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Research Article

Reliability estimation using expected and hierarchical Bayesian approach under progressive censoring with engineering application

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Abstract

In this present work, we propose the expected Bayesian (E- Bayesian) and hierarchical Bayesian (H- Bayesian) approaches to estimate the shape parameter and hazard rate under progressive Type-II censoring scheme. These estimates have been obtained using gamma priors based on various loss functions such as squared error, entropy, weighted balance, and minimum expected loss functions. Properties of E- and H- Bayesian estimates have been discussed. Further the relations between E- and H- Bayesian estimates based on different loss functions are provided. An investigation is carried out using Monte Carlo simulation to evaluate the effectiveness of the suggested estimators. The simulation provides a quantitative assessment of the estimates' accuracy and efficiency under various conditions by comparing them in terms of mean squared error. Additionally, the failure data set for ball bearings is examined to offer real-world examples of how the suggested estimations may be used and performed.

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1. Introduction

In the field of probability theory and statistical modeling, the choice of an appropriate probability distribution is crucial to accurately describing and analyzing real-world phenomena. One such distribution that has gained attention in recent years is the generalized inverted exponential distribution (GIED). Abouammoh and Alshingiti [1] are credited with pioneering the introduction of the GIED. This distribution provides a flexible and versatile framework for modeling a wide range of continuous random variables with skewed and heavy-tailed characteristics. GIED has emerged as a valuable tool in various areas of research, including engineering, economics, environmental sciences, and reliability analysis. It is a versatile distribution that extends the traditional exponential distribution,

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offering additional flexibility in capturing diverse real-world phenomena that exhibit complex behaviors such as asymmetry and heavy tails.
The exponential distribution is widely used to model events with constant hazard rates, such as radioactive decay and waiting times between events in a Poisson process. However, many real-world phenomena do not exhibit constant hazard rates and require more flexible distributions to accurately capture their statistical properties. The GIED provides an elegant solution by allowing a wide range of hazard rate shapes, including increasing, decreasing, bathtub-shaped, and multimodal hazard functions. The probability density function (PDF) of GIED is defined as

$$f(x;\alpha,\lambda) = \frac{\alpha\lambda}{x^2} e^{\frac{-\lambda}{x}} (1 - e^{\frac{-\lambda}{x}})^{\alpha-1};\ x,\alpha,\lambda > 0, \tag{1.1}$$

where α, and λ are the shape and scale parameters, respectively. The cumulative distribution function (CDF) of GIED is given by

$$F(x;\alpha,\lambda) = 1 - (1 - e^{\frac{-\lambda}{x}})^{\alpha},\ x,\alpha,\lambda > 0. \tag{1.2}$$

The reliability function of the GIED is obtained as

$$R(t) = (1 - e^{\frac{-\lambda}{t}})^{\alpha},\ t > 0, \alpha, \lambda > 0. \tag{1.3}$$

The hazard rate of the GIED is given as follows:

$$H(t) = \frac{\alpha\lambda}{t^2(e^{\frac{\lambda}{t}} - 1)}, t > 0, \alpha, \lambda > 0. \tag{1.4}$$

In recent years, GIED has gained a lot of attention from several researchers. [2] addressed the issue of reliability estimation in GIED using type II progressive censoring scheme (PCS). Dey et al. [3] focused on the estimation and prediction aspects of GIED under progressively censored data. In the study conducted by Dube et al. [4], they focused on GIED under progressive censorship of first-failure. Ahmed [5] developed estimation and prediction techniques for GIED based on progressively first-failure-censored data. Bayesian estimation involves updating prior beliefs with observed data to obtain a posterior distribution. In expected Bayesian (E-Bayesian), we impose certain restrictions on our hyperparameters, which provides more stable and reliable estimates, especially in situations where there is uncertainty about the appropriate prior specification. The hierarchical Bayesian (H-Bayesian) method is considered to be more robust because it involves a two-stage process to construct the prior distribution. However, a drawback of this approach lies in the intricate integrals within the estimator expression. These integrals often necessitate resolution through numerous numerical approximation techniques, making calculations tedious and time-consuming.

In recent past, E-Bayesian and H-Bayesian estimation have been considered for the parameters and reliability characteristics under different censored data. Yaghoobzadeh [6] conducted research on the E-Bayesian and H-Bayesian estimation of a scalar parameter in the Gompertz distribution under Type II censoring schemes based on fuzzy data. Their study focused on developing estimation techniques that account for uncertainty and imprecision in the data. Nassar et al. [7] considered the E-Bayesian estimation for the simple step–stress model based on type-II censoring scheme. Nagy et al. [7] provided an E-Bayesian estimation for an exponential model based on simple step stress with Type I hybrid censored data. They developed estimation methods to obtain reliable parameter estimates considering the specific censoring scheme used. Balakrishnan and Sandhu [8] discussed the best linear unbiased estimation (BLUE) and maximum likelihood estimation (MLE) techniques for exponential distributions under general progressive Type II censored samples. They provided statistical methodologies for parameter estimation under this type of censoring scenario. They developed methodologies to analyze this specific censoring scheme and its impact on the estimation of the distribution's parameters. Mohie El-Din et al. [9] proposed an E-Bayesian estimation approach for the parameters and hazard function of the Gompertz distribution using Type II PCS. Their research aimed to provide reliable estimation techniques for this specific censoring scheme and demonstrated the application of these methods in practical scenarios. Nassar et al. [10] conducted research on E-Bayesian estimation and associated properties of the simple step-stress model for the exponential distribution based on type-II censoring. Their study focused on developing estimation methods that account for the censoring scheme employed and investigating the properties of the estimated parameters.

Evaluations of quality utilizing lifetime data might run into problems in practice due to constraints in time, cost, and resources. Censorship schemes (CSs) have been used as a tool to handle these issues. Due to its adaptability and the variety of data it delivers, the type II PCS approach has become the standard. According to PCS, in an experiment having n number of experimental units, at the time of first failure R_1 surviving items are randomly removed from the remaining n-1 surviving items. In the experiment, the process continues until the m^{th} failure time at which all the remaining $R_m = n - m - \sum_i^{m-1} R_i$ surviving items are removed from the experiment at random. We denote the m ordered observed failure times by $(X_{1:m:n}^{(R_1,R_2,...,R_m)}, X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)})$ and call them type II progressively censored samples of size m using a sample of size n with Type PCS $(R_1, R_2, ..., R_m), m \leq n$. Now, assuming that the eliminated times follow an absolutely continuous distribution function (CDF) F(.) with probability density function (PDF) f(.), the joint probability density function based on Type II PCS with failure times $(X_{1:m:n}^R, X_{2:m:n}^R, ..., X_{m:m:n}^R)$, is given by

$$f_{(X_{1:m:n},X_{2:m:n},...,X_{m:m:n})}(x_1, x_2, ..., x_m) = D_{n,m-1} \prod_{i=1}^{m} f(x_i, \theta) \Big[(1 - F(x_i, \theta)) \Big]^{R_i}, \quad (1.5)$$

for $0 < x_1 < x_2 < ... < x_m < \infty$ where $D_{m,n-1} = (n - R_1 - 1)(n - R_1 - R_2 - 1)...(n - R_1 - R_2... - R_{m-1} - m + 1)$.

For more details on PCS, see Balakrishnan and Cramer [11]. The use of type II PCS is often motivated by practical considerations, such as cost constraints or limited resources, where it may not be feasible or efficient to monitor subjects continuously or until failure occurs. Instead, observations are taken periodically, and the study ends when a predetermined number of failures have been observed or a specific time period has elapsed. Analyzing data with Type II PCS poses challenges as the censoring mechanism depends on the observed failure times.

The time intervals between successive observations may vary, and the exact failure times beyond the current interval are unknown. This requires the development of specialized statistical methods for estimation and inference. In recent years, type II PCS has gained a lot of attention in reliability and survival analysis. For this instance, one may refer to Kim et al. [12], Kotb and Raqab [13], Maiti and Kayal [14], and Elshahhat et al. [15]. This effort is driven by the following essential goals: (a) recent research has demonstrated that E- and H-Bayesian estimates are more effective than both classical and Bayesian approaches, (b) to the best of our knowledge, no article can be found in the literature having both E- and H-Bayesian estimation for any lifetime distribution under progressive censoring scheme.

Motivated by the usefulness of PCS and E- and H-Bayesian estimation in reliability applications, this study has been considered to propose the E- and H-Bayesian estimation for GIED under type II PCS. Our main objective in this paper is to estimate the shape parameter and the hazard rate of GIED based on progressive censoring using E-Bayesian and H-Bayesian approach with four different loss functions. To obtain the estimates, a Monte Carlo simulation study was conducted. For showing the applicability in the real world phenomenon, ball bearing data have been analyzed for the proposed estimates. The work seeks to establish a guideline for selecting the best method of estimation if the item/subgroup quality characteristic follows GIED, which we believe would be of profound interest to applied statisticians and quality control engineers.

The structure of the article is as follows: Section 1 provides an introduction, outlining the research problem and objectives. Section 2 focuses on maximum likelihood estimation (MLE) and Bayesian estimation techniques for the shape parameter and hazard rate, considering different loss functions. Section 3 presents the derivation of E-Bayesian estimators for the shape parameter, also under different loss functions. In Section 4, H-Bayesian estimators are obtained, again considering different loss functions. The properties of the E- and H-Bayesian estimators are discussed in Section 5. To assess the performance of the estimators, a simulation study is conducted in Section 6, where different loss functions are considered and the estimators are compared using R software. Section 7 presents the analysis of a real-life dataset to demonstrate the practical application of the proposed estimators. Finally, in Section 8, the article concludes by summarizing the key findings and implications of the study.

2. Maximum Likelihood Estimation

In this section, the parameters of GIED will be estimated using the MLE method. The likelihood function is defined as

$$L \propto \prod_{i=1}^{m} \left(\frac{\alpha\lambda}{x_i^2} e^{-\frac{\lambda}{x_i}} (1 - e^{-\frac{\lambda}{x_{i=1}}})^{\alpha-1} \right) \left((1 - e^{-\frac{\lambda}{x_i}})^{\alpha} \right)^{R_i}. \tag{2.1}$$

Taking log on (2.1), we get the log-liklihood as

$$\log L = m\log(\alpha) + m\log(\lambda) - \sum_{i=1}^{m} \frac{\lambda}{x_i} + (\alpha-1)\sum_{i=1}^{m} \log(1 - e^{-\frac{\lambda}{x_i}}) + \sum_{i=1}^{m} R_i \log(1 - e^{-\frac{\lambda}{x_i}})^{\alpha}. \tag{2.2}$$

Differentiating (2.2) with respect to α and equating to zero, the MLE of α will be obtained as

$$\hat{\alpha} = -\frac{m}{\sum_{i=1}^{m}(1 + R_i)\log(1 - e^{-\frac{\lambda}{x_i}})}. \tag{2.3}$$

2.1. Bayesian Estimation

In this section, we developed Bayesian estimates for the shape parameter α and the hazard rate of the GIED using various loss functions. These loss functions include the squared error loss function (SELF), entropy loss function (ELF), weighted balance loss function (WBLF), and minimum expected loss function (MELF).

We consider gamma distribution as prior distribution of α as it is a conjugate prior. The gamma prior distribution with shape and scale parameter 'a' and 'b', respectively has the following PDF

$$g(\alpha|a, b) = \frac{b^a \alpha^{a-1}}{\Gamma(a)} e^{-b\alpha}; \alpha, a, b > 0. \tag{2.4}$$

Here, the hyperparameters a and b have been chosen based on a formula, which can be found in Dutta and Kayal [16]. Using (2.1) and (2.4), the posterior distribution can be expressed as

$$\begin{aligned} P_1(\alpha|x) &= \frac{L(\alpha|x)g(\alpha|a,b)}{\int_0^\infty L(\alpha|x)g(\alpha|a,b)d\alpha} \\ &= \frac{\alpha^{(m+a)-1}\exp\left[-\alpha\left(b-(1+R_i)\sum_{i=1}^m log\left(1-e^{\frac{-\lambda}{x_i}}\right)\right)\right]}{\int_0^\infty \alpha^{(m+a)-1}\exp\left[-\alpha\left(b-(1+R_i)\sum_{i=1}^m log\left(1-e^{-\frac{\lambda}{x_i}}\right)\right)\right]d\alpha} \\ &= \frac{T^{m+a}\alpha^{(m+a)-1}\exp(-\alpha T)}{\Gamma(m+a)}, \end{aligned}$$

where $T = b + P$, $P = -\sum_{i=1}^m (1+R_i)log\left(1-e^{-\frac{\lambda}{x_i}}\right)$.

2.2. Bayesian estimate with SELF

The squared Error loss function (SELF) is defined as

$$L(\alpha) = c(\alpha - \hat{\alpha})^2, \tag{2.5}$$

where $\hat{\alpha}$ is an estimator of α. Then the associated Bayes estimator of α is given by

$$\hat{\alpha}_{BS} = E[\alpha|x], \tag{2.6}$$

provided that $E[\alpha|x]$ exists and finite.

Theorem 2.1. *For a sample $\underline{x} = \{x_1, \cdots, x_n\}$, the Bayes estimator of α under SELF is given by*

$$\hat{\alpha}_{BS} = \frac{m+a}{T}.$$

Proof. By using SELF, the Bayes estimator is obtained by

$$\hat{\alpha}_{BS} = E[\alpha|x] = \int_0^\infty \alpha.P_1(\alpha|x)d\alpha = \frac{m+a}{T}. \tag{2.7}$$

□

The Bayes estimate of $H(t)$ for given λ is defined as

$$\hat{H}_{BS}(t) = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\hat{\alpha}_{BS}. \tag{2.8}$$

2.3. Bayesian estimate with ELF

Dey et al. [17] discussed the ELF of the form

$$L(\alpha) = \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right], \tag{2.9}$$

where $\hat{\alpha}$ is an estimator of α. Then the associated Bayes estimate of α is given by

$$\hat{\alpha}_{BE} = E[\alpha^{-1}|x]^{-1}, \tag{2.10}$$

provided that $E[\alpha^{-1}|x]^{-1}$ exists and finite.

Theorem 2.2. *For a sample $\underline{x} = \{x_1, \cdots, x_n\}$, the Bayes estimate of α under ELF is given by*

$$\hat{\alpha}_{BE} = \frac{m+a-1}{T}.$$

Proof. By using ELF, the Bayes estimator is given by

$$\hat{\alpha}_{BE} = E[\alpha^{-1}|x]^{-1} = \left[\int_0^\infty \frac{1}{\alpha}.P_1(\alpha|x)d\alpha\right]^{-1} = \frac{m+a-1}{T}. \tag{2.11}$$

□

The Bayes estimate of $H(t)$ for given λ under ELF is given as

$$\hat{H}_{BE}(t) = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\hat{\alpha}_{BE}. \tag{2.12}$$

2.4. Bayesian estimate with WBLF

The WBLF can be expressed as (see Nasir and Aslam [18])

$$L(\alpha) = \left(\frac{\alpha-\hat{\alpha}}{\hat{\alpha}}\right)^2, \tag{2.13}$$

and the associated Bayes estimate of α is given by

$$\hat{\alpha}_{BW} = \frac{E[\alpha^2|x]}{E[\alpha|x]}, \tag{2.14}$$

provided that $E[\alpha^2|x]$ and $E[\alpha|x]$ exist and finite.

Theorem 2.3. *For a sample* $\underline{x} = \{x_1, \cdots, x_n\}$*, the Bayes estimate of* α *under WBLF is given by*

$$\hat{\alpha}_{BW} = \frac{m+a+1}{T}.$$

Proof. Here, we have

$$E[\alpha^2|x] = \int_0^\infty \alpha^2.P_1(\alpha|x)d\alpha = \frac{(m+a)(m+a+1)}{T^2} \quad \text{and} \quad E[\alpha|x] = \frac{m+a}{T}.$$

By using MELF, the Bayes estimate is obtained as

$$\hat{\alpha}_{BW} = \frac{(m+a+1)}{T}. \tag{2.15}$$

□

The Bayes estimate of $H(t)$ for given λ under WBLF is given as

$$\hat{H}_{BW}(t) = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\hat{\alpha}_{BW}. \tag{2.16}$$

2.4.1. Bayesian estimate with MELF.

Tummala and Sathe [19] defined the MELF as follows:

$$L(\alpha) = \frac{(\hat{\alpha}-\alpha)^2}{\alpha^2}, \tag{2.17}$$

where $\hat{\alpha}$ is an estimator of α . The Bayes estimate can be obtained as:

$$\hat{\alpha}_{BM} = \frac{E[\alpha^{-1}|x]}{E[\alpha^{-2}|x]}, \tag{2.18}$$

provided that $E[\alpha^{-1}|x]$ and $E[\alpha^{-2}|x]$ exist and finite.

Theorem 2.4. *For a sample* $\underline{x} = \{x_1, \cdots, x_n\}$*, the Bayes estimate of* α *under MELF is given by*

$$\hat{\alpha}_{BM} = \frac{m+a-2}{T}.$$

Proof. Here, we have

$$E[\alpha^{-2}|x] = \int_0^\infty \frac{1}{\alpha^2}.P_1(\alpha|x)d\alpha = \frac{T^{m+a}}{\Gamma(m+a)}.\frac{\Gamma(m+a-2)}{T^{m+a-2}}, \quad \text{and} \quad E[\alpha^{-1}|x] = \frac{T}{m+a-1}.$$

By using MELF, the Bayes estimate is obtained as

$$\hat{\alpha}_{BM} = \frac{(m+a-2)}{T}. \tag{2.19}$$

□

The Bayes estimate of $H(t)$ for given λ under MELF is given as

$$\hat{H}_{BM}(t) = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\hat{\alpha}_{BM}. \tag{2.20}$$

3. E-Bayesian Estimation

According to Han [20], it is recommended to select the prior parameters 'a' and 'b' in such a way that the prior distribution in (2.4) is a decreasing function of α.

$$\frac{d}{d\alpha}g(\alpha|a,b) = \frac{(a^b)}{\Gamma(a)}\alpha^{(b-2)}e^{-\alpha a}((b-1)-a\alpha). \tag{3.1}$$

Specifically, it is suggested that for $0 < a < 1$ and $b > 0$, the prior distribution becomes a decreasing function of α. The E-Bayesian estimate of α is given by

$$\hat{\alpha}_{EB} = \int_0^1\int_0^k \hat{\alpha}_B\pi(\alpha,a,b)\ dadb, \tag{3.2}$$

The E-Bayesian estimate of α is then given by the integral of the Bayesian estimate of α, denoted as $\hat{\alpha}_B$, multiplied by the prior distribution, over suitable domains for the hyper parameters 'a' and 'b'. The domains for the first and second integrals correspond to the regions where the prior density function is a decreasing function of α. The specific distributions chosen for the hyper parameters 'a' and 'b' are as follows:

$$\pi_1(a,b) = \frac{2(k-b)}{k^2}, \quad 0 < a < 1, 0 < b < k, \tag{3.3}$$

$$\pi_2(a,b) = \frac{1}{k}, \quad 0 < a < 1, 0 < b < k, \tag{3.4}$$

and

$$\pi_3(a,b) = \frac{2b}{k^2}, \quad 0 < a < 1, 0 < b < k. \tag{3.5}$$

3.1. E-Bayesian estimation under SELF

Theorem 3.1. *For a sample $\underline{x} = \{x_1, \cdots, x_n\}$ of GIED, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of α under SELF are given by*

$$\hat{\alpha}_{EBS_1} = \frac{(2m+1)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right)-k\right],$$

$$\hat{\alpha}_{EBS_2} = \frac{(2m+1)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right],$$

$$\hat{\alpha}_{EBS_3} = \frac{(2m+1)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right)+k\right],$$

where $T = b + P$.

Proof. E-Bayes estimate of α with respect to (3.3), will be denoted by $\hat{\alpha}_{EBS_1}$ and can be expressed as

$$\begin{aligned}\hat{\alpha}_{EBS_1} &= \int_0^1 \int_0^k \hat{\alpha}_{BS} * \pi_1(a,b)\ dadb \\ &= \frac{2}{k^2}\int_0^1 \int_0^k \frac{n+a}{\left[b - \sum_{i=1}^{m}(1+R_i)log\left(1-e^{\frac{-\lambda}{x_i}}\right)\right]}.(k-b)\ dadb \\ &= \frac{(2m+1)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right]. \qquad (3.6)\end{aligned}$$

Similarly, the estimates $\hat{\alpha}_{EBS_2}$ and $\hat{\alpha}_{EBS_3}$ can be obtained easily. So, the proofs are omitted from here to maintain the brevity. □

Theorem 3.2. *For a time t, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of $H(t)$ under SELF are given by*

$$\begin{aligned}\hat{H}_{EBS_1} &= \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m+1)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right], \\ \hat{H}_{EBS_2} &= \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m+1)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right], \\ \hat{H}_{EBS_3} &= \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m+1)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right) + k\right],\end{aligned}$$

where $T = b + P$.

Proof. The proof is straightforward, so it is omitted from here. □

3.2. E-Bayesian estimation under ELF

Theorem 3.3. *For a sample $\underline{x} = \{x_1, \cdots, x_n\}$ of GIED, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of α under ELF are given by*

$$\begin{aligned}\hat{\alpha}_{EBE_1} &= \frac{(2m-1)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right], \\ \hat{\alpha}_{EBE_2} &= \frac{(2m-1)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right], \\ \hat{\alpha}_{EBE_3} &= \frac{(2m-1)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right) + k\right],\end{aligned}$$

where $T = b + P$.

Proof. E-Bayes estimate of α with respect to (3.3), will be denoted by $\hat{\alpha}_{EBE_1}$ can be expressed as

$$\begin{aligned}\hat{\alpha}_{EBE_1} &= \int_0^1 \int_0^k \hat{\alpha}_{BE} * \pi_1(a,b)\ dadb \\ &= \frac{2}{k^2}\int_0^1 \int_0^k \frac{m+a-1}{\left[b - \sum_{i=1}^{m}(1+R_i)log\left(1-e^{\frac{-\lambda}{x_i}}\right)\right]}.(k-b)\ dadb \\ &= \frac{(2m-1)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right]. \qquad (3.7)\end{aligned}$$

Similarly, the estimates $\hat{\alpha}_{EBE_2}$ and $\hat{\alpha}_{EBE_3}$ can be obtained easily. So, the proofs are omitted from here to maintain the brevity. □

Theorem 3.4. *For a time t, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of $H(t)$ under ELF are given by*

$$\hat{H}_{EBE_1} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m-1)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right)-k\right],$$
$$\hat{H}_{EBE_2} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m-1)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right],$$
$$\hat{H}_{EBE_3} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m-1)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right)+k\right],$$

where $T = b + P$.

Proof. The proof is straightforward, so it is omitted from here. □

3.3. E-Bayesian estimation under WBLF

Theorem 3.5. *For a sample $\underline{x} = \{x_1, \cdots, x_n\}$ of GIED, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of α under WBLF are given by*

$$\hat{\alpha}_{EBW_1} = \frac{(2m+3)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right)-k\right],$$
$$\hat{\alpha}_{EBW_2} = \frac{(2m+3)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right],$$
$$\hat{\alpha}_{EBW_3} = \frac{(2m+3)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right)+k\right],$$

where $T = b + P$.

Proof. E-Bayesian estimate of α under the WBLF with respect to (3.3), will be denoted by $\hat{\alpha}_{EBW_1}$ and can be expressed as

$$\begin{aligned}
\hat{\alpha}_{EBW_1} &= \int_0^1\int_0^k \hat{\alpha}_{BW}\pi_1(a,b)\ dadb \\
&= \frac{2}{k^2}\int_0^1\int_0^k \frac{m+a+1}{\left[b-\sum_{i=1}^m(1+R_i)log\left(1-e^{\frac{-\lambda}{x_i}}\right)\right]}.(k-b)\ dadb \\
&= \frac{2}{q^2}\int_0^1(n+a+1)da\int_0^k \frac{(k-b)}{b-\sum_{i=1}^m(1+R_i)log\left(1-e^{\frac{-\lambda}{x_i}}\right)}\ db \\
&= \frac{(2m+3)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right)-k\right]. \qquad (3.8)
\end{aligned}$$

Similarly, the estimates $\hat{\alpha}_{EBW_2}$ and $\hat{\alpha}_{EBW_3}$ can be obtained easily. So, the proofs are omitted from here to maintain the brevity. □

Theorem 3.6. *For a time t, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of $H(t)$ under WBLF are given by*

$$\hat{H}_{EBW_1} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m+3)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right)-k\right],$$
$$\hat{H}_{EBW_2} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m+3)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right],$$
$$\hat{H}_{EBW_3} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m+3)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right)+k\right],$$

where $T = b + P$.

Proof. The proof is straightforward, so it is omitted from here. □

3.4. E-Bayesian estimation under MELF

Theorem 3.7. *For a sample $\underline{x} = \{x_1, \cdots, x_n\}$ of GIED, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of α under MELF are given by*

$$\hat{\alpha}_{EBM_1} = \frac{(2m-3)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right],$$
$$\hat{\alpha}_{EBM_2} = \frac{(2m-3)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right],$$
$$\hat{\alpha}_{EBM_3} = \frac{(2m-3)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right) + k\right],$$

where $T = b + P$.

Proof. E-Bayesian estimate of α with respect to (3.3), will be denoted by $\hat{\alpha}_{EBM_1}$ and is given by

$$\begin{aligned}\hat{\alpha}_{EBM_1} &= \int_0^1\int_0^k \hat{\alpha}_{BM}\pi_1(a,b)\; dadb \\ &= \frac{2}{k^2}\int_0^1\int_0^k \frac{m+a-2}{\left[b - \sum_{i=1}^m (1+R_i)log\left(1 - e^{\frac{-\lambda}{x_i}}\right)\right]}.(k-b)\; dadb \\ &= \frac{(2m-3)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right]. \end{aligned} \tag{3.9}$$

Similarly, the estimates $\hat{\alpha}_{EBW_2}$ and $\hat{\alpha}_{EBW_3}$ can be obtained easily. So, the proofs are omitted from here to maintain the brevity. □

Theorem 3.8. *For a time t, using the priors given in (3.3), (3.4) and (3.5), the E-Bayesian estimators of $H(t)$ under MELF are given by*

$$\hat{H}_{EBM_1} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m-3)}{k^2}\left[(k+P)\log\left(\frac{k+P}{P}\right) - k\right],$$
$$\hat{H}_{EBM_2} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m-3)}{2k}\left[\log\left(\frac{k+P}{P}\right)\right],$$
$$\hat{H}_{EBM_3} = \frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\frac{(2m-3)}{k^2}.\left[P\log\left(\frac{P}{k+P}\right) + k\right],$$

where $T = b + P$.

Proof. The proof is straightforward, so it is omitted from here. □

4. H-Bayesian Estimation

In this part, the H-Bayesian estimates of the shape parameter of the GIED are obtained using different loss functions, namely SELF, ELF, WBLF, and MELF. Following the methodology proposed by Lindley and Smith [20], we introduce hyper parameters denoted as a and b in the prior distribution $g(\alpha|a,b)$ as given in Equation (2.4). The hyper prior distributions of a and b are defined in Equations (3.3), (3.4), and (3.5). Then, the corresponding hierarchical prior distributions of α are derived based on these hyper priors.

$$\pi_4(\alpha) = \int_0^1\int_0^k g(\alpha|a,b)\pi_1(a,b)dbda = \int_0^1\int_0^k \frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp\left(-b\alpha\right)\frac{2(k-b)}{k^2}\; dbda, \tag{4.1}$$

$$\pi_5(\alpha) = \int_0^1\int_0^k g(\alpha|a,b)\pi_2(a,b)dbda = \int_0^1\int_0^k \frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp\left(-b\alpha\right)\frac{1}{k}\; dbda, \tag{4.2}$$

and

$$\pi_6(\alpha) = \int_0^1 \int_0^k g(\alpha|a,b)\pi_3(a,b)dbda = \int_0^1 \int_0^k \frac{b^a \alpha^{a-1}}{\Gamma(a)} \exp(-b\alpha) \frac{2b}{k^2}\, dbda. \tag{4.3}$$

Using Bayes theorem, likelihood function and (4.1), (4.2) and (4.3), the hierarchical posterior distributions of α is written as

$$\pi_1(\alpha|x) = \frac{L(\alpha|x)\pi_4(\alpha)}{\int_0^\infty L(\alpha|x)\pi_4(\alpha)\, d\alpha} = \frac{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\alpha^{m+a-1}\exp(-\alpha T)\, dbda}{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}, \tag{4.4}$$

$$\pi_2(\alpha|x) = \frac{L(\alpha|x)\pi_5(\alpha)}{\int_0^\infty L(\alpha|x)\pi_5(\alpha)\, d\alpha} = \frac{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)}\alpha^{m+a-1}\exp(-\alpha T)\, dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}, \tag{4.5}$$

and

$$\pi_3(\alpha|x) = \frac{L(\alpha|x)\pi_6(\alpha)}{\int_0^\infty L(\alpha|x)\pi_6(\alpha)\, d\alpha} = \frac{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)}\alpha^{m+a-1}\exp(-\alpha T)dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}. \tag{4.6}$$

4.1. H-Bayesian estimation based on SELF

Using SELF and the H-posterior distributions which are defined in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimates $\hat{\alpha}_{HS_1}$, $\hat{\alpha}_{HS_2}$, $\hat{\alpha}_{HS_3}$ of α can be defined as

$$\hat{\alpha}_{HS_j} = E[(\alpha|x)];\, j = 1, 2, 3.$$

Here,

$$\hat{\alpha}_{HS_1} = E[(\alpha|x)] = \int_0^\infty \alpha.\pi_1(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\, dbda}{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}, \tag{4.7}$$

$$\hat{\alpha}_{HS_2} = E[(\alpha|x)] = \int_0^\infty \alpha.\pi_2(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}dbda}, \tag{4.8}$$

and

$$\hat{\alpha}_{HS_3} = E[(\alpha|x)] = \int_0^\infty \alpha.\pi_3(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\, dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}. \tag{4.9}$$

Theorem 4.1. *For a time t, using the posteriors given in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimators of $H(t)$ under SELF are given by*

$$\hat{H}_{HS_1} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\, dbda}{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}, \tag{4.10}$$

$$\hat{H}_{HS_2} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\, dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}, \tag{4.11}$$

$$\hat{H}_{HS_3} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)}\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\, dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\, dbda}. \tag{4.12}$$

4.2. H-Bayesian estimation based on ELF

Using ELF and the H-posterior distributions which are defined in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimates $\hat{\alpha}_{HE_1}$, $\hat{\alpha}_{HE_2}$, $\hat{\alpha}_{HE_3}$ of α can be defined as

$$\hat{\alpha}_{HE_j} = E[(\alpha^{-1}|x)]^{-1}; j = 1, 2, 3.$$

Here,

$$\hat{\alpha}_{HE_1} = E[(\alpha^{-1}|x)] = \int_0^{\infty} \frac{1}{\alpha}.\pi_1(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}}\, dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda}, \tag{4.13}$$

$$\hat{\alpha}_{HE_2} = E[(\alpha^{-1}|x)] = \int_0^{\infty} \frac{1}{\alpha}.\pi_2(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}} dbda}, \tag{4.14}$$

and

$$\hat{\alpha}_{HE_3} = E[(\alpha^{-1}|x)] = \int_0^{\infty} \frac{1}{\alpha}.\pi_3(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}}\, dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda} \tag{4.15}$$

Theorem 4.2. *For a time t, using the posteriors given in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimates of $H(t)$ under ELF are given by*

$$\hat{H}_{HE_1} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}}\, dbda}, \tag{4.16}$$

$$\hat{H}_{HE_2} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}}\, dbda}, \tag{4.17}$$

$$\hat{H}_{HE_3} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}}\, dbda}. \tag{4.18}$$

4.3. H-Bayesian estimation based on WBLF

Assuming WBLF and using the posteriors defined in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimates $\hat{\alpha}_{HW_1}$, $\hat{\alpha}_{HW_2}$, $\hat{\alpha}_{HW_3}$ of α are

$$\hat{\alpha}_{HW_j} = \frac{E[(\alpha^2|x)]}{E[(\alpha|x)]}; j = 1, 2, 3.$$

Here,

$$E[(\alpha^2|x)] = \int_0^{\infty} \alpha^2 \pi_1(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\, dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda},$$

and

$$E[(\alpha|x)] = \int_0^{\infty} \alpha \pi_1(\alpha|x)\, d\alpha = \frac{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\, dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\, dbda},$$

Then, we get

$$\hat{\alpha}_{HW_1} = \frac{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\, dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}}\, dbda}. \tag{4.19}$$

Similarly, one can easily obtain the following result

$$\hat{\alpha}_{HW_2} = \frac{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\ dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}}\ dbda}, \tag{4.20}$$

$$\hat{\alpha}_{HW_3} = \frac{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\ dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}}\ dbda}. \tag{4.21}$$

Theorem 4.3. *Using the posteriors given in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimators of $H(t)$ under WBLF are given by*

$$\hat{H}_{HW_1} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\ dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}}\ dbda}, \tag{4.22}$$

$$\hat{H}_{HW_2} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\ dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}}\ dbda}, \tag{4.23}$$

$$\hat{H}_{HW_3} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}}\ dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}}\ dbda}. \tag{4.24}$$

4.4. H-Bayesian estimation based on MELF

Assuming MELF and using the posterior defined in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimates $\hat{\alpha}_{HM_1}$, $\hat{\alpha}_{HM_2}$, $\hat{\alpha}_{HM_3}$ of α are defined as

$$\hat{\alpha}_{HM_j} = \frac{E[(\alpha^{-2}|x)]}{E[(\alpha^{-1}|x)]}; j = 1, 2, 3.$$

Here, we have

$$E[(\alpha^{-2}|x)] = \int_0^\infty \frac{1}{\alpha^2} \pi_1(\alpha|x)\ d\alpha = \frac{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\ dbda}$$

and

$$E[(\alpha^{-1}|x)] = \int_0^\infty \frac{1}{\alpha} \pi_1(\alpha|x)\ d\alpha = \frac{\int_0^1 \int_0^p (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a)}{T^{m+a}}\ dbda}.$$

Then, we obtain

$$\hat{\alpha}_{HM_1} = \frac{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}. \tag{4.25}$$

Similarly, we have

$$\hat{\alpha}_{HM_2} = \frac{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}, \tag{4.26}$$

$$\hat{\alpha}_{HM_3} = \frac{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}. \tag{4.27}$$

Theorem 4.4. *Using the posteriors given in Equations (4.4), (4.5) and (4.6), the H-Bayesian estimates of $H(t)$ under MELF are given by*

$$\hat{H}_{HM_1} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}, \tag{4.28}$$

$$\hat{H}_{HM_2} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}, \tag{4.29}$$

$$\hat{H}_{HM_3} = \frac{\frac{\lambda}{t^2(e^{\frac{\lambda}{t}}-1)} \int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} dbda}{\int_0^1 \int_0^k \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(m+a-2)}{T^{m+a-2}} dbda}. \tag{4.30}$$

5. Properties of E-Bayesian and H-Bayesian Estimation of α

In this section, the properties of E-Bayesian estimates and the relations among the E-Bayesian and H-Bayesian estimates are discussed.

5.1. The relations between the E-Bayesian estimates under different loss functions

Theorem 5.1. *It follows from Theorem 3.1 that*
(1) $\hat{\alpha}_{EBS_3} < \hat{\alpha}_{EBS_2} < \hat{\alpha}_{EBS_1}$.
(2) $\lim_{P\to\infty} \hat{\alpha}_{EBS_1} = \lim_{P\to\infty} \hat{\alpha}_{EBS_2} = \lim_{P\to\infty} \hat{\alpha}_{EBS_3}$.
(3) $\lim_{P\to\infty} \hat{\alpha}_{EBS_j} = 0, j = 1, 2, 3$.

Proof. (1) From Theorem 3.1, we have

$$\begin{aligned}\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBS_2} &= \frac{(2m+1)}{k^2}\left[(k+P)log\left(1+\frac{k}{P}\right) - k\right] - \frac{(2m-1)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right] \\ &= \frac{(2m+1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right) - 1\right],\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3} &= \frac{(2m+1)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right] - \frac{(2m+1)}{k^2}\left[P\log\left(\frac{P}{k+P}\right) + k\right] \\ &= \frac{(2m+1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right) - 1\right].\end{aligned}$$

Therefore, we define

$$\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBS_2} = \hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3} = \frac{(2m+1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right) - 1\right]$$

For $-1 < t \le 1$, we have $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - ... = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$.
Let $t = \frac{k}{P}$, when $0 < k < P$, $0 < \frac{k}{P} < 1$, we get

$$\begin{aligned}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right) - 1\right] &= \left(\frac{P}{k}+\frac{1}{2}\right)\left[\left(\frac{k}{P}\right) - \frac{1}{2}\left(\frac{k}{P}\right)^2 + \frac{1}{3}\left(\frac{k}{p}\right)^3 - \frac{1}{4}\left(\frac{k}{P}\right)^4 + \frac{1}{5}\left(\frac{k}{P}\right)^5 - \cdots\right] - 1 \\ &= \left[1 - \frac{1}{2}\left(\frac{k}{P}\right) + \frac{1}{3}\left(\frac{k}{P}\right)^2 - \frac{1}{4}\left(\frac{k}{P}\right)^3 + \frac{1}{5}\left(\frac{k}{P}\right)^4 - \frac{1}{6}\left(\frac{k}{P}\right)^5 + \cdots\right] \\ &\quad + \left[\frac{1}{2}\left(\frac{k}{P}\right) + \frac{1}{4}\left(\frac{k}{P}\right)^2 + \frac{1}{6}\left(\frac{k}{P}\right)^3 - \frac{1}{8}\left(\frac{k}{P}\right)^4 + \cdots\right] - 1 \\ &= \frac{1}{12}\left(\frac{k}{P}\right)^2\left[1 - \frac{k}{P}\right] + \frac{3}{40}\left(\frac{k}{P}\right)^4\left[1 - \frac{8}{9}\frac{k}{P}\right] + \cdots,\end{aligned}$$

where $0 < \frac{k}{P} < 1$. Thus,

$$\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBE_2} = \hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3} = \frac{(2m+1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] > 0,$$

which yields that, $\hat{\alpha}_{EBS_3} < \hat{\alpha}_{EBS_S} < \hat{\alpha}_{EBS_1}$.
(2) From (1), we get

$$\begin{aligned}\lim_{P\to\infty}(\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBS_2}) = & \lim_{P\to\infty}(\hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3}) \\ & = \frac{(2m+1)}{k} * \lim_{P\to\infty}\left[\frac{1}{12}\frac{k^2}{P^2}\left(1-\frac{k}{P}\right)+\frac{3}{40}\frac{k^4}{P^4}\left(1-\frac{8}{9}\frac{k}{P}\right)+\cdots\right].\end{aligned}$$

So it can be easily obtained that

$$\lim_{P\to\infty}\hat{\alpha}_{EBS_1} = \lim_{P\to\infty}\hat{\alpha}_{EBS_2} = \lim_{P\to\infty}\hat{\alpha}_{EBS_3}\ .$$

(3) From (3.6) and from the proof of (1), we have

$$\lim_{P\to\infty}\hat{\alpha}_{EBS_1} = \lim_{P\to\infty}\frac{2m+1}{k}\left[\frac{1}{2}\left(\frac{k}{P}\right)-\frac{1}{6}\left(\frac{k}{P}\right)^2+\frac{1}{12}\left(\frac{k}{P}\right)^3-\frac{1}{20}\left(\frac{k}{P}\right)^4+\cdots\right] = 0$$

Using (2), we get

$$\lim_{P\to\infty}\hat{\alpha}_{EBS_j} = 0, \text{ j=1,2,3.}$$

□

Theorem 5.2. *It follows from Theorem 3.3 that*
(1) $\hat{\alpha}_{EBE_3} < \hat{\alpha}_{EBE_2} < \hat{\alpha}_{EBE_1}$
(2) $\lim_{P\to\infty}\hat{\alpha}_{EBE_1} = \lim_{P\to\infty}\hat{\alpha}_{EBE_2} = \lim_{P\to\infty}\hat{\alpha}_{EBE_3}$
(3) $\lim_{P\to\infty}\hat{\alpha}_{EBE_j} = 0, j = 1,2,3.$

Proof. (1) From Theorem 3.3, we have

$$\begin{aligned}\hat{\alpha}_{EBE_1} - \hat{\alpha}_{EBE_2} &= \frac{(2m-1)}{k^2}\left[(k+P)log\left(1+\frac{k}{P}\right)-k\right] - \frac{(2m-1)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right] \\ &= \frac{(2m-1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right]\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}_{EBE_2} - \hat{\alpha}_{EBE_3} &= \frac{(2m-1)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right] - \frac{(2m-1)}{k^2}\left[P\log\left(\frac{P}{k+P}\right)+k\right] \\ &= \frac{(2m-1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right].\end{aligned}$$

Therefore, we obtain

$$\hat{\alpha}_{EBE_1} - \hat{\alpha}_{EBE_2} = \hat{\alpha}_{EBE_2} - \hat{\alpha}_{EBE_3} = \frac{(2m-1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right].$$

For $-1 < t \leq 1$, we have $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - ... = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$.
Let $t = \frac{k}{P}$, when $0 < k < P$, $0 < \frac{k}{P} < 1$, we get

$$\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] = \left(\frac{P}{k}+\frac{1}{2}\right)\left[\left(\frac{k}{P}\right)-\frac{1}{2}\left(\frac{k}{P}\right)^2+\frac{1}{3}\left(\frac{k}{P}\right)^3-\frac{1}{4}\left(\frac{k}{P}\right)^4+\frac{1}{5}\left(\frac{k}{P}\right)^5-\cdots\right]-1$$
$$= \left[1-\frac{1}{2}\left(\frac{k}{P}\right)+\frac{1}{3}\left(\frac{k}{P}\right)^2-\frac{1}{4}\left(\frac{k}{P}\right)^3+\frac{1}{5}\left(\frac{k}{P}\right)^4-\frac{1}{6}\left(\frac{k}{P}\right)^5+\cdots\right]$$
$$+\left[\frac{1}{2}\left(\frac{k}{P}\right)+\frac{1}{4}\left(\frac{k}{P}\right)^2+\frac{1}{6}\left(\frac{k}{P}\right)^3-\frac{1}{8}\left(\frac{k}{P}\right)^4+\cdots\right]-1$$
$$=\frac{1}{12}\left(\frac{k}{P}\right)^2\left[1-\frac{k}{P}\right]+\frac{3}{40}\left(\frac{k}{P}\right)^4\left[1-\frac{8}{9}\frac{k}{P}\right]+\cdots,$$

where $0 < \frac{k}{P} < 1$. Thus,

$$\hat{\alpha}_{EBE_1} - \hat{\alpha}_{EBE_2} = \hat{\alpha}_{EBE_2} - \hat{\alpha}_{EBE_3} = \frac{(2m-1)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] > 0,$$

which yields that, $\hat{\alpha}_{EBE_3} < \hat{\alpha}_{EBE_2} < \hat{\alpha}_{EBE_1}$.
(2) From (1), we get

$$\lim_{P\to\infty}(\hat{\alpha}_{EBE_1} - \hat{\alpha}_{EBE_2}) = \lim_{P\to\infty}(\hat{\alpha}_{EBE_2} - \hat{\alpha}_{EBE_3})$$
$$= \frac{(2m-1)}{p} * \lim_{P\to\infty}\left[\frac{1}{12}\frac{k^2}{P^2}\left(1-\frac{k}{P}\right)+\frac{3}{40}\frac{k^4}{P^4}\left(1-\frac{8}{9}\frac{k}{P}\right)+\cdots\right].$$

So it can be easily obtained that

$$\lim_{P\to\infty}\hat{\alpha}_{EBE_1} = \lim_{P\to\infty}\hat{\alpha}_{EBE_2} = \lim_{P\to\infty}\hat{\alpha}_{EBE_3}\ .$$

(3) From (3.7) and from the proof of (1), we have

$$\lim_{P\to\infty}\hat{\alpha}_{EBE_1} = \lim_{P\to\infty}\frac{2m-1}{k}\left[\frac{1}{2}\left(\frac{k}{P}\right)-\frac{1}{6}\left(\frac{k}{P}\right)^2+\frac{1}{12}\left(\frac{k}{P}\right)^3-\frac{1}{20}\left(\frac{k}{P}\right)^4+\cdots\right]=0$$

Using (2), we get

$$\lim_{P\to\infty}\hat{\alpha}_{EBE_j} = 0, \text{ j=1,2,3.}$$

□

Theorem 5.3. *It follows from Theorem 3.5 that*
(1) $\hat{\alpha}_{EBW_3} < \hat{\alpha}_{EBW_2} < \hat{\alpha}_{EBW_1}$,
(2) $\lim_{P\to\infty}\hat{\alpha}_{EBW_1} = \lim_{P\to\infty}\hat{\alpha}_{EBW_2} = \lim_{P\to\infty}\hat{\alpha}_{EBW_3}$,
(3) $\lim_{P\to\infty}\hat{\alpha}_{EBW_j} = 0, j = 1, 2, 3.$

Proof. (1) From Theorem 3.5, we have

$$\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2} = \frac{(2m+3)}{k^2}\left[(k+P)log\left(1+\frac{k}{P}\right)-k\right] - \frac{(2m+3)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right]$$
$$= \frac{(2m+3)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right],$$

and

$$\hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3} = \frac{(2m+3)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right] - \frac{(2m+3)}{k^2}\left[P\log\left(\frac{P}{k+P}\right)+k\right]$$
$$= \frac{(2m+3)}{k}\left[\log\left(1+\frac{q}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right].$$

Therefore, we define

$$\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2} = \hat{\alpha}_{EBW_2}\hat{\alpha}_{EBW_3} = \frac{(2m+3)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right].$$

For $-1 < t \leq 1$, we have $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - ... = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$.
Let $t = \frac{k}{P}$, when $0 < k < F$, $0 < \frac{k}{P} < 1$, we get

$$\begin{aligned}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] &= \left(\frac{P}{k}+\frac{1}{2}\right)\left[\left(\frac{k}{P}\right)-\frac{1}{2}\left(\frac{k}{P}\right)^2+\frac{1}{3}\left(\frac{k}{P}\right)^3-\frac{1}{4}\left(\frac{k}{P}\right)^4+\frac{1}{5}\left(\frac{k}{P}\right)^5-\cdots\right]-1\\ &= \left[1-\frac{1}{2}\left(\frac{k}{P}\right)+\frac{1}{3}\left(\frac{k}{P}\right)^2-\frac{1}{4}\left(\frac{k}{P}\right)^3+\frac{1}{5}\left(\frac{K}{P}\right)^4-\frac{1}{6}\left(\frac{k}{P}\right)^5+\cdots\right]\\ &\quad+\left[\frac{1}{2}\left(\frac{k}{P}\right)+\frac{1}{4}\left(\frac{k}{P}\right)^2+\frac{1}{6}\left(\frac{k}{P}\right)^3-\frac{1}{8}\left(\frac{k}{P}\right)^4+\cdots\right]-1\\ &= \frac{1}{12}\left(\frac{k}{P}\right)^2\left[1-\frac{k}{P}\right]+\frac{3}{40}\left(\frac{k}{P}\right)^4\left[1-\frac{8}{9}\frac{k}{P}\right]+\cdots,\end{aligned}$$

Then,

$$\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2} = \hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3} = \frac{(2n+3)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] > 0,$$

This shows that

$$\hat{\alpha}_{EBW_3} < \hat{\alpha}_{EBW_2} < \hat{\alpha}_{EBW_1}.$$

(2) From (1), we get

$$\begin{aligned}\lim_{P\to\infty}(\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2}) = \lim_{P\to\infty}(\hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3}) = \frac{(2m+3)}{k} * \lim_{P\to\infty}\Bigg[\frac{1}{12}\frac{k^2}{P^2}\left(1-\frac{k}{P}\right)\\ +\frac{3}{40}\frac{k^4}{P^4}\left(1-\frac{8}{9}\frac{k}{P}\right)+\cdots\Bigg]\end{aligned}$$

This yields that,

$$\lim_{P\to\infty}\hat{\alpha}_{EBW_1} = \lim_{P\to\infty}\hat{\alpha}_{EBW_2} = \lim_{P\to\infty}\hat{\alpha}_{EBW_3}.$$

(3) From (3.8) and from the proof of (1), we have $\lim_{P\to\infty}\hat{\alpha}_{EBW_1} = 0$
Using (2), we get

$$\lim_{P\to\infty}\hat{\alpha}_{EBW_j} = 0, \text{ for } j = 1, 2, 3.$$

□

Theorem 5.4. *It follows from Theorem 3.7 that*
(1) $\hat{\alpha}_{EBM_3} < \hat{\alpha}_{EBM_2} < \hat{\alpha}_{EBM_1}$
(2) $\lim_{P\to\infty}\hat{\alpha}_{EBM_1} = \lim_{P\to\infty}\hat{\alpha}_{EBM_2} = \lim_{P\to\infty}\hat{\alpha}_{EBM_3}$
(3) $\lim_{P\to\infty}\hat{\alpha}_{EBM_j} = 0, j = 1, 2, 3.$

Proof. (1) From Theorem 3.7, we have

$$\begin{aligned}\hat{\alpha}_{EBM_1} - \hat{\alpha}_{EBM_2} &= \frac{(2m-3)}{p^2}\left[(k+P)log\left(1+\frac{k}{P}\right)-k\right] - \frac{(2m+3)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right]\\ &= \frac{(2m-3)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right]\end{aligned}$$

and

$$\hat{\alpha}_{EBM_2} - \hat{\alpha}_{EBM_3} = \frac{(2m-3)}{2k}\left[\log\left(1+\frac{k}{P}\right)\right] - \frac{(2m-3)}{k^2}\left[P\log\left(\frac{k}{k+P}\right)+k\right]$$

$$= \frac{(2m-3)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right]$$

$$\hat{\alpha}_{EBM_1} - \hat{\alpha}_{EBM_2} = \hat{\alpha}_{EBM_2} - \hat{\alpha}_{EBM_3} = \frac{(2m-3)}{k}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right]$$

For $-1 < t \leq 1$, we have

$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - ... = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$.

Let $t = \frac{k}{P}$, when $0 < k < P$, $0 < \frac{k}{P} < 1$, we get

$$\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] = \left(\frac{P}{k}+\frac{1}{2}\right)\left[\left(\frac{k}{P}\right) - \frac{1}{2}\left(\frac{k}{P}\right)^2 + \frac{1}{3}\left(\frac{k}{P}\right)^3 - \frac{1}{4}\left(\frac{k}{P}\right)^4 + \frac{1}{5}\left(\frac{k}{P}\right)^5 - ...\right] - 1$$

$$= \left[1 - \frac{1}{2}\left(\frac{k}{P}\right) + \frac{1}{3}\left(\frac{k}{P}\right)^2 - \frac{1}{4}\left(\frac{k}{P}\right)^3 + \frac{1}{5}\left(\frac{k}{P}\right)^4 - \frac{1}{6}\left(\frac{k}{P}\right)^5 + \cdots\right] + \left[\frac{1}{2}\left(\frac{k}{P}\right) + \frac{1}{4}\left(\frac{k}{P}\right)^2\right.$$

$$\left. + \frac{1}{6}\left(\frac{k}{P}\right)^3 - \frac{1}{8}\left(\frac{k}{P}\right)^4 + ...\right] - 1$$

$$= \frac{1}{12}\left(\frac{k}{P}\right)^2\left[1 - \frac{k}{P}\right] + \frac{3}{40}\left(\frac{k}{P}\right)^4\left[1 - \frac{8}{9}\frac{k}{P}\right] + ...$$

where

$$0 < \tfrac{k}{P} < 1,$$

Thus, we have

$$\hat{\alpha}_{EBM_1} - \hat{\alpha}_{EBM_2} = \hat{\alpha}_{EBM_2} - \hat{\alpha}_{EBM_3} = \frac{(2m-3)}{p}\left[\log\left(1+\frac{k}{P}\right)\left(\frac{P}{k}+\frac{1}{2}\right)-1\right] > 0, \quad (5.1)$$

Then, we define

$$\hat{\alpha}_{EBM_3} < \hat{\alpha}_{EBM_2} < \hat{\alpha}_{EBM_1}.$$

(2) From (1), we get

$$\lim_{P\to\infty}(\hat{\alpha}_{EBM_1} - \hat{\alpha}_{EBM_2}) = \lim_{P\to\infty}(\hat{\alpha}_{EBM_2} - \hat{\alpha}_{EBM_3})$$

$$= \frac{(2m-3)}{k} * \lim_{P\to\infty}\left[\frac{1}{12}\frac{k^2}{P^2}\left(1-\frac{k}{P}\right) + \frac{3}{40}\frac{k^4}{P^4}\left(1 - \frac{8}{9}\frac{k}{P}\right) + ...\right] \quad (5.2)$$

Then, we get

$$\lim_{P\to\infty}\hat{\alpha}_{EBM_1} = \lim_{P\to\infty}\hat{\alpha}_{EBM_2} = \lim_{P\to\infty}\hat{\alpha}_{EBM_3}\ .$$

(3) From (3.9) and from the proof of (1), we have $\lim_{P\to\infty}\hat{\alpha}_{EBM_1} = 0$

Using (2), we get

$$\lim_{P\to\infty}\hat{\alpha}_{EBM_j} = 0, \text{ j=1,2,3.}$$

□

Similar relationships holds for the E-Bayesian estimates of $H(t)$ under different loss functions.

5.2. The relations between the H-Bayesian estimates under different loss functions

Theorem 5.5. *It follows from (4.7), (4.8) and (4.9) that*
$$\lim_{P\to\infty}\hat{\alpha}_{HS_j}=0,\ \textit{for}\ j=1,2,3.$$

Proof. Based on SELF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HS_1}=\frac{\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\,dbda}{\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\,dbda}.$$

Using the result $\Gamma(m+a+1)=(m+a)\Gamma(m+a)$, we have,

$$\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\,dbda=\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{(m+a)\Gamma(m+a)}{T^{n+a+1}}\,dbda.$$

As $T=b+P$, and For $a\in(0,1)$, $b\in(0,q)$, $(m+a)(b+P)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}>0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_2\in(0,1)$ and $b_2\in(0,k)$ such that,

$$\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a+1)}{T^{m+a+1}}\,dbda=\frac{(m+a_2)}{(b_2+P)}\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\,dbda.$$

Therefore,

$$\hat{\alpha}_{HS_1}=\frac{\frac{(m+a_2)}{(b_2+P)}\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\,dbda}{\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\,dbda}=\frac{(m+a_2)}{(b_2+P)}.$$

Taking limit as $P\to\infty$, it has been noticed that

$$\lim_{P\to\infty}\hat{\alpha}_{HS_1}=0.$$

Similarly, we have

$$\lim_{P\to\infty}\hat{\alpha}_{HS_2}=\lim_{P\to\infty}\hat{\alpha}_{HS_3}=0.$$

Thus,

$$\lim_{P\to\infty}\hat{\alpha}_{HS_j}=0,\ \text{for}\ j=1,2,3.$$

□

Theorem 5.6. *It follows from Equations (4.13), (4.14) and (4.15) that*
$$\lim_{P\to\infty}\hat{\alpha}_{HE_j}=0,\ \textit{for}\ j=1,2,3.$$

Proof. Based on ELF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HE_1}=\frac{\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{n+a}}\,dbda}{\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-1)}{T^{m+a-1}}\,dbda}.$$

Using the result $\Gamma(m+a)=(m+a-1)\Gamma(m+a-1)$, we have

$$\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\,dbda=\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{(m+a-1)\Gamma(m+a-1)}{T^{m+a}}\,dbda.$$

As $T=b+P$, and For $a\in(0,1)$, $b\in(0,k)$, $(m+a-1)(b+P)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-1)}{T^{m+a-1}}>0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_5\in(0,1)$ and $b_5\in(0,k)$ such that

$$\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a)}{T^{m+a}}\,dbda=\frac{(m+a_5-1)}{(b_5+P)}\int_0^1\int_0^k(k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-1)}{T^{n+a-1}}\,dbda.$$

Therefore,

$$\hat{\alpha}_{HE_1} = \frac{\frac{(m+a_5-1)}{(b_5+P)} \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} \; dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a-1)}{T^{m+a-1}} \; dbda} = \frac{(m+a_5-1)}{(b_5+P)}.$$

Taking limit as $P \to \infty$, it has been noticed that

$$\lim_{P\to\infty} \hat{\alpha}_{HE_1} = 0.$$

Similarly, we have

$$\lim_{P\to\infty} \hat{\alpha}_{HE_2} = \lim_{P\to\infty} \hat{\alpha}_{HE_3} = 0.$$

Thus,

$$\lim_{P\to\infty} \hat{\alpha}_{HE_j} = 0, \text{ for } j = 1, 2, 3.$$

□

Theorem 5.7. *It follows from Equations (4.19), (4.20) and (4.21) that*

$$\lim_{P\to\infty} \hat{\alpha}_{HW_j} = 0, \textit{ for } j = 1, 2, 3.$$

Proof. Under WBLF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HW_1} = \frac{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}} \; dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}} \; dbda}.$$

Using the result, $\Gamma(m+a+2) = (m+a+1)\Gamma(m+a+1)$, we have

$$\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}} \; dbda = \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{(m+a+1)\Gamma(m+a+1)}{T^{m+a+2}} \; dbda.$$

As $T = b + P$, and For $a \in (0,1)$, $b \in (0,k)$, $(m+a+1)(b+P)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}} > 0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_8 \in (0,1)$ and $b_8 \in (0,k)$ such that

$$\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+2)}{T^{m+a+2}} \; dbda = \frac{(m+a_8+1)}{(b_8+P)} \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}} \; dbda.$$

Therefore, we obtain

$$\hat{\alpha}_{HW_1} = \frac{\frac{(m+a_8+1)}{(b_8+P)} \int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}} \; dbda}{\int_0^1 \int_0^k (k-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(m+a+1)}{T^{m+a+1}} \; dbda} = \frac{(m+a_8+1)}{(b_8+P)}.$$

Taking limit as $P \to \infty$, it has been obtained that

$$\lim_{P\to\infty} \hat{\alpha}_{HW_1} = 0.$$

Similarly, we have

$$\lim_{P\to\infty} \hat{\alpha}_{HW_2} = \lim_{P\to\infty} \hat{\alpha}_{HW_3} = 0$$

Thus,

$$\lim_{P\to\infty} \hat{\alpha}_{HW_j} = 0, \text{ for } j = 1, 2, 3.$$

□

Theorem 5.8. *It follows from Equations (4.25), (4.26) and (4.27) that*

$$\lim_{P\to\infty} \hat{\alpha}_{HM_j} = 0, j = 1, 2, 3.$$

Proof. Under MELF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HM_1} = \frac{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-1)}{T^{m+a-1}}dbda}{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-2)}{T^{m+a-2}}dbda}$$

Using the result $\Gamma(m+a-1) = (m+a-2)\Gamma(m+a-2)$, we have,

$$\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-1)}{T^{m+a-1}}dbda = \int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{(m+a-2)\Gamma(m+a-2)}{T^{m+a-1}}dbda$$

As $T = b + P$, and For $a \in (0,1)$, $b \in (0,k)$, $(m+a-2)(b+P)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-2)}{T^{m+a-2}} > 0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_{11} \in (0,1)$ and $b_{11} \in (0,k)$ such that

$$\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-1)}{T^{m+a-1}}dbda = \frac{(m+a_{11}-2)}{(b_{11}+P)}\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-2)}{T^{m+a-2}}dbda$$

$$\hat{\alpha}_{HM_1} = \frac{\frac{(m+a_{11}-2)}{(b_{11}+P)}\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-2)}{T^{m+a-2}}dbda}{\int_0^1 \int_0^k (k-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(m+a-2)}{T^{m+a-2}}dbda}$$

$$\hat{\alpha}_{HM_1} = \frac{(n+a_{11}-2)}{(b_{11}+P)}. \tag{5.3}$$

Taking limit as $P \to \infty$

$$\lim_{P\to\infty} \hat{\alpha}_{HM_1} = 0$$

Similarly, we have

$$\lim_{P\to\infty} \hat{\alpha}_{HM_2} = \lim_{P\to\infty} \hat{\alpha}_{HM_3} = 0$$

Thus,

$$\lim_{P\to\infty} \hat{\alpha}_{HM_j} = 0, j = 1,2,3$$

□

Similar relationships holds for the H-Bayesian estimates of hazard rate under different loss functions.

6. Simulation Study

In this section, Monte Carlo simulation study is performed to compare the performance of the proposed estimates based on Type II PCS for the GIED. The simulation study has been conducted using the R software. To generate the data, the initial true values of α and λ are assumed as 1.5 and 1.2, and k=1. Here, the value of hazard rate $H(t)$ is considered as 0.71833 with $t = 0.5$. In this simulation study, $10,000$ progressively censored data have been generated using the following three progressive schemes:

- **Scheme I:** $R_1 = \cdots = R_{m-1} = 0$, $R_m = n - m$;
- **Scheme II:** $R_1 = n - m$, $R_2 = \cdots = R_m = 0$;
- **Scheme III:** $R_1 = \cdots = R_{n-m} = 1$, $R_{n-m+1} = \cdots = R_m = 0$.

The progressively censored data have been generated by using the following algorithm:

Algorithm

---

(1) Define the values of n and m.
(2) Simulate m random variables from uniform $(0, 1)$ as $V_1, V_2 ... V_m$.
(3) Determine the value of censoring scheme R_i, i=1,2,...,m.
(4) Set $U_i = \frac{1}{\sum_{j=m+i-1}^{m}} R_j$ for i=1,2,...m.
(5) Obtain the progressive Type II right censored sample $(V^*_{1:m:n}, V^*_{2:m:n} ... V^*_{m:m:n})$, where $V^*_{I:m:n} = 1 - \prod_{j=i+m-1}^{m} U_i$, i=1,2,...m.
(6) Set $X_{i:m:n} = F^{-1}(V^*_{i:m:n})$ for i=1,2,... m, where $F^{-1}(V^*_{i:m:n})$represent the quantile function of the GIED distribution.

Thus, $\{X_{1:m:n}, \cdots, X_{m:m:n}\}$, becomes the needed progressive type-II right censored sample from the specified distribution $F(.)$ by using the inverse transformation method. Using gamma prior the Bayes, E-Bayesian and H-Bayesian estimates are obtained under four different loss functions, as SELF, ELF, WBLF, and MELF. For these gamma priors, the hyperparameters (a, b) are considered as $(3, 2)$. The performance of the proposed estimates has been assessed using the value of mean squared error (MSE). The values of the average estimates (AEs) and the associated MSEs are simulated and presented in tables $1 - 8$. From the tables, the following conclusions have been made

- As the value of n increases, the MSE decreases.
- The Bayesian, E-Bayesian and H-Bayesian estimates outperform MLE in terms of MSE.
- For any fixed loss function, the E-Bayesian estimates have a smaller MSE than the Bayesian and H-Bayesian estimates.
- For a fixed value of n, when m increases, AEs come closer to their true value.
- In most of the cases, the estimates under ELF perform better than the estimates using other loss functions.

Combining all the above results, it is recommended to use the E-Bayesian technique to estimate the parameters and the hazard rate for type II PCS based on ELF, due to the better performance than other estimates in terms of MSE.

Table 1. AE of α and their MSE in parentheses based on SELF.

(n,m)	(CS)	MLE	Bayes	E-Bayesian			H-Bayesian		
			$\hat{\alpha}_{BS}$	$\hat{\alpha}_{EBS_1}$	$\hat{\alpha}_{EBS_2}$	$\hat{\alpha}_{EBS_3}$	$\hat{\alpha}_{HS_1}$	$\hat{\alpha}_{HS_2}$	$\hat{\alpha}_{HS_3}$
(40,30)	I	1.5640	1.5525	1.5506	1.5372	1.5239	1.5476	1.5469	1.5457
		(0.0889)	(0.0854)	(0.0830)	(0.0790)	(0.0755)	(0.0846)	(0.0845)	(0.0843)
	II	1.5676	1.5646	1.5549	1.5513	1.5377	1.5559	1.5552	1.5541
		(0.0896)	(0.0872)	(0.0842)	(0.0836)	(0.0810)	(0.0856)	(0.0854)	(0.0848)
	III	1.5642	1.5612	1.5418	1.5286	1.5154	1.5486	1.5479	1.5467
		(0.0878)	(0.0855)	(0.0833)	(0.0824)	(0.0789)	(0.0846)	(0.0835)	(0.0793)
(40,35)	I	1.5406	1.5388	1.5385	1.5272	1.5160	1.5308	1.5318	1.5324
		(0.7377)	(0.0681)	(0.0663)	(0.0636)	(0.0612)	(0.0630)	(0.0631)	(0.0632)
	II	1.5480	1.5459	1.5236	1.5126	1.5015	1.5321	1.5331	1.5337
		(0.0753)	(0.0715)	(0.0708)	(0.0684)	(0.0662)	(0.0663)	(0.0664)	(0.0665)
	III	1.5365	1.5347	1.5326	1.5215	1.5103	1.5266	1.5276	1.5282
		(0.0728)	(0.0692)	(0.0678)	(0.0652)	(0.0630)	(0.0774)	(0.0776)	(0.0777)
(60,40)	I	1.5440	1.5424	1.5422	1.5323	1.5224	1.5447	1.5442	1.5433
		(0.0645)	(0.0617)	(0.0610)	(0.0587)	(0.0567)	(0.0626)	(0.0624)	(0.0592)
	II	1.5491	1.5489	1.5473	1.5457	1.5356	1.5474	1.5469	1.5460
		(0.0728)	(0.0666)	(0.0649)	(0.0632)	(0.0617)	(0.0652)	(0.0653)	(0.0624)
	III	1.5458	1.5442	1.5411	1.5312	1.5214	1.5416	1.5410	1.5402
		(0.0632)	(0.0605)	(0.0597)	(0.0575)	(0.0555)	(0.0599)	(0.0586)	(0.0582)
(60,45)	I	1.5315	1.5303	1.5249	1.5163	1.5077	1.5268	1.5275	1.5280
		(0.0578)	(0.0556)	(0.0490)	(0.0475)	(0.0462)	(0.0499)	(0.0500)	(0.0500)
	II	1.5295	1.5248	1.5270	1.5184	1.5097	1.5221	1.5228	1.5233
		(0.0517)	(0.0498)	(0.0544)	(0.0528)	(0.0513)	(0.0508)	(0.0509)	(0.0509)
	III	1.5352	1.5341	1.5280	1.5194	1.5107	1.5240	1.5248	1.5253
		(0.0527)	(0.0507)	(0.0559)	(0.0542)	(0.0527)	(0.0578)	(0.0579)	(0.0580)
(90,50)	I	1.5351	1.5341	1.5217	1.5140	1.5062	1.5321	1.5317	1.5310
		(0.0555)	(0.0536)	(0.0490)	(0.0477)	(0.0465)	(0.0506)	(0.0501)	(0.0486)
	II	1.5350	1.5340	1.5330	1.5251	1.5173	1.5353	1.5348	1.5341
		(0.0485)	(0.0479)	(0.0461)	(0.0460)	(0.0453)	(0.0477)	(0.0476)	(0.0466)
	III	1.5300	1.5291	1.5290	1.5212	1.5134	1.5259	1.5252	1.5248
		(0.0484)	(0.0478)	(0.0474)	(0.0460)	(0.0448)	(0.0483)	(0.0472)	(0.0462)
(90,55)	I	1.5215	1.5208	1.5178	1.5108	1.5038	1.5252	1.5257	1.5612
		(0.0446)	(0.0433)	(0.0405)	(0.0396)	(0.0387)	(0.0431)	(0.0431)	(0.0432)
	II	1.5220	1.5213	1.5198	1.5128	1.5058	1.5186	1.5192	1.5196
		(0.0454)	(0.0440)	(0.0425)	(0.0415)	(0.0405)	(0.0469)	(0.0469)	(0.0470)
	III	1.5213	1.5206	1.5199	1.5129	1.5059	1.5167	1.5173	1.5177
		(0.0436)	(0.0423)	(0.0407)	(0.0397)	(0.0388)	(0.0390)	(0.0390)	(0.0390)

Table 2. AE of α and their MSE in parentheses based on ELF.

(n,m)	**(CS)**	**Bayes**	**E-Bayesian**			**H-Bayesian**		
		$\hat{\alpha}_{BE}$	$\hat{\alpha}_{EBE_1}$	$\hat{\alpha}_{EBE_2}$	$\hat{\alpha}_{EBE_3}$	$\hat{\alpha}_{HE_1}$	$\hat{\alpha}_{HE_2}$	$\hat{\alpha}_{HE_3}$
(40,30)	I	1.4700	1.4964	1.4957	1.4829	1.4952	1.4963	1.4970
		(0.0798)	(0.0786)	(0.0760)	(0.0755)	(0.0769)	(0.0770)	(0.0770)
	II	1.5137	1.5136	1.5005	1.4873	1.5032	1.5051	1.5051
		(0.0829)	(0.0798)	(0.0751)	(0.0771)	(0.0796)	(0.0795)	(0.0785)
	III	1.4912	1.4985	1.49573	1.50863	1.49617	1.49732	1.49803
		(0.0791)	(0.0768)	(0.0748)	(0.0732)	(0.0756)	(0.0766)	(0.0762)
(40,35)	I	1.4733	1.4957	1.4951	1.4842	1.4878	1.4888	1.4894
		(0.0630)	(0.0612)	(0.0590)	(0.0583)	(0.0598)	(0.0588)	(0.0588)
	II	1.4700	1.5027	1.4907	1.4825	1.4890	1.4900	1.4906
		(0.0668)	(0.0656)	(0.0653)	(0.0642)	(0.0647)	(0.0648)	(0.0668)
	III	1.4786	1.4977	1.4918	1.4895	1.4837	1.4847	1.4853
		(0.0643)	(0.0631)	(0.0626)	(0.0604)	(0.0627)	(0.0628)	(0.0628)
(60,40)	I	1.5046	1.5041	1.4944	1.4908	1.5067	1.5061	1.5053
		(0.0570)	(0.0563)	(0.0549)	(0.0537)	(0.0566)	(0.0565)	(0.0564)
	II	1.5174	1.5138	1.5075	1.4977	1.5093	1.5088	1.5079
		(0.0617)	(0.0610)	(0.0592)	(0.0575)	(0.0599)	(0.0598)	(0.0598)
	III	1.5063	1.5031	1.4934	1.4918	1.5036	1.5031	1.5022
		(0.0557)	(0.0552)	(0.0538)	(0.0526)	(0.0563)	(0.0553)	(0.0552)
(60,45)	I	1.4869	1.4994	1.4950	1.4946	1.4933	1.4940	1.4945
		(0.0523)	(0.0463)	(0.0455)	(0.0448)	(0.0461)	(0.0461)	(0.0462)
	II	1.4850	1.4955	1.4930	1.4906	1.4887	1.4894	1.4899
		(0.0514)	(0.0504)	(0.0495)	(0.0469)	(0.0502)	(0.0503)	(0.0503)
	III	1.4860	1.5005	1.4945	1.4915	1.4906	1.4913	1.49185
		(0.0527)	(0.0517)	(0.0508)	(0.0494)	(0.0528)	(0.0529)	(0.0529)
(90,50)	I	1.5038	1.4992	1.4985	1.4979	1.5008	1.5014	1.5019
		(0.0504)	(0.0467)	(0.0459)	(0.0452)	(0.0447)	(0.0458)	(0.0454)
	II	1.4970	1.5026	1.4949	1.4972	1.5038	1.5045	1.5049
		(0.0496)	(0.0491)	(0.0481)	(0.0472)	(0.0497)	(0.0497)	(0.0487)
	III	1.5046	1.4987	1.4970	1.4934	1.4957	1.4953	1.4946
		(0.04540)	(0.0447)	(0.0438)	(0.0431)	(0.0459)	(0.0441)	(0.0443)
(90,55)	I	1.4904	1.4967	1.4936	1.4935	1.4976	1.4982	1.4986
		(0.0414)	(0.0388)	(0.0383)	(0.0379)	(0.0408)	(0.0410)	(0.0410)
	II	1.4944	1.4962	1.4955	1.4922	1.4913	1.4919	1.4923
		(0.0420)	(0.0406)	(0.0400)	(0.0395)	(0.0497)	(0.0497)	(0.0487)
	III	1.4925	1.4988	1.5092	1.4934	1.4965	1.4972	1.4975
		(0.0404)	(0.0384)	(0.0383)	(0.0379)	(0.0402)	(0.0402)	(0.0402)

Table 3. AE of α and their MSE in parentheses based on WBLF.

(n,m)	(CS)	Bayes	E-Bayesian			H-Bayesian		
		$\hat{\alpha}_{BW}$	$\hat{\alpha}_{EBW_1}$	$\hat{\alpha}_{EBW_2}$	$\hat{\alpha}_{EBW_3}$	$\hat{\alpha}_{HW_1}$	$\hat{\alpha}_{HW_2}$	$\hat{\alpha}_{HW_3}$
(40,30)	I	1.6030	1.5971	1.5834	1.5697	1.5981	1.5979	1.5974
		(0.1024)	(0.0990)	(0.0933)	(0.0880)	(0.0101)	(0.0992)	(0.0996)
	II	1.6163	1.6154	1.6022	1.5881	1.6068	1.6060	1.6049
		(0.1079)	(0.0987)	(0.1015)	(0.0955)	(0.0965)	(0.0967)	(0.0957)
	III	1.6120	1.5923	1.5787	1.5999	1.5985	1.4973	1.5973
		(0.0965)	(0.0987)	(0.0932)	(0.0881)	(0.0943)	(0.0946)	(0.0938)
(40,35)	I	1.5818	1.5818	1.5702	1.5587	1.5739	1.5749	1.5755
		(0.07717)	(0.0752)	(0.0714)	(0.0679)	(0.0710)	(0.0713)	(0.0714)
	II	1.5892	1.5665	1.5552	1.5438	1.5752	1.5762	1.5768
		(0.0813)	(0.0787)	(0.0752)	(0.0719)	(0.0746)	(0.0748)	(0.0750)
	III	1.5776	1.5758	1.5643	1.5528	1.5695	1.5705	1.5711
		(0.0779)	(0.0763)	(0.0726)	(0.0693)	(0.0769)	(0.0762)	(0.0763)
(60,40)	I	1.5835	1.5803	1.5701	1.5600	1.5834	1.5822	1.5828
		(0.0683)	(0.0686)	(0.0655)	(0.0626)	(0.0691)	(0.0690)	(0.0689)
	II	1.5942	1.5900	1.5839	1.5735	1.5855	1.5850	1.5842
		(0.0759)	(0.0752)	(0.0723)	(0.0689)	(0.07321)	(0.0728)	(0.0719)
	III	1.5821	1.5792	1.5691	1.5589	1.5795	1.5790	1.5781
		(0.0682)	(0.0672)	(0.0641)	(0.0612)	(0.0677)	(0.0679)	(0.0670)
(60,45)	I	1.5638	1.5585	1.5497	1.5409	1.5603	1.5610	1.56154
		(0.0611)	(0.0539)	(0.0518)	(0.0499)	(0.0550)	(0.0551)	(0.0552)
	II	1.561	1.5606	1.5518	1.5429	1.5555	1.5563	1.5567
		(0.0550)	(0.0597)	(0.0574)	(0.0553)	(0.0566)	(0.0567)	(0.0568)
	III	1.5676	1.5616	1.5528	1.5439	1.5575	1.5582	1.5587
		(0.0563)	(0.0614)	(0.0590)	(0.0569)	(0.0601)	(0.0602)	(0.0603)
(90,50)	I	1.5643	1.5518	1.5439	1.5624	1.5008	1.5620	1.5613
		(0.05870)	(0.0531)	(0.0513)	(0.0497)	(0.0513)	(0.0512)	(0.0511)
	II	1.5643	1.5633	1.5553	1.5473	1.5656	1.5655	1.5645
		(0.0587)	(0.0571)	(0.0551)	(0.0532)	(0.0575)	(0.0575)	(0.0563)
	III	1.5604	1.5592	1.5513	1.5433	1.5561	1.5556	1.5549
		(0.05391)	(0.0519)	(0.0500)	(0.0483)	(0.0494)	(0.0493)	(0.0483)
(90,55)	I	1.5481	1.5472	1.5401	1.5329	1.5525	1.5531	1.5535
		(0.0467)	(0.0437)	(0.0423)	(0.0411)	(0.0467)	(0.0468)	(0.0495)
	II	1.5486	1.5472	1.5401	1.5329	1.5459	1.5466	1.5464
		(0.0475)	(0.0458)	(0.0444)	(0.0431)	(0.0503)	(0.0504)	(0.0505)
	III	1.5472	1.5473	1.5402	1.5330	1.5440	1.5446	1.5450
		(0.0457)	(0.0440)	(0.0426)	(0.0413)	(0.0420)	(0.0421)	(0.0421)

Table 4. AE of α and their MSE in parentheses based on MELF.

(n,m)	(CS)	Bayes	E-Bayesian			H-Bayesian		
		$\hat{\alpha}_{BM}$	$\hat{\alpha}_{EBM_1}$	$\hat{\alpha}_{EBM_2}$	$\hat{\alpha}_{EBM_3}$	$\hat{\alpha}_{HM_1}$	$\hat{\alpha}_{HM_2}$	$\hat{\alpha}_{HM_3}$
(40,30)	I	1.4334	1.4445	1.4326	1.4302	1.4464	1.4457	1.4446
		(0.0796)	(0.0744)	(0.0752)	(0.0786)	(0.0768)	(0.0768)	(0.0758)
	II	1.4469	1.4623	1.4577	1.4469	1.4542	1.4535	1.4524
		(0.0849)	(0.0758)	(0.0770)	(0.0829)	(0.0832)	(0.0833)	(0.0815)
	III	1.4340	1.4407	1.4483	1.4460	1.4474	1.4467	1.4455
		(0.07946)	(0.07577)	(0.07637)	(0.0751)	(0.0751)	(0.0760)	(0.0750)
(40,35)	I	1.4445	1.4605	1.4498	1.4390	1.4448	1.4457	1.4463
		(0.0636)	(0.0620)	(0.0594)	(0.0591)	(0.0593)	(0.0593)	(0.0593)
	II	1.4574	1.4595	1.4488	1.4381	1.4460	1.4469	1.4475
		(0.0685)	(0.0664)	(0.0660)	(0.0658)	(0.0610)	(0.0610)	(0.0610)
	III	1.4597	1.4616	1.4508	1.4401	1.4407	1.4417	1.4423
		(0.0652)	(0.0633)	(0.0618)	(0.0616)	(0.0718)	(0.0718)	(0.0718)
(60,40)	I	1.4472	1.4660	1.4566	1.4511	1.4686	1.4672	1.4681
		(0.0559)	(0.0535)	(0.0540)	(0.0549)	(0.0542)	(0.0532)	(0.0531)
	II	1.4508	1.4790	1.4694	1.4598	1.4712	1.4707	1.4698
		(0.0550)	(0.0530)	(0.0541)	(0.0549)	(0.0535)	(0.0532)	(0.0521)
	III	1.4462	1.4650	1.4556	1.4474	1.4656	1.4651	1.4642
		(0.05326)	(0.0512)	(0.0520)	(0.0530)	(0.0518)	(0.0529)	(0.0526)
(60,45)	I	1.4534	1.4679	1.4497	1.4415	1.4597	1.4605	1.4610
		(0.0533)	(0.0470)	(0.0467)	(0.0456)	(0.0466)	(0.0466)	(0.0466)
	II	1.4616	1.4699	1.4516	1.4434	1.4553	1.4560	1.4565
		(0.0510)	(0.0506)	(0.0503)	(0.0500)	(0.0480)	(0.0479)	(0.0479)
	III	1.4670	1.4609	1.4526	1.4438	1.4571	1.4579	1.4584
		(0.0523)	(0.0519)	(0.0514)	(0.0511)	(0.0542)	(0.0541)	(0.0541)
(90,50)	I	1.4580	1.4813	1.4737	1.4661	1.4716	1.4712	1.4705
		(0.0477)	(0.0454)	(0.0459)	(0.0460)	(0.0458)	(0.0456)	(0.0448)
	II	1.4515	1.4723	1.4677	1.4642	1.4746	1.4742	1.4746
		(0.0430)	(0.0411)	(0.0413)	(0.0411)	(0.0417)	(0.0418)	(0.0418)
	III	1.45342	1.47385	1.46096	1.45991	1.46564	1.46521	1.46453
		(0.0438)	(0.0415)	(0.0420)	(0.0439)	(0.0436)	(0.0440)	(0.0441)
(90,55)	I	1.4662	1.4631	1.4564	1.4496	1.4702	1.4708	1.4712
		(0.0419)	(0.0389)	(0.0385)	(0.0385)	(0.0406)	(0.0403)	(0.0401)
	II	1.4607	1.4650	1.4583	1.4515	1.4639	1.4646	1.4650
		(0.0436)	(0.0413)	(0.0401)	(0.0400)	(0.0448)	(0.0447)	(0.0446)
	III	1.4621	1.4651	1.4584	1.4516	1.4691	1.4697	1.4701
		(0.0410)	(0.0396)	(0.0385)	(0.0384)	(0.0397)	(0.0397)	(0.0397)

Table 5. AE of Hazard rate at t=0.5 and their MSE in parentheses based on SELF, $H(0.5) = 0.71833$.

(n,m)	(CS)	MLE	Bayes	E-Bayesian			H-Bayesian		
			$\hat{H}_{BS}$	$\hat{H}_{EBS_1}$	$\hat{H}_{EBS_2}$	$\hat{H}_{EBS_3}$	$\hat{H}_{HS_1}$	$\hat{H}_{HS_2}$	$\hat{H}_{HS_3}$
(40,30)	I	0.7489	0.7435	0.7425	0.7361	0.7297	0.7402	0.7408	0.7411
		(0.0198)	(0.0203)	(0.0190)	(0.0181)	(0.0173)	(0.0193)	(0.0198)	(0.0194)
	II	0.7507	0.7494	0.7492	0.7429	0.7364	0.7442	0.7447	0.7451
		(0.0225)	(0.0212)	(0.0193)	(0.0201)	(0.0191)	(0.0210)	(0.02115)	(0.0201)
	III	0.7490	0.7476	0.7383	0.7320	0.7257	0.7407	0.7412	0.7416
		(0.02014)	(0.0189)	(0.0197)	(0.0189)	(0.0181)	(0.0198)	(0.0196)	(0.0197)
(40,35)	I	0.7377	0.7369	0.7367	0.7313	0.7260	0.7331	0.73359	0.7338
		(0.0164)	(0.0156)	(0.0152)	(0.0145)	(0.0140)	(0.0146)	(0.0149)	(0.0145)
	II	0.7413	0.7403	0.7296	0.7243	0.7190	0.7337	0.7342	0.7344
		(0.0172)	(0.0164)	(0.0162)	(0.0156)	(0.0151)	(0.0154)	(0.0153)	(0.0152)
	III	0.7358	0.7349	0.7339	0.7286	0.7232	0.7310	0.7315	0.7310
		(0.0167)	(0.0158)	(0.0155)	(0.0149)	(0.0144)	(0.0166)	(0.0160)	(0.0161)
(60,40)	I	0.7494	0.7386	0.7385	0.7338	0.7290	0.7391	0.7395	0.7397
		(0.0157)	(0.0141)	(0.0139)	(0.0134)	(0.0130)	(0.0141)	(0.0142)	(0.0142)
	II	0.7550	0.7440	0.7432	0.7402	0.7354	0.7404	0.7408	0.7410
		(0.02175)	(0.01545)	(0.0153)	(0.0147)	(0.0141)	(0.0158)	(0.0157)	(0.0156)
	III	0.7402	0.7395	0.7380	0.7333	0.7285	0.7376	0.7380	0.7382
		(0.0155)	(0.0138)	(0.0136)	(0.0131)	(0.0127)	(0.0130)	(0.0129)	(0.0128)
(60,45)	I	0.7334	0.7328	0.7303	0.7261	0.7220	0.7311	0.7315	0.7317
		(0.0132)	(0.0127)	(0.0112)	(0.0109)	(0.0106)	(0.0115)	(0.0117)	(0.0118)
	II	0.7324	0.7319	0.7313	0.7271	0.7230	0.7289	0.7293	0.7295
		(0.0118)	(0.0114)	(0.0124)	(0.01211)	(0.01177)	(0.0116)	(0.0117)	(0.0118)
	III	0.7352	0.7346	0.7317	0.7276	0.7235	0.7298	0.7302	0.7304
		(0.0120)	(0.0116)	(0.0128)	(0.0124)	(0.0120)	(0.0131)	(0.0132)	(0.0133)
(90,50)	I	0.7351	0.7346	0.7287	0.7250	0.7213	0.7333	0.7335	0.7337
		(0.0127)	(0.0123)	(0.0112)	(0.0109)	(0.0106)	(0.0117)	(0.0118)	(0.0119)
	II	0.7351	0.7346	0.7341	0.7303	0.7266	0.7347	0.7350	0.7352
		(0.0111)	(0.0107)	(0.0119)	(0.0116)	(0.0113)	(0.0116)	(0.0117)	(0.0119)
	III	0.7327	0.7322	0.7322	0.7284	0.7247	0.7302	0.7305	0.7307
		(0.0111)	(0.0107)	(0.0108)	(0.0105)	(0.0102)	(0.0108)	(0.0109)	(0.0107)
(90,55)	I	0.7286	0.7283	0.7268	0.7235	0.7201	0.7303	0.7306	0.73308
		(0.0102)	(0.0099)	(0.0092)	(0.0090)	(0.0088)	(0.0098)	(0.0099)	(0.0010)
	II	0.7288	0.7285	0.7278	0.7244	0.7211	0.7272	0.7275	0.7277
		(0.0104)	(0.0101)	(0.0097)	(0.0095)	(0.0093)	(0.0106)	(0.0107)	(0.0108)
	III	0.7285	0.7282	0.7278	0.7245	0.7211	0.7298	0.7301	0.7301
		(0.0100)	(0.0097)	(0.0093)	(0.0091)	(0.0089)	(0.0097)	(0.0098)	(0.0099)

Table 6. AE of Hazard rate at t=0.5 their MSE in parentheses based on ELF.

(n,m)	(CS)	Bayes	E-Bayesian			H-Bayesian		
		$\hat{H}_{BE}$	$\hat{H}_{EBE_1}$	$\hat{H}_{EBE_2}$	$\hat{H}_{EBE_3}$	$\hat{H}_{HE_1}$	$\hat{H}_{HE_2}$	$\hat{H}_{HE_3}$
(40,30)	I	0.7233	0.7163	0.7101	0.7039	0.7221	0.7225	0.7229
		(0.0208)	(0.0180)	(0.0174)	(0.0169)	(0.0174)	(0.0176)	(0.0178)
	II	0.7249	0.7248	0.7185	0.7122	0.7201	0.7205	0.7207
		(0.0190)	(0.0172)	(0.0183)	(0.0176)	(0.0185)	(0.0188)	(0.0179)
	III	0.7233	0.7141	0.7080	0.7192	0.7165	0.7170	0.7173
		(0.0194)	(0.0181)	(0.01761)	(0.0171)	(0.0182)	(0.0178)	(0.0173)
(40,35)	I	0.7163	0.7160	0.7107	0.7055	0.7169	0.7129	0.7132
		(0.0144)	(0.0140)	(0.0136)	(0.0133)	(0.0138)	(0.0139)	(0.0135)
	II	0.7196	0.7091	0.7039	0.7098	0.7131	0.7135	0.7138
		(0.0150)	(0.0153)	(0.0149)	(0.0147)	(0.0146)	(0.0147)	(0.0148)
	III	0.7144	0.7133	0.7081	0.7029	0.7105	0.7110	0.7112
		(0.0147)	(0.0144)	(0.0141)	(0.0138)	(0.0146)	(0.0147)	(0.0148)
(60,40)	I	0.7205	0.7203	0.7156	0.7110	0.7160	0.7165	0.7169
		(0.0130)	(0.0122)	(0.0125)	(0.0121)	(0.0123)	(0.0124)	(0.0125)
	II	0.7266	0.7249	0.7219	0.7172	0.7198	0.7204	0.7207
		(0.0141)	(0.0140)	(0.0135)	(0.0132)	(0.0135)	(0.0138)	(0.0139)
	III	0.7213	0.7198	0.7152	0.7105	0.7194	0.7198	0.7200
		(0.0127)	(0.0126)	(0.0123)	(0.0120)	(0.0117)	(0.0113)	(0.0114)
(60,45)	I	0.7168	0.7142	0.7102	0.7071	0.7151	0.7154	0.7157
		(0.0120)	(0.01063)	(0.01044)	(0.01028)	(0.01184)	(0.01182)	(0.01181)
	II	0.7159	0.7152	0.7111	0.7071	0.7129	0.7133	0.7135
		(0.0127)	(0.0117)	(0.0115)	(0.0113)	(0.0119)	(0.0118)	(0.0116)
	III	0.7186	0.7157	0.7116	0.7076	0.7138	0.7142	0.7144
		(0.0128)	(0.0121)	(0.0118)	(0.0116)	(0.0122)	(0.0119)	(0.0117)
(90,50)	I	0.72019	0.71430	0.71068	0.70705	0.71905	0.71905	0.71925
		(0.0115)	(0.0107)	(0.0105)	(0.0103)	(0.0110)	(0.0114)	(0.0117)
	II	0.7169	0.7196	0.7159	0.7128	0.7157	0.7161	0.7163
		(0.00955)	(0.0090)	(0.0089)	(0.0086)	(0.0091)	(0.0098)	(0.0092)
	III	0.7205	0.7177	0.7140	0.7104	0.7156	0.7150	0.7150
		(0.0099)	(0.0088)	(0.0087)	(0.00879)	(0.0084)	(0.0088)	(0.0086)
(90,55)	I	0.7152	0.7137	0.7104	0.7072	0.7172	0.7175	0.7177
		(0.0094)	(0.0089)	(0.0089)	(0.0086)	(0.0093)	(0.0094)	(0.0095)
	II	0.7154	0.7147	0.7114	0.7081	0.7141	0.7144	0.7146
		(0.0096)	(0.0093)	(0.0091)	(0.0090)	(0.0101)	(0.0104)	(0.0107)
	III	0.7151	0.7147	0.7114	0.7081	0.7167	0.7170	0.7178
		(0.0092)	(0.0089)	(0.0088)	(0.0086)	(0.0093)	(0.0091)	(0.0094)

Table 7. AE of Hazard rate at t=0.5 and their MSE in parentheses based on WBLF.

(n,m)	(CS)	Bayes	E-Bayesian			H-Bayesian		
		$\hat{H}_{BW}$	$\hat{H}_{EBW_1}$	$\hat{H}_{EBW_2}$	$\hat{H}_{EBW_3}$	$\hat{H}_{HW_1}$	$\hat{H}_{HW_2}$	$\hat{H}_{HW_3}$
(40,30)	I	0.7676	0.7648	0.7582	0.7517	0.7644	0.7650	0.7653
		(0.0234)	(0.0227)	(0.0214)	(0.0209)	(0.0222)	(0.0220)	(0.0223)
	II	0.7740	0.7736	0.7672	0.7605	0.7685	0.7691	0.7694
		(0.02475)	(0.0226)	(0.0222)	(0.0219)	(0.0235)	(0.0236)	(0.0237)
	III	0.7719	0.7625	0.7560	0.7495	0.7649	0.7655	0.7658
		(0.0226)	(0.0221)	(0.0213)	(0.0202)	(0.0217)	(0.0226)	(0.0219)
(40,35)	I	0.7575	0.75753	0.75200	0.74646	0.75374	0.75420	0.75449
		(0.0176)	(0.0172)	(0.0163)	(0.0155)	(0.0160)	(0.0165)	(0.0168)
	II	0.7610	0.7502	0.7447	0.7393	0.7543	0.7548	0.7551
		(0.01866)	(0.0180)	(0.0172)	(0.0164)	(0.0171)	(0.0171)	(0.0172)
	III	0.7555	0.7546	0.7491	0.7436	0.7516	0.7521	0.7524
		(0.0178)	(0.0175)	(0.0166)	(0.0158)	(0.0181)	(0.0187)	(0.0188)
(60,40)	I	0.7583	0.7567	0.7519	0.7470	0.7573	0.7577	0.7579
		(0.0158)	(0.0147)	(0.0142)	(0.0143)	(0.0148)	(0.0143)	(0.0146)
	II	0.7634	0.7614	0.7585	0.7535	0.7586	0.7590	0.7593
		(0.0174)	(0.0173)	(0.0164)	(0.0158)	(0.0174)	(0.0165)	(0.0168)
	III	0.7576	0.7562	0.7514	0.7465	0.7557	0.7561	0.7564
		(0.0156)	(0.0149)	(0.0146)	(0.0140)	(0.0147)	(0.0146)	(0.0149)
(60,45)	I	0.7489	0.7463	0.7421	0.7379	0.7472	0.7475	0.7478
		(0.0140)	(0.0123)	(0.0118)	(0.0114)	(0.0122)	(0.0125)	(0.0127)
	II	0.7479	0.7473	0.7431	0.7589	0.7449	0.7452	0.7455
		(0.0146)	(0.0137)	(0.0131)	(0.0127)	(0.0126)	(0.0127)	(0.0128)
	III	0.7507	0.7478	0.7436	0.7394	0.7458	0.7462	0.7464
		(0.0142)	(0.0140)	(0.0135)	(0.0130)	(0.0148)	(0.0141)	(0.0143)
(90,50)	I	0.7491	0.7431	0.7393	0.7356	0.7477	0.7480	0.7482
		(0.0136)	(0.0121)	(0.0117)	(0.0114)	(0.0122)	(0.0119)	(0.0117)
	II	0.7457	0.7448	0.7486	0.7410	0.7492	0.7495	0.7497
		(0.0134)	(0.0126)	(0.0121)	(0.0122)	(0.0125)	(0.0128)	(0.0126)
	III	0.7467	0.7467	0.7429	0.7391	0.7446	0.7456	0.7452
		(0.0120)	(0.0119)	(0.0114)	(0.0110)	(0.0106)	(0.0103)	(0.0105)
(90,55)	I	0.7413	0.7399	0.7365	0.7331	0.7463	0.7438	0.7439
		(0.0107)	(0.0100)	(0.0097)	(0.0094)	(0.0107)	(0.0105)	(0.0106)
	II	0.7416	0.7409	0.7375	0.7341	0.7403	0.7406	0.7408
		(0.0109)	(0.0105)	(0.0101)	(0.0098)	(0.0115)	(0.0117)	(0.0118)
	III	0.7413	0.7410	0.7375	0.7341	0.7394	0.7397	0.7399
		(0.0104)	(0.0100)	(0.0097)	(0.0094)	(0.0094)	(0.0096)	(0.0097)

Table 8. AE of Hazard rate at t=0.5 and their MSE in parentheses based on MELF.

(n,m)	(CS)	Bayes	E-Bayesian			H-Bayesian		
		$\hat{H}_{BM}$	$\hat{H}_{EBM_1}$	$\hat{H}_{EBM_2}$	$\hat{H}_{EBM_3}$	$\hat{H}_{HM_1}$	$\hat{H}_{HM_2}$	$\hat{H}_{HM_3}$
(40,30)	I	0.6925	0.6959	0.6899	0.6838	0.6918	0.7165	0.6927
		(0.0196)	(0.0173)	(0.0175)	(0.0176)	(0.0174)	(0.0176)	(0.0175)
	II	0.6981	0.70031	0.69422	0.68812	0.69554	0.72045	0.69644
		(0.0199)	(0.0180)	(0.0176)	(0.0173)	(0.0185)	(0.0198)	(0.0188)
	III	0.6867	0.6899	0.6840	0.6781	0.6922	0.7170	0.6931
		(0.0177)	(0.0175)	(0.0177)	(0.0174)	(0.0173)	(0.0172)	(0.0170)
(40,35)	I	0.6917	0.6945	0.6943	0.6891	0.6919	0.6923	0.6926
		(0.0149)	(0.0137)	(0.0137)	(0.0135)	(0.0138)	(0.01337)	(0.0136)
	II	0.6979	0.6989	0.6938	0.6886	0.6954	0.6929	0.6932
		(0.01657)	(0.0151)	(0.0151)	(0.01510)	(0.0150)	(0.0152)	(0.0156)
	III	0.6905	0.6999	0.6948	0.6896	0.6899	0.6905	0.6907
		(0.0165)	(0.0142)	(0.0141)	(0.0142)	(0.0158)	(0.0154)	(0.0157)
(60,40)	I	0.6997	0.7020	0.6975	0.6930	0.7026	0.7213	0.7033
		(0.0136)	(0.0125)	(0.0123)	(0.0122)	(0.0131)	(0.0135)	(0.0125)
	II	0.6957	0.7082	0.7036	0.6990	0.7039	0.7225	0.7045
		(0.0135)	(0.0133)	(0.0131)	(0.0129)	(0.0123)	(0.0134)	(0.0129)
	III	0.6926	0.7016	0.6971	0.6931	0.7012	0.7198	0.7019
		(0.0123)	(0.0121)	(0.0121)	(0.0119)	(0.0116)	(0.0122)	(0.0116)
(60,45)	I	0.7008	0.6981	0.6942	0.6903	0.6990	0.6994	0.6996
		(0.0127)	(0.0115)	(0.0114)	(0.0117)	(0.0119)	(0.0110)	(0.0116)
	II	0.6999	0.6991	0.6952	0.6912	0.6969	0.6973	0.6975
		(0.0126)	(0.0116)	(0.0118)	(0.0114)	(0.0116)	(0.0118)	(0.0114)
	III	0.7025	0.6996	0.6956	0.6916	0.6978	0.6981	0.6984
		(0.0126)	(0.0119)	(0.0118)	(0.0117)	(0.0123)	(0.0124)	(0.0121)
(90,50)	I	0.6982	0.7094	0.6963	0.6927	0.7042	0.7190	0.7047
		(0.0108)	(0.0105)	(0.0104)	(0.0103)	(0.0106)	(0.0105)	(0.0104)
	II	0.6999	0.7050	0.7014	0.6978	0.7056	0.7105	0.7061
		(0.0109)	(0.0108)	(0.0108)	(0.0107)	(0.0112)	(0.0115)	(0.0119)
	III	0.6960	0.7032	0.6996	0.6960	0.7013	0.7161	0.7018
		(0.0101)	(0.0100)	(0.0099)	(0.0093)	(0.0109)	(0.0908)	(0.0102)
(90,55)	I	0.7021	0.7006	0.6988	0.6942	0.70407	0.7043	0.7045
		(0.0095)	(0.0090)	(0.0088)	(0.0083)	(0.0091)	(0.0095)	(0.0092)
	II	0.7024	0.7016	0.6983	0.6951	0.7010	0.7013	0.7015
		(0.0097)	(0.0094)	(0.0092)	(0.0091)	(0.0102)	(0.0101)	(0.0102)
	III	0.7021	0.7016	0.6984	0.6952	0.7035	0.7038	0.7040
		(0.0093)	(0.0082)	(0.0089)	(0.0087)	(0.0092)	(0.0091)	(0.0090)

7. Application

In this section, a real-life endurance test failure data of ball bearings reported in Lawless [22] have been considered to analyze for further illustrative purposes. The set is given below:

17.88, 28.92, 33.0, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

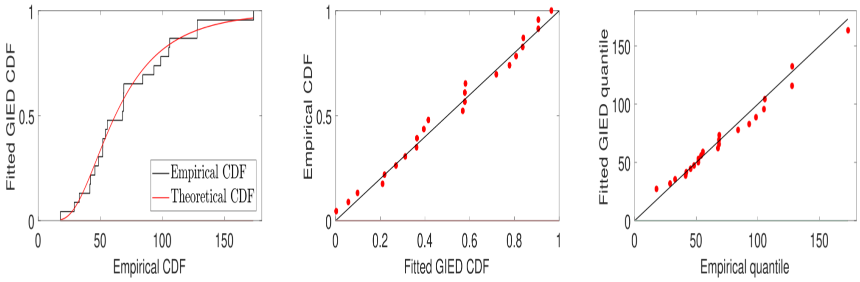
Before analyzing the data set, we have to check whether the data set is fit or not with GIED. For the purpose of the goodness of fit test to check whether the data set fits the GR distribution, Kolmogorov-Smirnov (K-S) test has been employed here. From K-S test, we get the K-S distance as 0.0916 and the associated p-value as 0.9806. In Figure 1, the empirical CDF (ECDF), probability-probability (P-P) plot, and quantile-quantile (Q-Q) plot are represented for the purpose of goodness-of-fit test of the given data with GIED. These results yields that the ball bearings data set fits the GIED quiet well.

Table 9. Average estimates of the parameter α.

(n,m)	(CS)	MLE	Loss function	Bayes	E-Bayesian			H-Bayesian		
				$\hat{\alpha}_B$	$\hat{\alpha}_{EB_1}$	$\hat{\alpha}_{EB_2}$	$\hat{\alpha}_{EB_3}$	$\hat{\alpha}_{H_1}$	$\hat{\alpha}_{H_2}$	$\hat{\alpha}_{H_3}$
15	I	5.8064	SELF	5.6650	5.3478	5.0717	4.7956	4.3766	4.3821	4.3854
	II	9.3323		7.7857	8.1004	7.4982	6.8960	6.0213	6.0284	6.0326
	III	11.7665		8.9238	9.8376	8.9761	8.1145	6.9044	6.9124	6.9171
20	I	5.2166		5.2338	4.9346	4.7513	4.5679	4.2786	4.2826	4.2851
	II	7.0768		6.6124	6.5235	6.2104	5.8973	5.4087	5.4137	5.4166
	III	7.1514		6.6638	6.5856	6.2668	5.9480	5.4508	5.4558	5.4588
15	I		ELF	5.3860	5.0028	4.7445	4.4862	4.0965	4.1021	4.1054
	II			7.4022	7.5778	7.0144	6.4511	5.6364	5.6437	5.6479
	III			8.4842	9.2029	8.3970	7.5910	6.4634	6.4714	6.4762
20	I			5.0269	4.6939	4.5195	4.3451	4.0711	4.0752	4.0777
	II			6.3510	6.2052	5.9074	5.6096	5.1467	5.1517	5.1546
	III			6.4004	6.2643	5.9611	5.6578	5.1868	5.1918	5.1948
15	I		WBLF	5.9441	5.6928	5.3989	5.1050	4.6566	4.6621	4.6653
	II			8.1693	8.6230	7.9819	7.3409	6.4061	6.4131	6.4173
	III			9.3634	1.0472	9.5552	8.6380	7.3454	7.3533	7.3579
20	I			5.4407	5.1753	4.9830	4.7907	4.4860	4.4900	4.4924
	II			6.8738	5.6707	5.6756	5.6785	5.6707	5.6756	5.6785
	III			6.9272	6.9068	6.5725	6.2381	5.7149	5.7198	5.7227
15	I		MELF	5.1069	4.6578	4.4173	4.1768	3.8164	3.8220	3.8253
	II			7.0187	7.05521	6.5307	6.0062	5.2514	5.2588	5.2631
	III			8.0446	8.5682	7.8179	7.0675	8.5682	7.8179	7.0675
20	I			4.8201	4.4532	4.2877	4.1222	3.8637	3.8678	3.8703
	II			6.0897	5.8870	5.6045	5.3219	5.8870	5.6045	5.3219
	III			6.1370	5.9431	5.6554	5.3677	4.9226	4.9277	4.9307

Table 10. Average estimates ($*e^{-110}$) of the Hazard Rate,true value H(0.5)= $3.37131e^{-110}$.

(n,m)	(CS)	MLE	Loss function	Bayes	E-Bayesian			H-Bayesian		
				$\hat{H}_B$	$\hat{H}_{EB_1}$	$\hat{H}_{EB_2}$	$\hat{H}_{EB_3}$	$\hat{H}_{H_1}$	$\hat{H}_{H_2}$	$\hat{H}_{H_3}$
15	I	3.6881	SELF	3.5983	3.3968	3.2215	3.0461	2.7799	2.7834	2.7855
	II	5.9277		4.9454	5.1452	4.7627	4.3802	3.8246	3.8291	3.8318
	III	7.4739		5.6682	6.2487	5.7014	5.1542	4.3856	4.3906	4.3936
20	I	3.3135		3.3244	3.1344	3.0179	2.9014	2.7177	2.7202	2.7218
	II	4.4951		4.2001	4.1436	3.9447	3.4355	3.4387	3.4405	3.7458
	III	4.5425		4.2327	4.1830	3.9805	3.7780	3.4623	3.4654	3.4673
15	I		ELF	3.4211	3.1777	3.0136	2.8496	2.6020	2.6056	2.6077
	II			4.7017	4.8133	4.4554	4.0976	3.5801	3.5847	3.5875
	III			5.3890	5.8455	5.3336	4.8217	4.1054	4.1105	4.1136
20	I			3.1930	2.9815	2.8707	2.7599	2.5859	2.5885	2.5901
	II			4.3929	3.9414	3.7523	3.5631	3.2691	3.2722	3.2741
	III			4.0654	3.9790	3.7864	3.5937	3.2945	3.2977	3.2996
15	I		WBLF	3.7756	3.6160	3.4293	3.2426	2.9578	2.9612	2.9633
	II			5.1890	5.4772	5.0700	4.6628	4.0690	4.0735	4.0761
	III			5.9475	6.6518	6.0693	5.4867	4.6657	4.6707	4.6736
20	I			3.4558	3.2873	3.1651	3.0430	2.8494	2.8520	2.8535
	II			4.3661	4.3457	4.1371	3.9286	3.6019	3.6051	3.6069
	III			4.4000	4.3871	4.1747	3.9623	3.6300	3.6331	3.6350
15	I		MELF	3.2438	2.9585	2.8058	2.6530	2.4241	2.4276	2.4298
	II			4.4581	4.4813	4.1481	3.8150	3.3356	3.3403	3.3430
	III			5.1098	5.4424	4.9658	4.4891	3.8251	3.8304	3.8334
20	I			3.0616	2.8286	2.7235	2.6184	2.4541	2.4568	2.4583
	II			3.8680	3.7393	3.5598	3.3804	3.1026	3.1058	3.1077
	III			3.8981	3.7749	3.5922	3.4095	3.1267	3.1300	3.1319



(a) (b) (c)

Figure 1. (a) ECDF and CDF comparison, (b) Q-Q plot, and (c) P-P plot for the GIED fitted to ball bearings data set.

The MLE, Bayes, E-Bayes, and H-Bayes estimates values and the corresponding hazard rate values are obtained for the real data set and tabulated in Tables 9 and 10. From these tables, it has been observed that when m increases, the value of the estimates becomes closer to the estimated values for the entire sample. The values of the estimates under Scheme I outperform the other two schemes. Similarly to simulation studies, the E-Bayes estimates perform better than other estimates in real data set analysis.

8. Conclusion

In this study, E-Bayesian and H-Bayesian estimation for the unknown shape parameter, and hazard rate functions of GIED are proposed with Type II PCS. To obtain the Bayes, E- and H-Bayes estimates four different types of loss functions are introduced. A Monte Carlo simulation study has been performed to compare the performance of the estimates of parameters such as the shape parameter, and hazard function. In terms of MSE, the simulation study yields that E-Bayes estimates outperform all other estimates. Finally, a ball bearings data set has been analyzed to illustrate the applicability of the proposed estimates. After analyzing this data set, it is concluded that E-Bayes estimates for the parameters and the hazard function perform better than other estimates. the limitation of this study lies in the assumption of the parameter λ to be known. For future work, one may consider the estimation of reliability characteristics under E- and H-Bayesian approach under different censoring schemes and having different lifetime distributions.

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