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Traveling Wave Solutions of the RLW-Burgers Equation and Potential Kdv Equation by Using the $\left(\frac{G'}{G}\right)$ - Expansion Method

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Abstract

In this paper, we implemented the $\left(\frac{G'}{G}\right)$- expansion method for the traveling wave solutions of the RLW-Burgers equation and potential KdV equation. By using this scheme, we found some traveling wave solutions of the above-mentioned equations.

Key Words. RLW-Burgers equation, Potential KdV equation, $\left(\frac{G'}{G}\right)$- expansion method, Traveling wave solutions.

Özet

Bu çalışmada, RLW-Burgers ve potansiyel KdV denklemlerinin hareket eden dalga çözümleri için $\left(\frac{G'}{G}\right)$- açılım metodu sunulur. Bu metot yardımı ile yukarıda bahsedilen denklemlerin bazı hareket eden dalga çözümleri bulunur.

Anahtar kelimeler. RLW-Burgers denklemi, Potansiyel KdV denklemi, $\left(\frac{G'}{G}\right)$- açılım metot, Hareket eden dalga çözümler.

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1. Introduction

In this work, we will consider to solve the traveling wave solutions of the RLW-Burgers equation and potential KdV equation by using the $\left(\frac{G'}{G}\right)$- expansion method which is introduced by Wang, Li and Zhang [1]. Many authors applied this method to various equations [2-7].

Nonlinear phenomena play a crucial role in applied mathematics and physics. Calculating exact and numerical solutions, in particular, traveling wave solutions, of nonlinear equations in mathematical physics plaices an important role in soliton theory [8,9]. Recently, it has become more interesting that obtaining exact solutions of nonlinear partial differential equations through using symbolical computer programs such as Maple, Matlab, Mathematica that facilitate complex and tedious algebraical computations. It is too important to find exact solutions of nonlinear partial differential equations. These equations are mathematical models of complex physical occurrences that arise in engineering, chemistry, biology, mechanics and physics. Various effective methods have been developed to understand the mechanisms of these physical models, to help physicians and engineers and to ensure knowledge for physical problems and its applications. Many explicit exact and numerical methods have been introduced in literature [10–29]. Some of them are: Bäcklund transformation, generalized Miura transformation, Darboux transformation, Cole–Hopf transformation, tanh method, sine–cosine method, Painlevé method, homogeneous balance method, similarity reduction method and so on.

Traveling wave solutions of many nonlinear differential equations can be stated with tanh function terms [30, 31]. The tanh function terms firstly were used on base *ad hoc* in 1990 and 1991 [32, 33]. Then, Malfliet [34] formalized the tanh method in 1992 and illustrated it with several examples, Parkes and Duffy presented the automatic tanh method [35] in 1996, after, Fan defined the extended tanh method [36] in 2000, later Elwakil presented the modified extended tanh method [37] in 2002, separately, the generalized extended tanh method [38] by Zheng in 2003, the improved extended tanh method [39] by Yomba in 2004, the tanh function method [40] by Chen and Zhang in 2004.

2. An Analysis of the Method and applications

Before starting to give the $\left(\frac{G'}{G}\right)$- expansion method, we will give a simple description of the $\left(\frac{G'}{G}\right)$- expansion method. For doing this, one can consider in a two-variables general form of nonlinear PDE

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0, \tag{1}$$

and transform Eq.(1) with $u(x,t) = u(\xi)$, $\xi = x - wt$ where w is constant. After transformation, we get a nonlinear ODE for $u(\xi)$

$$Q'(u', u'', u''', \dots) = 0. \tag{2}$$

The solution of the equation (2) we are looking for is expressed as

$$u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \cdots, \tag{3}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{4}$$

where $\alpha_m, \dots, \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$, the positive integer m can be determined by balancing the highest order derivative and with the highest nonlinear terms into equation (2). Substituting solution (3) into equation (2) and using (4) yields a set of algebraic equations for same order of $\left(\frac{G'}{G}\right)$; then all coefficients same order of $\left(\frac{G'}{G}\right)$ have to vanish. After this separated algebraic equation, we can find $\alpha_m, \dots, w, \lambda$ and μ constants. General solutions of the equation (4) have been well known us, then substituting $\alpha_m, \dots, w$ and the general solutions of equation (4) into (3) we have more traveling wave solutions of equation (1) [1].

Example 1. Consider the RLW-Burgers equation,

$$u_t + u_x + 12uu_x - au_{xx} - bu_{xxt} = 0. \tag{5}$$

For doing this example, we can use transformation with Eq. (1) then Eq. (5) become

$$-wu' + u' + 12uu' - au'' + wbu''' = 0. \tag{6}$$

When balancing uu' with u''' then gives *m=2*. Therefore, we may choose

$$u(\xi) = \alpha_2 \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \tag{7}$$

Substituting equation (7) into (6) yields a set of algebraic equations for $\alpha_0, \alpha_1, \alpha_2$. These systems are

$$-\alpha_1\mu + w\alpha_1\mu - 12\alpha_0\alpha_1\mu - a\alpha_1\lambda\mu - bw\alpha_1\lambda^2\mu - 2bw\alpha_1\mu^2 - 2a\alpha_2\mu^2 - 6bw\alpha_2\lambda\mu^2 = 0,$$

$$-\alpha_1\lambda + w\alpha_1\lambda - 12\alpha_0\alpha_1\lambda - a\alpha_1\lambda^2 - bw\alpha_1\lambda^3 - 2a\alpha_1\mu - 12\alpha_1^2\mu - 2\alpha_2\mu + 2w\alpha_2\mu$$
$$-24\alpha_0\alpha_2\mu - 8bw\alpha_1\lambda\mu - 6a\alpha_2\lambda\mu - 14bw\alpha_2\lambda^2\mu - 16bw\alpha_2\mu^2 = 0,$$
$$-\alpha_1 + w\alpha_1 - 12\alpha_0\alpha_1 - 3a\alpha_1\lambda - 12\alpha_1^2\lambda - 2\alpha_2\lambda + 2w\alpha_2\lambda - 24\alpha_0\alpha_2\lambda - 7bw\alpha_1\lambda^2 - 4a\alpha_2\lambda^2$$
$$-8bw\alpha_2\lambda^3 - 8bw\alpha_1\mu - 8a\alpha_2\mu - 36\alpha_1\alpha_2\mu - 52bw\alpha_2\lambda\mu = 0,$$
$$-2a\alpha_1 - 12\alpha_1^2 - 2\alpha_2 + 2w\alpha_2 - 24\alpha_0\alpha_2 - 12bw\alpha_1\lambda - 10a\alpha_2\lambda - 36\alpha_1\alpha_2\lambda - 38bw\alpha_2\lambda^2$$
$$-40bw\alpha_2\mu - 24\alpha_2^2\mu = 0,$$
$$-6bw\alpha_1 - 6a\alpha_2 - 36\alpha_1\alpha_2 - 54bw\alpha_2\lambda - 24\alpha_2^2\lambda = 0,$$
$$-24bw\alpha_2 - 24\alpha_2^2 = 0. \qquad (8)$$

From the solutions system, we obtain the following with the aid of Mathematica.

$$\alpha_0 = \frac{1}{60}\left(-5 - 6a\lambda + \frac{a}{b\sqrt{\lambda^2-4\mu}} - \frac{12a\mu}{\sqrt{\lambda^2-4\mu}}\right), \alpha_1 = \frac{1}{5}\left(-a - \frac{a\lambda}{\sqrt{\lambda^2-4\mu}}\right),$$
$$\alpha_2 = -\frac{a}{5\sqrt{\lambda^2-4\mu}}, w = \frac{a}{5b\sqrt{\lambda^2-4\mu}}, a \neq 0, b \neq 0, \lambda^2 - 4\mu \neq 0. \qquad (9)$$

Substituting (9) into (7) we have three types' solutions of equation (5):

(i) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions,

$$u_1(\xi) = -\frac{a}{5}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 Sinh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi] + C_2 Cosh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}{C_1 Cosh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi] + C_2 Sinh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}\right)\right) -$$
$$-\frac{a}{5\sqrt{\lambda^2-4\mu}}\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1 Sinh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi] + C_2 Cosh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}{C_1 Cosh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi] + C_2 Sinh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}\right)\right)^2 + \frac{a}{60b\sqrt{\lambda^2-4\mu}} +$$
$$+\frac{a\lambda^2}{20\sqrt{\lambda^2-4\mu}} - \frac{a\mu}{5\sqrt{\lambda^2-4\mu}} - \frac{1}{12}$$

where $\xi = \left(x + \frac{a}{5b\sqrt{\lambda^2-4\mu}}t\right)$, C_1 and C_2 are arbitrary constants.

(ii) When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions,

$$u_2(\xi)=-\frac{a}{5}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 Sin[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]+C_2 Cos[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]}{C_1 Cos[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]+C_2 Sin[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}\right)\right)-$$

$$-\frac{a}{5\sqrt{\lambda^2-4\mu}}\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1 Sin[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]+C_2 Cos[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]}{C_1 Cos[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]+C_2 Sin[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]}\right)\right)^2+\frac{a}{60b\sqrt{\lambda^2-4\mu}}+$$

$$+\frac{a\lambda^2}{20\sqrt{\lambda^2-4\mu}}-\frac{a\mu}{5\sqrt{\lambda^2-4\mu}}-\frac{1}{12}$$

where $\xi=\left(x+\frac{a}{5b\sqrt{\lambda^2-4\mu}}t\right)$, C_1 and C_2 are arbitrary constants.

Example 2. Consider the potential KdV equation,

$$u_t+3u_x^2+u_{xxx}=0. \tag{10}$$

For doing this example, we can use transformation with Eq. (1) then Eq. (10) become

$$-wu'+3(u')^2+u'''=0, \tag{11}$$

when balancing $(u')^2$ with u''' then gives *m=1*. Therefore, we may choose

$$u(\xi)=\alpha_1\left(\frac{G'}{G}\right)+\alpha_0, \tag{12}$$

substituting equation (12) into (11) yields a set of algebraic equations for α_0,α_1. These systems are

$$\begin{gathered}w\alpha_1\mu-\alpha_1\lambda^2\mu-2\alpha_1\mu^2+3\alpha_1^2\mu^2=0,\ w\alpha_1\lambda-\alpha_1\lambda^3-8\alpha_1\lambda\mu+6\alpha_1^2\lambda\mu=0,\\ w\alpha_1-7\alpha_1\lambda^2+3\alpha_1^2\lambda^2-8\alpha_1\mu+6\alpha_1^2\mu=0,\ -12\alpha_1\lambda+6\alpha_1^2\lambda=0,\ -6\alpha_1+3\alpha_1^2=0.\end{gathered} \tag{13}$$

From the solutions system, we obtain the following with the aid of Mathematica.

$$\alpha_0=\alpha_0,\quad \alpha_1=2,\ w=\lambda^2-4\mu. \tag{14}$$

Substituting (14) into (12) we have three types’ solutions of equation (10)

(i) When $\lambda^2-4\mu>0$, we obtain the hyperbolic function traveling wave solutions,

$$u_1(\xi)=2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A_1 Sinh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]+A_2 Cosh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}{A_1 Cosh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]+A_2 Sinh[\frac{\sqrt{\lambda^2-4\mu}}{2}\xi]}\right)\right)+\alpha_0-\lambda.$$

where $\xi=\left[x-\left(\lambda^2-4\mu\right)t\right]$, A_1 and A_2 are arbitrary constants.

(ii) When $\lambda^2-4\mu$<0, we obtain the trigonometric function traveling wave solutions,

$$u_2(\xi)=2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-A_1 Sin[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]+A_2 Cos[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]}{A_1 Cos[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]+A_2 Sin[\frac{\sqrt{4\mu-\lambda^2}}{2}\xi]}\right)\right)+\alpha_0-\lambda.$$

where $\xi=\left[x-\left(\lambda^2-4\mu\right)t\right]$ A_1 and A_2 are arbitrary constants.

(iii) When $\lambda^2-4\mu$=0, we obtain the rational function solutions,

$$u_3(\xi)=2\left(\frac{A_2}{A_1+A_2x}\right)+\alpha_0-\lambda.$$

where A_1 and A_2 are arbitrary constants.

3. Conclusions

In this work, we consider to solve the traveling wave solutions of the RLW-Burgers equation and potential KdV equation by using the $\left(\frac{G'}{G}\right)$- expansion method. The method [1] can be used to many other nonlinear equations or coupled ones. In addition, this method is also computerizable, which allows us to perform complicated and tedious algebraic calculation on a computer.

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