrow g the source(tail) is tg and the target(head) is hg. Arrows g, g' are composable only when $hg = t g'$

$$\begin{tikzcd}
tg \arrow[r, bend left, "g"] & hg=t g' \arrow[r, bend left, "g'"] & h g'
\end{tikzcd}$$

$$\begin{tikzcd}
r \arrow[r, "(r,s)"] \arrow[rr, bend right, "(r,s s')"'] & r(hs)=r' \arrow[r, "(r',s')"] & r'(h s')=r(hs)(h s')=r(h(s s'))
\end{tikzcd}$$

So we have a composition of composable arrows:

$$(r,s)*(r',s')=(r,s s')$$

when $r(hs)=r'$. Applying θ to determine the composition rule for $g* g'$

$$g* g'=g(et g')^{-1} g'=g(ehg)^{-1} \tag{5}$$

with tail tg and head $h g'$. Also $*$ is associative:

$$((r_1,s_1)*(r_2,s_2))*(r_3,s_3)=(r_1,s_1)*((r_2,s_2)*(r_3,s_3)) \tag{6}$$

In order to find an inverse $\tilde{g}$ (equivalently $(\tilde{r},\tilde{s})$) for $*$, we require

$$
\begin{tikzcd}
r \arrow[r, bend left, "(r,s)"] & r(hs) \arrow[l, bend left, "(r',s')"]
\end{tikzcd}
$$

$$(r,s)*(\tilde{r},\tilde{s})=1_r=e(r)=(r,1), \tag{7}$$
$$(\tilde{r},\tilde{s})*(r,s)=1_{r(hs)}=e(r(hs))=(r(hs),1), \tag{8}$$

$$
\begin{aligned}
(r,s\tilde{s}) &= (r,1)\\
(\tilde{r},\tilde{s}s) &= (r(hs),1)
\end{aligned}
$$

and these are satisfied provided $\tilde{s}=s^{-1}$, $\tilde{r}=r(hs)$. Thus (r,s) has inverse $(r(hs),s^{-1})$ under $*$. The inverse $\tilde{g}$ of g for this composition is given by

$$\tilde{g}=(ehg)g^{-1}(etg)$$
(9)

The homomorphism $g \mapsto \tilde{g}$ on G is the identity map on eR and provides a cat[1]-isomorphism from C to $\tilde{C}=(e;h,t:G\to R)$.

The set of arrows out of 1_R are the elements of $\ker t$ while the arrows in to 1_R are the elements of $\ker h$, so $\ker\partial$ is the set of loops at 1_R. The set of objects in the component of G connected to 1_R is the image of ∂, so G is discrete when $\partial=0$.

A section ξ defines a morphism $\Delta(\xi):G\to G$ as follows. Consider the diagram

$$
\begin{tikzcd}
h\xi tg \arrow[r, "\Delta(\xi)(g)"] & h\xi hg \\
tg \arrow[u, "\xi tg"] \arrow[ur] \arrow[r, "g"'] & hg \arrow[u, "\xi hg"']
\end{tikzcd}
$$

where $\xi\, tg$ has tail $t\xi\, tg{=}tg$ and head $h\xi\, tg$, while $\xi\, hg$ has tail $t\xi\, hg$ $=hg$ and tail $h\xi\, hg$. Then we define as follows:

$$\Delta(\xi)(g) = (\xi\, tg) * (g * \xi\, hg) \tag{10}$$

where

$$g * \xi\, hg = g(et\xi\, hg)^{-1}(\xi\, hg) = g(ehg)^{-1}(\xi\, hg) \tag{11}$$

It follows that

$$\theta^{-1}\,\Delta(\xi)(g) = \left(h\xi\, tg, (\xi\, tg)^{-1} g(ehg)^{-1}(\xi\, hg)\right). \tag{12}$$

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