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On the Synchronization of Bidirectionally Coupled Nonidentical Systems via Output Feedback

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ABSTRACT

We investigate the synchronization of bidirectionally coupled nonidentical chaotic systems, addressing a critical challenge in nonlinear dynamics. Unlike traditional master-slave or unidirectional synchronization approaches, we propose a novel synchronization scheme based on output feedback linearization that ensures identical synchronization even in the presence of parameter mismatches and structural differences between systems. Our approach incorporates a nonlinear switching feedback law, which enhances stability and robustness in bidirectionally coupled configurations. We analyze the synchronization conditions using Lyapunov stability theory and illustrate our results through numerical simulations on well-known benchmark chaotic systems, including the Lorenz and Sprott systems. Our findings demonstrate that the proposed method can achieve stable synchronization in both identical and nonidentical configurations, even when the systems exhibit piecewise nonlinearities. These results extend the applicability of synchronization techniques to a broader class of chaotic systems and lay the groundwork for future research in networked dynamical systems.

KEYWORDS
Synchronization
Bidirectional coupling
Piece-wise linear systems
Relative degree

INTRODUCTION

Significant progress has been made since Pecora and Carroll introduced a synchronization scheme in their groundbreaking work (Pecora and Carroll 1990). In that study, they addressed the synchronization problem using a master-slave configuration, where a signal from the master system drives a slave system with an identical structure. Since its introduction, this methodology has seen broad adoption. For instance, Alvarez (1996) established formal synchronizability conditions and demonstrated their application using the Lorenz system (Lorenz 1963). Similarly, Zhu and Zhou (2008) employed this framework in the context of dual inverted pendulum systems. Research has also explored non-forced synchronization, including bifurcation patterns in non-identical Duffing oscillators (Vincent and Kenfack 2008). Over time, numerous synchronization strategies have emerged to address various aspects of the problem. One key challenge is achieving synchronization in chaotic systems without relying on a master-slave setup. Hong *et al.* (2001) presented an adaptive synchronization method in this context, and Sarasola *et al.* (2003) proposed a technique based on linear feedback coupling. A robust approach using sliding mode control was introduced by (Alvarez *et al.* 2010). More recently, the scope of synchronization has expanded to encompass complex networks (Duan *et al.* 2007), with notable examples including consensus and pinning strategies (Olfati-Saber *et al.* 2007).

In the most general sense, two or more dynamical systems are said to be synchronized if through a subtle interaction their states become correlated in time Rulkov *et al.* (1995). From this point of view, many different types of synchronized behavior can be defined, including identical, phase, and generalized synchronization to mention but a few (Pikovsky *et al.* 2001; Boccaletti *et al.* 2002). In the simplest case, two systems are *unidirectionally* connected, which is usually called *drive-response* configuration (Pecora and Carroll 1990). Designing the interaction between these systems poses a significant challenge, particularly in formulating the coupling term within the response subsystem. This term is typically developed using diverse control techniques such as robust control (Almeida *et al.* 2006), adaptive schemes (Hong *et al.* 2001), and optimal control approaches (Pan and Yin 1997). Another approach involves establishing a bidirectional coupling between the two systems. In this setup, synchronization becomes more intricate,

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as both subsystems mutually influence one another through their dynamic interplay (Boccaletti *et al.* 2002). Research in this area has naturally progressed toward the study of dynamical networks (Boccaletti *et al.* 2006), giving rise to prominent challenges such as consensus achievement and pinning control (Su and Wang 2013).

The study of synchronization in dynamical systems has evolved significantly in the last decade, with a growing focus on nonidentical systems, bidirectional coupling, and networked dynamics. Recent works have explored more complex configurations. In Arreola-Delgado and Barajas-Ramírez (2021), they investigated the controllability of networks with nonidentical linear nodes, providing theoretical conditions under which heterogeneous systems can be synchronized. Their findings highlight the impact of network topology on synchronization feasibility, offering insights into designing control strategies for complex networks. Bidirectional coupling presents additional challenges, particularly when systems exhibit multistability. In (Ruiz-Silva *et al.* 2022), they analyzed bidirectionally coupled multistable systems, showing that synchronization depends on initial conditions and parameter mismatches. This work builds on previous research by (Ruiz-Silva *et al.* 2021), which explored the emergence of synchronous behavior in chaotic multistable systems. Their results emphasize the role of dynamical stability and bifurcation structures in determining synchronization outcomes, suggesting that synchronization can be highly sensitive to system parameters.

The influence of network topology on synchronization patterns has also been studied in the context of network motifs, Uriostegui-Legorreta *et al.* (2024) examined the synchronization of three piecewise Rössler systems coupled in a ring configuration. Their study demonstrates that different coupling configurations can lead to phase-locking, generalized synchronization, or desynchronization, depending on the system parameters and interaction strengths. These findings highlight the importance of structural connectivity in determining collective dynamics. Time delays in coupling can significantly affect synchronization behavior, either facilitating or disrupting synchronization. In (Serrano and Ghosh 2022), they proposed a robust stabilization and synchronization strategy for chaotic systems with time-varying delays, showing that adaptive control techniques can mitigate the negative effects of delays. Their approach is particularly relevant for applications where communication constraints or biological rhythms introduce inherent time-dependent perturbations.

Beyond classical integer-order systems, synchronization in fractional-order and neural networks has gained attention as can be observed in (Jahanshahi *et al.* 2022); they studied the synchronization of variable-order fractional Hopfield-like neural networks, revealing that parameter adaptation techniques can effectively synchronize such systems. Their work suggests that fractional dynamics introduce additional flexibility in synchronization, making them applicable to biological and artificial neural networks.These studies collectively emphasize the significance of bidirectional interactions, network topology, and system heterogeneity in synchronization dynamics. While recent advances have provided a deeper understanding of these factors, challenges remain in achieving robust synchronization in complex networks, particularly in the presence of uncertainties and structural mismatches. Future research may explore hybrid synchronization strategies, integrate learning-based approaches, and extend these concepts to multi-agent and cyber-physical systems.

This work addresses the synchronization challenge in nonidentical dynamical systems configured both in *drive-response* form (Assali 2021) and through *bidirectional* coupling. Our focus is directed toward chaotic systems characterized by simple quadratic dynamics, as well as more tractable chaotic models incorporating piecewise linear (PWL) elements (Delgado-Aranda *et al.* 2020; Escalante-González and Campos 2021). To achieve asymptotic identical synchronization, we develop an interconnection framework based on output feedback control. Nonetheless, attaining synchronized behavior in non-identical systems remains nontrivial due to potential amplitude suppression arising from the coupling effects. To address this, we introduce a synchronization strategy tailored for bidirectionally coupled, piecewise smooth nonlinear systems exhibiting full relative degree. The approach relies on a nonlinear switching feedback mechanism capable of handling parameter mismatches and structural disparities, including nonsmooth components.

The remainder of this paper is organized as follows. In Section 2, we define the synchronization problem for both *drive-response* and *bidirectional* coupling scenarios, and introduce an output feedback-based design methodology. Section 3 details the proposed synchronization scheme, and Section 4 showcases numerical simulations that validate our findings. The paper concludes with a summary of key insights and suggestions for future research directions.

PROBLEM FORMULATION

In this contribution, we consider two nonidentical systems bidirectionally coupled, that is,

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t)) + \mathcal{G}_{12}(y_1(t), y_2(t)), \\ y_1(t) &= h_1(x_1(t)), \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{x}_2(t) &= f_2(x_2(t)) + \mathcal{G}_{21}(y_1(t), y_2(t)), \\ y_2(t) &= h_2(x_2(t)), \end{aligned} \tag{2}$$

where $x_1(t), x_2(t) \in \mathbf{R}^n$ are the state variables, $y_1(t), y_2(t) \in \mathbf{R}^m$ ($m \leq n$) are output variables with measurement functions $h_1(\cdot), h_2(\cdot) : \mathbf{R}^n \to \mathbf{R}^m$, and $f_1(\cdot), f_2(\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ describe the dynamics of each isolated system, respectively. The coupling functions of system 2 to 1; $\mathcal{G}_{12}(\cdot,\cdot) : \mathbf{R}^{2m} \to \mathbf{R}^n$, and system 1 to 2; $\mathcal{G}_{21}(\cdot,\cdot) : \mathbf{R}^{2m} \to \mathbf{R}^n$ are to be designed.

For simplicity, we consider both systems to be three-dimensional, the outputs scalar, and their measurement functions linear ($h_1(x_1(t)) = C_1x_1(t)$, $h_2(x_2(t)) = C_2x_2(t)$ with $C_1, C_2 \in \mathbf{R}^{1\times 3}$). Furthermore, the systems are interconnected through a linear function based on the difference in their outputs, that is, diffusive output coupling:

$$\begin{aligned} \mathcal{G}_{12}(y_1(t), y_2(t)) &= K_{12}(C_2x_2(t) - C_1x_1(t)), \\ \mathcal{G}_{21}(y_1(t), y_2(t)) &= K_{21}(C_1x_1(t) - C_2x_2(t)), \end{aligned} \tag{3}$$

where the coupling gains $K_{12} \in \mathbf{R}^{3\times 1}$ and $K_{21} \in \mathbf{R}^{3\times 1}$ are chosen such that *identical synchronization* is achieved, that is, the following conditions are satisfied:

$$\lim_{t\to\infty} x_1(t) - x_2(t) = 0, \text{ and } \lim_{t\to\infty} x_2(t) - x_1(t) = 0. \tag{4}$$

To assess whether *identical synchronization* occurs in systems (1)-(3), as described by the condition in (4), we introduce the corresponding error variables:

$$e_{12}(t) = x_2(t) - x_1(t),$$
$$e_{21}(t) = x_1(t) - x_2(t). \quad (5)$$

Then, the error dynamics are given by

$$\begin{aligned} \dot{e}_{12}(t) &= f_{12}(t) + K_{12}(C_2x_2(t) - C_1x_1(t)) \\ &\quad -K_{21}(C_1x_1(t) - C_2x_2(t)), \\ \dot{e}_{21}(t) &= f_{21}(t) + K_{21}(C_1x_1(t) - C_2x_2(t)) \\ &\quad -K_{12}(C_2x_2(t) - C_1x_1(t)), \end{aligned} \quad (6)$$

where $f_{12}(t) = f_2(x_2(t)) - f_1(x_1(t))$ and $f_{21}(t) = f_1(x_1(t)) - f_2(x_2(t))$. Letting the output functions be identical ($C = C_1 = C_2 \in \mathbf{R}^{1\times3}$), the error dynamics can be rewritten as:

$$\begin{aligned} \dot{e}_{12}(t) &= f_{12}(t) + K_{12}Ce_{12}(t) - K_{21}Ce_{21}(t), \\ \dot{e}_{21}(t) &= f_{21}(t) - K_{12}Ce_{12}(t) + K_{21}Ce_{21}(t). \end{aligned} \quad (7)$$

Let the coupling gains be chosen as: $K_{12} = K_{21} = k\,[1,1,1]^\top \in \mathbf{R}^3$, then the error dynamics in vector form becomes:

$$\dot{\mathbf{e}}(t) = \mathbf{f}(t) + k\,[\mathbf{1}(6,2) \otimes C]\,\mathbf{e}(t), \quad (8)$$

where $\mathbf{e}(t) = [e_{12}(t), e_{21}(t)]^\top \in \mathbf{R}^6$, $\mathbf{f}(t) = [f_{12}(t), f_{21}(t)]^\top \in \mathbf{R}^6$, $\mathbf{1}(6,2)$ is matrix of one entries with six rows and two columns, and $\otimes$ represents the Kronecker product.

Finding a coupling gain k such that (8) has zero as its asymptotically stable equilibrium point can be very difficult. In general, when $f_1(\cdot) \neq f_2(\cdot)$ the *identical synchronization* solution for (1)-(3) can not be stabilized by a choice of k. An alternative way to describe the conditions in (4) is to assume that an identical synchronization solution exist (Duane *et al.* 2007), that is,

$$\lim_{t\to\infty} x_1(t) = x_2(t) = s(t), \quad (9)$$

where $s(t) \in \mathbf{R}^3$ is the identical synchronization solution. Further, we assume that the dynamics of $s(t)$ are also known, such that we have:

$$\dot{s}(t) = f_s(s(t)), \quad (10)$$

with $f_s(\cdot) : \mathbf{R}^3 \to \mathbf{R}^3$.

In the case of identical dynamics the synchronized solution $s(t)$ is realized since the differential coupling term vanish. Therefore, $s()t$ is actually the dynamics of either node in isolation. Furthermore, identical synchronization can be achieved by an appropriately chosen constant gain k. For the case of non-identical nodes, the diffusive coupling does not vanishes so $s(t)$ is not exactly the dynamics of an isolated node. Yet if the coupling function forces the nodes to synchronize to the same behavior as in (10), the synchronization solution, either impose as desired behavior $\dot{s}(t) = f_s(s(t))$, or as the average dynamics of the nodes $\dot{s}(t) = f_s(s(t)) = (\frac{1}{2})(f_1(s(t)) + f_2(s(t)))$, can be assume to exit. Additionally, it can only be made stable by an appropriate design of the coupling function, which as describe in the remainder of this contribution it will require nonlinear terms to guaranty its stability. From the above, we do not argue that $\dot{s}(t) = f_1(s(t))$ nor $\dot{s}(t) = f_2(s(t))$, but that $s(t)$ is a solution where the coupled nodes are synchronized despite their differences, therefore $\dot{s}(t) = f_s(s(t))$ is our control objective dynamics. In this sense, we are solving a controlled synchronization problem that becomes a bit more interesting since the systems are bidirectionally coupled.

In terms of this desired synchronization solution the errors are:

$$e_1(t) = x_2(t) - s(t),$$
$$e_2(t) = x_1(t) - s(t). \quad (11)$$

Then, the error dynamics are found from (1)-(10) to be:

$$\begin{aligned} \dot{e}_1(t) &= f_{1s}(t) + \mathcal{G}_{12}(y_1(t), y_2(t)), \\ \dot{e}_2(t) &= f_{2s}(t) + \mathcal{G}_{21}(y_1(t), y_2(t)), \end{aligned} \quad (12)$$

where $f_{1s}(t) = f_1(x_1(t)) - f_s(s(t))$ and $f_{2s}(t) = f_2(x_2(t)) - f_s(s(t))$.

In this contribution, instead of using a purely linear component for the coupling function, like (3), we propose to add a nonlinear component $\mathcal{F}_i(\cdot), (i = 1, 2)$, such that:

$$\begin{aligned} \mathcal{G}_{12}(y_1(t), y_2(t)) &= KC(e_2(t) - e_1(t)) + \mathcal{F}_1(y_1(t), y_2(t)), \\ \mathcal{G}_{21}(y_1(t), y_2(t)) &= KC(e_1(t) - e_2(t)) + \mathcal{F}_2(y_1(t), y_2(t)), \end{aligned} \quad (13)$$

with $K \in \mathbf{R}^{3\times1}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are designed to have

$$\lim_{t\to\infty} e_1(t) = 0, \text{ and } \lim_{t\to\infty} e_2(t) = 0. \quad (14)$$

Again, a general solution for the design problem described above is very complex. However, since the error and output functions in (13) can be expressed in terms of the state variable $x(t)$, each system (1) and (2) may be rewritten as a control-input/affine system. Using this notation will allow a better design for inputs. Therefore, we consider three-dimensional systems of the form:

$$\begin{aligned} \dot{x}(t) &= F(x(t)) + G(x(t))u(t), \\ y(t) &= H(x(t)), \end{aligned} \quad (15)$$

with $F(\cdot)$ used to represent functions like $f_1(\cdot)$ and $f_2(\cdot)$ in (1) and (2) and the control-input/affine function $G(\cdot)u(t)$ represent functions $\mathcal{G}_{12}(\cdot,\cdot)$ and $\mathcal{G}_{21}(\cdot,\cdot)$. Also, $x(t) \in \mathbf{R}^3$ and $u(t) \in \mathbf{R}$ are the state variables and input to the system, respectively. In particular, we focus on the case of vector fields $F(x(t))$ and $G(x(t))$ such that (15) has *full relative degree*. That is, the following conditions are satisfied (Isidori 1985):

i) $L_G L_F^k H(x) = 0, k = 0, 1,$
ii) $L_G L_F^2 H(x) \neq 0,$

where $L_F H(x) = \frac{\partial H(x)}{\partial x} F(x)$ represents the Lie derivative of $H(x)$ along the vector field $F(x)$, where $L_F^0 H(x) = H(x)$ by definition.

Using a coordinate transformation $z_1(t) = H(x(t))$, $z_2(t) = L_F H(x(t))$, $z_3(t) = L_F^2 H(x(t))$. The system in (15) can be rewritten in its normal form:

$$
\begin{aligned}
\dot{z}_1(t) &= z_2(t), \\
\dot{z}_2(t) &= z_3(t), \\
\dot{z}_3(t) &= L_F^3 H(x(t)) + L_G L_F^2 H(x(t)) u(t), \\
y_z(t) &= z_1(t).
\end{aligned} \tag{16}
$$

Different three-dimensional chaotic systems may have a full relative degree.

We also consider, dynamical systems with an structure based on the Jerk equation

$$
\dddot{x}(t) = a_1 \ddot{x}(t) + a_2 \dot{x}^2(t) + a_3 x(t) + NL(x(t)). \tag{17}
$$

Using $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, and $x_3(t) = \ddot{x}(t)$ in vector form of (17) is rewritten as:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= a_1 x_3(t) + a_2 x_2^2(t) + a_3 x_1(t) + NL(x(t)), \\
y_x(t) &= x_1(t).
\end{aligned} \tag{18}
$$

As shown in (Sprott 2000), for many different choices of parameters and nonlinear function $NL(x(t))$, the system (18) exhibits chaotic behavior.

In the following Section, we propose a design for the bidirectional coupling of systems in the form of (16) and (18), such that identical synchronization is achieved in the sense of (14).

SYNCHRONIZATION STRATEGY

In this Section the controlled synchronization problem is addressed departing form the version of theproblem formulated above, in our proposed strategy the error dynamics are described under the assumption that a known desired synchronization solution exist, that is, instead of solving for the errors $x_1(t) - x_2(t)$ as the problem is originally formulated, we focus on the reformulated errors as $x_1(t) - s(t)$ and $x_2(t) - s(t)$. In this way, the *bidirectionally* coupled systems have the error dynamics:

$$
\begin{aligned}
\dot{e}_1(t) &= F_1 + \kappa_1 C e_2(t) - \kappa_2 C e_1(t), \\
\dot{e}_2(t) &= F_2 + \kappa_2 C e_1(t) - \kappa_1 C e_2(t),
\end{aligned} \tag{19}
$$

for $F_2 = F(y(t)) - F(x(t))$. For simplicity, we assume $\kappa_1 = \kappa_2 = \kappa \in \mathbf{R}^3$. Under successful synchronization, the coupled systems admit a common solution of the form:

$$
x(t) = y(t) = s(t). \tag{20}
$$

This implies that when the systems are synchronized, the coupling terms in (11) vanish, and each node evolves according to:

$$
\dot{s}(t) = F(s(t)). \tag{21}
$$

To characterize the deviation from this synchronized behavior, we define the following error variables:

$$
\begin{aligned}
\epsilon_1(t) &= x(t) - s(t), \\
\epsilon_2(t) &= y(t) - s(t).
\end{aligned} \tag{22}
$$

The dynamics of these error variables are then given by:

$$
\begin{aligned}
\dot{\epsilon}_1(t) &= F_{1s} + \kappa C \epsilon_2(t) - \kappa C \epsilon_1(t), \\
\dot{\epsilon}_2(t) &= F_{2s} + \kappa C \epsilon_1(t) - \kappa C \epsilon_2(t),
\end{aligned} \tag{23}
$$

where $F_{1s} = F(x(t)) - F(s(t))$ and $F_{2s} = F(y(t)) - F(s(t))$. In vector form, these equations can be rewritten as:

$$
\dot{E}(t) = \mathbf{F} + (A \otimes \kappa C) E(t), \tag{24}
$$

with $E(t) = \begin{bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \end{bmatrix} \in \mathbf{R}^6$, $\mathbf{F} = \begin{bmatrix} F_{1s} \\ F_{2s} \end{bmatrix} : \mathbf{R}^6 \to \mathbf{R}^6$, $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ representing the Laplacian matrix associated with the *bidirectional* coupling. The symbol $\otimes$ denotes the Kronecker product.

Bidirectional synchronization of the coupled systems described by (1)-(3) is achieved if the error dynamics in (24) is, at least, locally asymptotically stable at the origin.

To analyze stability, we linearize equation (24) at the zero solution, yielding:

$$
\dot{E}(t) = [D\mathbf{F}(s(t)) + (A \otimes \kappa C)] E(t), \tag{25}
$$

where $D\mathbf{F}(s(t)) = [DF(s(t)), DF(s(t))]^\top$ and $DF(\cdot)$ the Jacobian of the nonlinear dynamics. Given that A is a Laplacian matrix, a change of coordinates $E(t) = \Phi[v_1(t), v_2(t)]^\top$, where Φ built with eigenvectors of A, allows us to decouple the linearized error dynamics into:

$$
\begin{aligned}
\dot{v}_1(t) &= [DF(s(t)) + \lambda_1 \kappa C] v_1(t), \\
\dot{v}_2(t) &= [DF(s(t)) + \lambda_2 \kappa C] v_2(t),
\end{aligned} \tag{26}
$$

where $\lambda_1 = 0$ and $\lambda_2 = -2$ are the eigenvalues of A. Since λ_1 corresponds to the synchronized motion $x(t) = y(t)$ it suffices to ensure that $\dot{v}_2(t) = [DF(s(t)) + \lambda_2 \kappa C] v_2(t)$ is asymptotically stable. This can be established using the Lypaunov function

$$
V(v_2(t)) = v_2(t)^\top \Pi v_2(t), \tag{27}
$$

where $\Pi = \Pi^\top > 0$ is a positive definite matrix of suitable dimension. The derivative of this function along the trajectories of the second equation in (26) yields:

$$
\begin{aligned}
\dot{V}(v_2(t)) &= v_2(t)^\top ([DF(s(t)) + \lambda_2 \kappa C]^\top \Pi \\
&\quad + \Pi [DF(s(t)) + \lambda_2 \kappa C]) v_2(t).
\end{aligned} \tag{28}
$$

The function $\dot{V}(v_2(t))$ is strictly negative if the following inequality holds:

$$[DF(s(t)) + \lambda_2 \kappa C]^\top \Pi + \Pi [DF(s(t)) + \lambda_2 \kappa C] \leq -\tau_2 I_3, \quad (29)$$

for some $\tau_2 > 0$. Therefore, selecting κ such that the aforementioned inequality (29) is satisfied, ensures that the coupled system (1)-(3) will achieve bidirectionally synchronization.

To identically synchronize the systems (16) and (18) with a bidirectional coupling in the form of (13). We start by defining the desired synchronization solution to have the structure of (18), that is:

$$\begin{aligned} \dot{s}_1(t) &= s_2(t), \\ \dot{s}_2(t) &= s_3(t), \\ \dot{s}_3(t) &= a_1 s_3(t) + a_2 s_2(t) + a_3 s_1(t) + NL(s(t)), \\ y_s(t) &= s_1(t). \end{aligned} \quad (30)$$

The errors are defined as:

$$\begin{aligned} e_z(t) = z(t) - s(t). \\ e_x(t) = x(t) - s(t). \end{aligned} \quad (31)$$

Thus:

$$\begin{aligned} \dot{e}_{z1}(t) &= e_{z2}(t) + \mathcal{G}_{zs1}(y_z(t), y_s(t)), \\ \dot{e}_{z2}(t) &= e_{z3}(t) + \mathcal{G}_{zs2}(y_z(t), y_s(t)), \\ \dot{e}_{z3}(t) &= L_F^3 H(x(t)) + L_G L_F^2 H(x(t)) u(t) \\ &\quad -[a_1 s_3(t) + a_2 s_2(t) + a_3 s_1(t) + NL(s(t))] \\ &\quad +\mathcal{G}_{zs3}(y_z(t), y_s(t)), \\ \dot{e}_{x1}(t) &= e_{x2} + \mathcal{G}_{xs1}(y_x(t), y_s(t)), \\ \dot{e}_{x2}(t) &= e_{x3} + \mathcal{G}_{xs2}(y_x(t), y_s(t)), \\ \dot{e}_{x3}(t) &= a_1 x_3(t) + a_2 x_2(t) + a_3 x_1(t) + NL(x(t)), \\ &\quad -[a_1 s_3(t) + a_2 s_2(t) + a_3 s_1(t) + NL(s(t))], \\ &\quad +\mathcal{G}_{xs3}(y_x(t), y_s(t)). \end{aligned} \quad (32)$$

With:

$$\begin{aligned} \mathcal{G}_{zs3}(y_1(t), y_2(t)) &= [0, 0, K_{z3}]^\top C(e_x(t) - e_z(t)), \\ &\quad +\mathcal{F}_z(y_z(t), y_x(t)) \\ \mathcal{G}_{xs3}(y_1(t), y_2(t)) &= [0, 0, K_{x3}]^\top C(e_z(t) - e_x(t)), \\ &\quad +\mathcal{F}_x(y_z(t), y_x(t)). \end{aligned} \quad (33)$$

Then, by property designing the coupling gains K_{z3}, K_{z3} and the nonlinear functions $\mathcal{F}_z(y_z(t), y_x(t))$, or $\mathcal{F}_x(y_z(t), y_x(t))$ we can make the error dynamics asymptotically stable.

SIMULATION RESULTS

In the following subsections, we present simulation results demonstrating the application of the proposed synchronization law to several well-known chaotic systems.

Example 1

For the simulations, we employed a slightly adapted variant of a circuit originally introduced by Sprott (2000), known to exhibit chaotic dynamics for specific parameter values. The structure of the selected circuit is given by:

$$\dddot{x} = -\mu \ddot{x} + \dot{x}^2 - x + \beta u, \quad (34)$$

The system shows chaotic dynamics for $\mu = -2.017$ and $\beta = 0$, with Lyapunov exponents of $(0.055, 0, -2.072)$. By selecting the output $y = x$, it is straightforward to show that equation (34) has full relative degree. Consequently, a pair of Sprott-like circuits can be described as follows:

$$\begin{aligned} \dot{x}_1^j &= x_2^j, \\ \dot{x}_2^j &= x_3^j, \\ \dot{x}_3^j &= -x_1^j + (x_2^j)^2 + \mu_j x_3^j + \beta_j u, \\ y_j &= x_1^j, \end{aligned} \quad (35)$$

for $j = 1, 2$. We define the synchronization error as $e_1 = x_1^1 - x_1^2 = y_1 - y_2$, leading to the following error dynamics:

$$\begin{aligned} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= e_3, \\ \dot{e}_3 &= -e_1 + e_2(e_2 + 2x_2^2) + \mu_1 x_3^1 - \mu_2 x_3^2 + \tilde{\beta} u, \end{aligned} \quad (36)$$

with $\tilde{\beta} = \beta_1 - \beta_2$ and $\beta_1 \neq \beta_2$. The systems described in (35) exhibit chaotic dynamics for parameter values $\mu_j = -2.017$, $\beta_j = 0$, $(j = 1, 2)$ and initial conditions given by $x^j(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

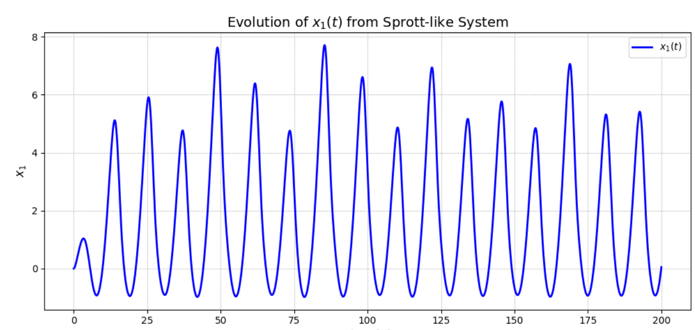

Figure 1 Temporal evolution of state variable x_1 from system (35) for parameter $\mu_1 = -2.017$

It is evident that if $\mu_1 = \mu_2$ and the initial conditions satisfy $x^1(0) = x^2(0)$, the circuits will remain perfectly synchronized. To validate our approach, we consider $\mu_1 = -2.017$ and $\mu_2 = -2.02$. Figure 1 displays the solution for x_1 in system 34. Despite the

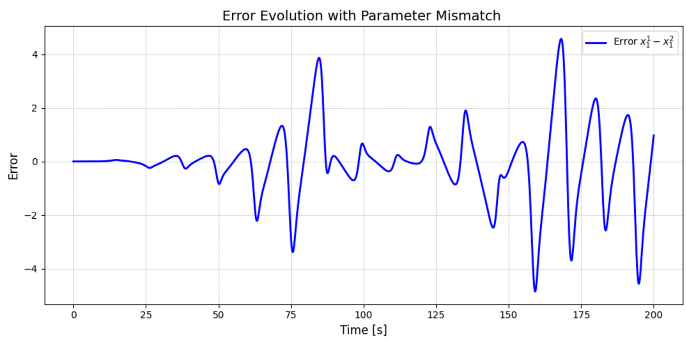

Figure 2 Time evolution for the synchronization error in state x_1 of systems (35) under parameter mismatch conditions.

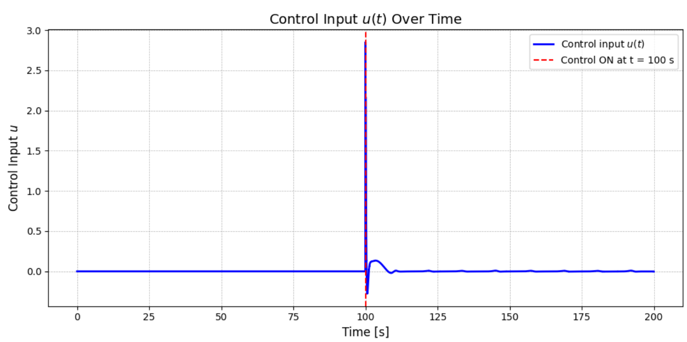

Figure 3 Time response for the synchronization law used for system (35) in Example 1.

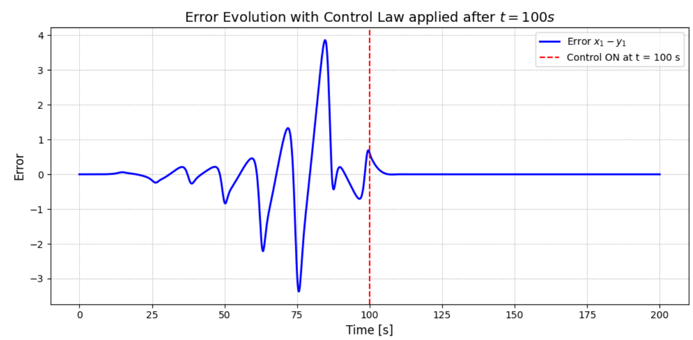

Figure 4 Time response of the synchronization error for system (35) in Example 1.

seemingly minor parameter mismatch, the time responses of the two systems diverge significantly, as illustrated in Figure 2.

To ensure that the system states synchronize, i.e., $e \to 0$, we introduce the following synchronization control law:

$$u = \frac{1}{\beta}[a^T e + e_1 - e_2(e_2 + 2x_2^2) - \mu_1 x_3^1 + \mu_2 x_3^2]. \tag{37}$$

For demonstration purposes, we select $a = \begin{bmatrix} -6 & -11 & -6 \end{bmatrix}^T$, corresponding to desired pole locations at $\lambda = 1, 2, 3$. With this synchronization law, the closed-loop system is expected to exhibit an asymptotically stable equilibrium point.

The synchronization control was activated at $t = 70s$, and its time evolution is illustrated in Fig. 3. The corresponding synchronization error over time is presented in Fig. 4.

Example 2

This example focuses on synchronizing two distinct dynamical systems using the proposed approach: a normal-form Lorenz system and a Sprott system. The Lorenz system is defined by the following dynamics:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 + u, \\ y &= x_1. \end{aligned} \tag{38}$$

It is well known that the system exhibits chaotic behavior for the parameter values $\sigma = 10$, $\rho = 28$, and $\beta = \frac{8}{3}$ Lorenz (2017). System (38) can be expressed in normal form via the coordinate transformation defined in (16), i.e., $x^1 = \varphi(x)$. Accordingly, the new coordinates for the Lorenz system are:

$$\begin{aligned} \varphi_1(x) &= x_1, \\ \varphi_2(x) &= \sigma(x_2 - x_1), \\ \varphi_3(x) &= -\sigma^2(x_2 - x_1) + \sigma(\rho x_1 - x_2 - 20 x_1 x_3), \end{aligned} \tag{39}$$

leading to the following dynamics

$$\begin{aligned} \dot{x}_1^1 &= x_2^1, \\ \dot{x}_2^1 &= x_3^1, \\ \dot{x}_3^1 &= f^1(x^1) + g^1(x^1)u, \\ y &= x_1^1, \end{aligned} \tag{40}$$

for

$$\begin{aligned} f^1(x^1) = {} & (\rho - 1)\sigma x_2^1 - \\ & (\sigma + 1)x_3^1 - (x_1^1)^2(x_2^1 + \sigma x_1^1) - \\ & (\beta x_1^1 - x_2^1) \\ & \left[\frac{\sigma(1-\rho)x_1^1 + (\sigma+1)x_2^1 + x_3^1}{x_1^1}\right], \end{aligned} \tag{41}$$

and

$$g^1(x^1) = -\sigma x_1^1. \tag{42}$$

System (40) is now expressed in normal form. Next, we consider another system to synchronize with:

$$\dddot{x} = -\mu \ddot{x} - \dot{x} + x - x^3 + \beta u. \tag{43}$$

Equation (43) models an electronic circuit described in Sprott (2000), which exhibits chaotic behavior for $\mu = 0.7$ and $\beta = 0$. The corresponding normal form of this system is given by:

$$\begin{aligned}\dot{x}_1^2 &= x_2^2,\\ \dot{x}_2^2 &= x_3^2,\\ \dot{x}_3^2 &= -\mu x_3^2 - x_2^2 + x_1^2 - (x_1^2)^3 + \beta u,\\ y^2 &= x_1^2.\end{aligned} \tag{44}$$

The synchronization error is defined as $e_1 = y^1 - y^2 = x_1^1 - x_1^2$, which leads to the following error dynamics:

$$\begin{aligned}\dot{e}_1 &= e_2,\\ \dot{e}_2 &= e_3,\\ \dot{e}_3 &= f(x^1, x^2) + g(x^1, x^2)u,\end{aligned} \tag{45}$$

where $f(x^1, x^2) = f^1(x^1) - f^2(x^2)$ and $g(x^1, x^2) = \beta - \sigma x_1^1$. Accordingly, a synchronization law can be formulated as follows:

$$u = \frac{a^T e - f(x^1, x^2)}{g(x^1, x^2)}. \tag{46}$$

For demonstration purposes, we apply the synchronization law in $t = 100s$, Fig. 5 shows the time response for e_1.

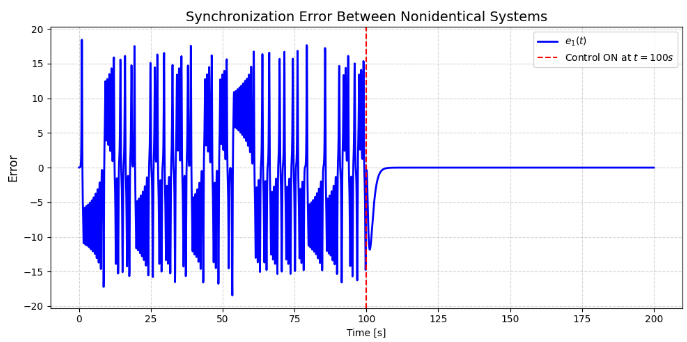

Figure 5 Error e_1 between (40) and (44) with synchronization law applied at $t = 100s$.

CONCLUSION

This work presented a synchronization strategy for bidirectionally coupled, nonidentical chaotic systems using an output feedback design framework. The approach leverages the concept of full relative degree and applies a coordinate transformation to bring the systems into normal form, enabling the design of nonlinear feedback coupling laws. Unlike traditional master-slave schemes, this method addresses mutual interactions and allows for synchronization even under structural differences and parameter mismatches. The proposed method demonstrated successful synchronization of both identical and non-identical systems, including combinations such as Lorenz and Sprott-type models. Numerical simulations confirmed the stability and convergence of the synchronization errors under the proposed coupling scheme. The inclusion of both linear and nonlinear feedback components contributes to the robustness and flexibility of the control design, making it suitable for a wide range of chaotic systems, including those with nonsmooth or piecewise linear dynamics. This contribution provides a promising direction for practical applications in engineering systems where exact matching of models is not feasible. It opens avenues for synchronization in real-world scenarios involving imperfect information, model uncertainties, and heterogeneous components.

Future research will aim to generalize the proposed method to networks with more than two nodes, including those found in complex systems and multi-agent coordination problems. Additional work will focus on the development of robust synchronization schemes by incorporating internal model control and adaptive techniques capable of handling uncertainty and external disturbances. Lastly, experimental validation will be pursued using physical platforms such as electronic circuits or robotic systems, in order to assess the real-time performance and practical feasibility of the approach under realistic conditions. Overall, this work contributes a scalable and effective strategy for the synchronization of nonidentical chaotic systems, with potential applications across a variety of fields in applied sciences and engineering.

Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Availability of data and material

Not applicable.

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