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ON THE MOTION OF THE FRENET VECTORS AND TIMELIKE RULED SURFACES IN THE MINKOWSKI 3-SPACE.

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ABSTRACT

In this paper, we obtained the distribution parameter of a timelike ruled surface generated by a timelike straight line in Frenet trihedron moving along a space-like curve. We show that the timelike ruled surface is developable if and only if the base curve is a helix (inclened curve). Furthermore, some theorems are given for the special cases which the line is being the principal normal and binormal of the başe curve. In addition, it is shown that when the base curve is the same as the striction curve, the ruled surface is not developable.

ÖZET

MINKOWSKI 3-UZAYINDA FRENET EKTÖRLERİNİN HAREKETİ VE TIME-LIKE REGLE YÜZEYLER

Bu çalışmada bir space-like eğri boyunca, Frenet üç yüzlüsünde alınan sabit bir doğrunun hareketiyle oluşan time-like regle yüzeyin dağılma parametresi hesaplandı. Regle yüzeyin dayanak eğrisinin helis olması halinde yüzeyin açılabilir olduğunu gösterdik. Ayrıca, sabit doğrunun; dayanak eğrisinin teğeti, normali, binormali v.s. olması halinde bazı teoremler verdik. Daha fazlası, dayanak eğrisinin, striksiyon çizgisi olması halinde, regle yüzeyinin açılabilir olmadığını gösterdik.

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Introduction

A surface in the 3 dimensional Minkowski space $IR_1^3 = (IR_1^3,\ dx^2 + dy^2 - dz^2)$ is called a timelike surface if the induced metric on the surface is a Lorentz metric (Beem,1981) A ruled surface is a surface swept out by a straight line X moving along a curve α. The various positions of the generating line X are called the rulings of the surface. Such a surface, thus, has a parametrization in ruled form as follows,

$$\varphi(t,v) = \alpha(t) + vX(t)$$

We call α to be the base curve, and X to be the director curve. If the tangent plane is constant along a fixed ruling, then the ruled surface is called a developable surface. The remaining ruled surface are called skew surfaces (Hacısalihoğlu,1997). If there exists a common perpendicular to two preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction

The timelike ruled surface M is given by the parametrization

$$\varphi: I \times IR \longrightarrow IR_1^3$$
$$(t,v) \longrightarrow \varphi(t,v) = \alpha(t) + vX(t)$$

in IR_1^3 where $\alpha: IR \longrightarrow IR_1^3$ is a differentiable spacelike curve paremetrized by its arc-length in IR_1^3 that is, $(<\alpha'(t),\alpha'(t)>= 1)$ and X(t) is the director vector of the director curve such that X is orthogonal to the tangent vector field T of the base curve α. $\{T,\overline{N},X\}$ is an orthonormal frame field along α in IR_1^3 where $\overline{N}$ is the normal vector field of M along α, $\overline{N}$ and T are space like and X is timelike. Thus

$$<T,T>=<\overline{N},\overline{N}>=1,\ <X,X>=-1$$

The curve of striction of a skew timelike surface is given by

$$\overline{\alpha}(t) = \alpha(t) - \frac{<T,D_TX>}{<D_TX,D_TX>}X(t) \tag{1.1}$$

and $\overline{\alpha}$ is a spacelike curve (Hacısalihoğlu,1997). Let P_x be distribution parameter of timelike ruled surface, then

$$P_x = -\frac{\det(T,X,D_TX)}{<D_TX,D_TX>} \tag{1.2}$$

(Hacısalihoğlu,1997). Where D is Levi-Civita connection on IR_1^3

Theorem 1.1. A timelike ruled surface is a developable surface if and only if the distribution parameter of the timelike ruled surface is zero (Hacısalihoğlu,1997).

2. The Frenet Vectors for Spacelike Curves

If the principal vector field N of a spacelike curve $\alpha(t)$ is timelike and the binormal vector field B is spacelike, then we have the following Frenet formula along $\alpha(t)$:

$$\begin{aligned} \alpha'(t) &= T \\ D_T T &= \frac{dT}{dt} = k_1(t)N \\ D_T N &= \frac{dN}{dt} = k_1(t)T + k_2(t)B \\ D_T B &= \frac{dB}{dt} = k_2(t)N \end{aligned} \tag{2.1}$$

(Ikama ,1985).

If the principal vector field N of a spacelike curve $\alpha(t)$ is spacelike and the binormal vector field B is timelike, then we have the following Frenet formula along $\alpha(t)$:

$$\begin{aligned} D_T T &= k_1 N \\ D_T N &= -k_1 T + k_2 B \\ D_T B &= k_2 N \end{aligned} \tag{2.2}$$

(Uğurlu ,1996).

3. One-Paremeter Spatial Motion in IR_1^3

Let $\alpha: I \longrightarrow IR_1^3$ be a spacelike curve and {T,N,B} be Frenet vector where T,N and B are the tangent, Principal normal and binormal vectors of the curve, respectively. T is spacelike and N or B is timelike vectors.

The two coordinate sytems {0;T,N,B} and $\{0'; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are orthogonal coordinate sytems in IR_1^3 which represent the moving space H and the fixed space H', respectively. Let us express the displacaments (H / H') of H with respect to H'. During the one paremeter spatial motion H / H', each fixed line X of the moving space H, generates, in generally, a timelike ruled surface in the fixed space H'.

Let X be a unit timelike vector and fixed.Thus

$$X \in Sp\{T, N, B\} \text{ and } X = x_1 T + x_2 N + x_3 B$$

such that

$$< X, X >= -1 \quad \text{and all } x_i,\ 1 \le i \le 3, \text{ are fixed} \tag{3.1.}$$

We can obtain the distribution parameter of the timelike ruled surface generated by line X of the moving space H. Let N be a timelike vector. T and B are spacelike. From (3.1)

$$D_T X = x_1 T' + x_2 N' + x_3 B' \tag{3.2}$$

Substituting (2.1) into (3.2)

$$D_T X = x_2 k_1 T + (x_1 k_1 + x_3 k_2) N + x_2 k_2 B$$

From (1.2) we obtain

$$P_x = -\frac{\det(T, X, D_T X)}{< D_T X, D_T X >}$$

$$= \frac{-k_2 x_2^2 + k_2 x_3^2 + x_1 k_1 x_3}{k_1^2 x_2^2 - (k_1 x_1 + k_2 x_3)^2 + k_2^2 x_2^2}, \quad x_1^2 - x_2^2 + x_3^2 = -1 \tag{3.3}$$

$$P_x = \frac{x_1 k_1 x_3 - k_2 (1 + x_1^2)}{x_2^2 (k_1^2 + k_2^2) - (x_1 k_1 + x_3 k_2)^2}$$

Let B be timelike vector then T and N are spacelike vectors. Substitung (2.2) into (3.2)

$$D_T X = -k_1 x_2 T + (x_1 k_1 + x_3 k_2) N + x_2 k_2 B$$

from (1.2) we obtain

$$P_x = \frac{x_1 k_1 x_3 + k_2 (x_3^2 - x_2^2)}{x_2^2 (k_1^2 - k_2^2) + (x_1 k_1 + x_3 k_2)^2}, \ x_1^2 + x_2^2 - x_3^2 = -1 \tag{3.4}$$

The ruled surface developebale if and only if P_x is zero.(Hacısalihoğlu,1997)

From (3.3) (or (3.4)) $P_x = 0$ if and only if $\dfrac{k_1}{k_2} = \dfrac{x_2^2 - x_3^2}{x_1 x_3}$.

Hence we state the following theorem

Theorem 3.1: During the one-parameter spatial motion H / H' the timelike ruled surface in the fixed space H' generated by a fixed line X of the moving space H is developable if and only if $\alpha(t)$ is a helix such that the harmonic curvature h of the base curve $\alpha(t)$ satisfies the equality

$$h = \frac{k_1}{k_2} = \frac{x_2^2 - x_3^2}{x_1 x_3}$$

4. Special Cases

4.1 The Case X=T

The case can not be hold. Because X is timelike and T is spacelike

4.2 The Case X=N (Timelike):

In this case, $x_1 = x_3 = 0$ and $x_2 = 1$

Thus from (3.3)

$$P_N = \frac{-k_2}{k_1^2 + k_2^2} \tag{4.1}$$

4.3 The Case X=B (Timelike)

In this case, $x_1 = x_2 = 0$ and $x_3 = 1$.Thus from (3.4)

$$P_B = \frac{k_2}{k_2^2} = \frac{-1}{k_2} \tag{4.2}$$

By (4.1) and (4.2) we can give the relation between P_B and P_N

$$\left(\frac{k_1}{k_2}\right)^2 = \left(-1 + \frac{P_B}{P_N}\right) \tag{4.3}$$

Hence the following theorem is hold.

Theoerem 4.1 During the one-parameter spatial motion H / H' the curve of base $\alpha : I \longrightarrow IR_1^3$ is helix (inclened curve) if and only if P_N / P_B is constant, where P_N and P_B are the distribution parameters of the surfaces genereted by the principal normal and binormal.

4.4. The Case, X is in the normal plane.

In this case x_1 is zero. The timelike surface is developable surface since from (3.3) (or(3.4))

$$k_2(x_3^2 - x_2^2) = 0 \qquad (4.4)$$

$X = x_2 N + x_3 B$ and if N is timelike then $-x_2^2 + x_3^2 = -1$ if B is timelike then $x_2^2 - x_3^2 = -1$. Hence, from (4.4) $k_2 = 0$. Thus $\alpha(t)$ is a planer curve. i.e $\alpha(t)$ is a Lorentzian circle in osculating plane. Hence the following theorem is hold.

Theorem 4.2 During the one-parameter spatial motion H/H' the timelike ruled surface in the fixed space H' generated by a fixed line X in the normal plane of the base curve $\alpha(t)$ in H is developable if and only if $\alpha(t)$ is a Lorentzian circle in osculating plane.

4.5. The Case, X is in the Osculating plane.

In this case x_3 is zero. Thus the timelike ruled surface is develapable since from (3.3) (or(3.4))

$$k_2 x_2^2 = 0 \qquad (4.5)$$

if $x_2 = 0$ then $X = x_1 T$, $x_1^2 = -1$ which is not possible. Thus $k_2 = 0$. Hence $\alpha(t)$ is a planar curve, i.e, $\alpha(t)$ is a Lorentzian circle in osculating plane. Therefore Theorem 4.2 can be restated as the following

4.6. The Case X is in the rectifying plane.

In this case x_2 is zero. From (3.3) (or(3.4)) for distribution parameter we can write that

$$P_X = \mp \frac{x_3}{k_1 x_1 + k_2 x_3} \qquad (4.6)$$

Thus,$P_X = 0$ if and only if x_3 is zero. Then $X = x_1 T$ and $x_1^2 = -1$ which is not possible. Thus the timelike ruled surface is not developable. Hence the following theorems can be stated.

Theorem 4.3 During the one parameter spatial motion H/H' the timelike ruled surface in the fixed space H' generated by a fixed line X in the osculator plane of the base curve α in H is developable iff α is a Lorentzian circle in osculator plane.

Theorem 4.4 During the one-parameter spatial motion H/H' the distribution parameters of the timelike ruled surfaces in the fixed space H' generated by a fixed line X in the rectifying plane of base curve in H are the same if and only if the base curves have Bertrand couples.

Proof. If $\alpha(s)$ has a Bertrand couple then

$$x_1k_1+x_3k_2 = \text{constant}$$

(Hacısalihoğlu,1994).Thus from (4.6) P_X is constant.

If P_X is constant then from (4.6), $x_1k_1+x_3k_2$ is constant. Hence $\alpha(s)$ has a Bertrand couple.

4.7. The Case, the curve $\alpha(t)$ of base is the striction curve $\overline{\alpha}(t)$

In this case, from (1.1), $<T, D_TX>=0$

$$D_TX = \mp k_1x_2T+(x_1k_1+x_3k_2)N+x_2k_2B$$

we have $k_1x_2=0$, $k_1\neq 0$ x_2 is zero.

Hence the following theorem is hold.

Theorem 4.5. If the curve $\alpha(t)$ of base is the same as the striction curve $\overline{\alpha}(t)$ then the director curves of the timelike ruled surfaces lies in the rectifying plane of $\overline{\alpha}(t)$.

If $\alpha(t)=\overline{\alpha}(t)$ then X(t) lies in the rectifying plane of α. Thus theorem (4.3) and (4.4) can be repeated for the striction curve $\overline{\alpha}(t)$.Thus, in the case of that $\alpha(t)=\overline{\alpha}(t)$ we can say that from (4.3) the timelike ruled surface is not developable.

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